ON THE MACINTYRE CONJECTURE

BY
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1. Introduction

Let

$$f(z) = \sum_{k=0}^{\infty} c_k z^{n_k} \tag{1}$$

where $\{n_k\}$ is an increasing sequence of non-negative integers satisfying

$$n_0 = 0 \quad \text{and} \quad \sum_{k=1}^{\infty} 1/n_k < \infty. \tag{2}$$

Macintyre [10] conjectured that if $f(z)$ is an entire function of the form (1) with the gap condition (2), then f has no finite asymptotic values. Macintyre proved the conjecture for radial paths, and Fuchs [5] implicitly proved the conjecture for functions of finite order. Kovari [8], [9], Gaier [6], and most recently Anderson and Binmore [1] have shown the nonexistence of finite asymptotic values for functions with stronger gap conditions than (2). In this paper we obtain the desired conclusion for the gap condition (2), but we must restrict the rate of growth of the function. We have

THEOREM 1. *Let $f(z)$ be an entire function of the form (1) for which $\{n_k\}$ satisfies (2). Suppose $f(z)$ has finite lower order μ. Then f has no finite asymptotic value.*

The ideas of the proof of Theorem 1 also yield two theorems which are related to theorems of Gaier [7] and Anderson and Binmore [1].

THEOREM 2. *Let $f(z)$ be an entire function of the form (1) for which $\{n_k\}$ satisfies (2). Suppose $f(z)$ has finite lower order μ. If, for some positive integer n,*

$$|f(z)| = O(|z|^n)$$

on a path Γ receding to ∞ (a Jordan curve joining zero to infinity), then f is a polynomial of degree at most n.

THEOREM 3. *Let $f(z)$ be an entire function of the form (1) for which $\{n_k\}$ satisfies (2). Suppose $f(z)$ has finite lower order μ. If, for some $\alpha > 0$,*

$$|f(z)| = O(e^{|z|^\alpha})$$

on a path Γ receding to ∞, then f is of order at most α.

Gaier proved Theorems 2 and 3 for radial paths Γ without the growth assumption on f, and Anderson and Binmore proved Theorems 2 and 3 for a stronger gap condition than (2) without the growth assumption on f.

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2. Statements of preliminary lemmas

The following lemmas are needed to obtain the stated theorems. Lemma 1 is closely related to Lemma 2 in [11] and will be proved in Section 5.

LEMMA 1. *Let* $\{n_k\}$ *be a strictly increasing sequence of integers satisfying* (2). *For each sequence* $s_0, s_1, s_2, \cdots$, *with* $s_k = \pm 1$ *there exists a real-valued function* $g(t)$ *in* $0 \leq t \leq 1$ *such that*

(i) $s_k \int_0^1 g(t) t^{n_k}\, dt = B_k > 0$,

(ii) $\int_0^1 | g(t) |\, dt \leq \frac{1}{2}$,

(iii) $\inf_k B_k \rho^{-n_k} > \Lambda(\rho, \varepsilon)$

for each fixed ε *in* $0 < \varepsilon < 1$ *and* $0 < \rho < e^{-\varepsilon}$ *where*

$$\Lambda(\rho, \varepsilon) = K(\log \rho^{-1} - \varepsilon)^2 (\exp\{-\varepsilon 2(\log \rho^{-1} - \varepsilon)^{-1}\})\, \rho^{-2(\log \rho^{-1} - \varepsilon)^{-1}}.$$

(K *is a constant independent of the sequence* $\{s_k\}$).

LEMMA 2 (Edrei [3]). *Let* f *be an entire function for which*

$$\liminf_{r \to \infty} (\log \log M(r))/\log r = \mu < +\infty,$$

where $M(r) = \max_{|z|=r} |f(z)|$. *Then there exists a sequence of Polya peaks for* $\log M(r)$. *That is, there exists an unbounded positive sequence* $r_1, r_2, r_3, \cdots$ *which is strictly increasing and four sequences of nonnegative terms*

$$\{\varepsilon_k\}, \{a_k\}, \{\xi_k\}, \{A_k\}$$

such that

$$\lim_{k \to \infty} \varepsilon_k = \lim_{k \to \infty} a_k = \lim_{k \to \infty} \xi_k = 0, \qquad \lim_{k \to \infty} A_k = \lim_{k \to \infty} a_k r_k = +\infty,$$

and such that the inequalities $a_k r_k \leq x \leq A_k r_k$ *imply*

$$\log M(x) \leq (1 + \xi_k)(x/r_k)^{\mu + \varepsilon_k} \log M(r_k).$$

LEMMA 3. (Edrei [4]). *Let* $G(t)$ *be a real, continuous, non-decreasing function defined for* $t \geq t_0 > 0$. *Assume that there exist* σ, τ *such that* $0 < \sigma < \tau$ *and such that*

$$\limsup_{t \to \infty} \frac{G(t)}{t^\tau} = +\infty, \qquad \liminf_{t \to \infty} \frac{G(t)}{t^\sigma} = 0.$$

Then there exist arbitrarily large values r' *such that simultaneously*

(i) $(r')^\tau \leq G(r')$,

(ii) $G(t)/t^\tau \leq G(r')/(r')^\tau$, $(t_0 \leq t \leq (r')^{\tau/\sigma})$.

Lemma 3 forms part of the proof of Lemma 2.

3. Proof of Theorems 1 and 2

Suppose the theorem is false, and let Γ be a path tending to ∞ along which $|f(z)|$ is bounded by $L < \infty$. Choose $0 < \varepsilon < 1$ such that $2\varepsilon\mu < \log 4/3$. Let z_0 be the first point at which Γ intersects $|z| = r$, and let $\arg z = \theta$ be the ray joining the origin and the point z_0.

We assume first $\theta = 0$ and estimate

$$U(r) = \sup_{0 \le u \le r} | f(u) |$$

from below. Let

$$s_k = \text{sign}(\text{Re } c_k) = 1 \quad \text{if Re } c_k \ge 0,$$
$$= -1 \quad \text{if} \quad \text{Re } c_k < 0.$$

Construct the function $g(t)$ of Lemma 2 for this choice of $\{s_k\}$. Then, for $0 < \rho < e^{-\varepsilon}$,

$$\Lambda(\rho, \varepsilon) \cdot \sum | \text{Re } c_k(\rho r)^{n_k} | \le \{\inf_k B_k \rho^{-n_k}\} \cdot \sum | \text{Re } c_k(\rho r)^{n_k} |$$
$$\le \sum | \text{Re } c_k(\rho r)^{n_k} | B_k \rho^{-n_k}.$$

The right-hand side is equal to

$$\sum \text{Re } c_k r^{n_k} \int_0^1 t^{n_k} g(t)\, dt = \text{Re} \int_0^1 f(rt) g(t)\, dt$$
$$\le \int_0^1 | f(rt) | \, | g(t) | \, dt$$
$$\le U(r) \int_0^1 | g(t) | \, dt \le \tfrac{1}{2} U(r).$$

Hence, for $0 < \rho < e^{-\varepsilon}$,

$$\Lambda(\rho, \varepsilon) \cdot \sum | \text{Re } c_k(\rho r)^{n_k} | \le \tfrac{1}{2} U(r). \tag{3}$$

By the same argument, for $0 < \rho < e^{-\varepsilon}$,

$$\Lambda(\rho, \varepsilon) \cdot \sum | \text{Im } c_k(\rho r)^{n_k} | \le \tfrac{1}{2} U(r). \tag{4}$$

But

$$\sum | \text{Re } c_k(\rho r)^{n_k} | + \sum | \text{Im } c_k(\rho r)^{n_k} | \ge \sum | c_k(\rho r)^{n_k} | \ge M(\rho r).$$

Thus, adding (3) and (4), we obtain, for $0 < \rho < e^{-\varepsilon}$,

$$U(r) \ge \Lambda(\rho, \varepsilon) \cdot M(\rho r). \tag{5}$$

By considering $f(ue^{i\theta})$ instead of $f(u)$ and noting that K in $\Lambda(\rho, \varepsilon)$ is independent of the sequence $\{s_k\}$, we see, for $0 < \rho < e^{-\varepsilon}$,

$$\sup_{0 \le u \le r} | f(ue^{i\theta}) | \ge \Lambda(\rho, \varepsilon) M(\rho r). \tag{6}$$

We now use a reflection trick of Polya. Let $\bar{\Gamma}$ be the reflection of Γ across the radius joining the origin and z_0. Since $| f(z) | \le M(r)$ for z on $\bar{\Gamma}$, the inequality

$$| f(z') | < M(r)^{1/2} \cdot L^{1/2}$$

holds for z' on the radius joining the origin and z_0. Hence, (6) becomes

$$M(\rho r) \cdot \Lambda(\rho, \varepsilon) \le M(r)^{1/2} \cdot L^{1/2} \tag{7}$$

for $0 < \rho < e^{-\varepsilon}$.

Take ρ to be $e^{-2\varepsilon}$. By Lemma 2 there exists a sequence $\{r_k\}$ of Polya peaks for $\log M(r)$. Since a_k approaches zero and A_k approaches infinity as k approaches infinity, there exists an integer k_0 such that for $k > k_0$,

$$a_k r_k < r_k < r_k/\rho < A_k r_k .$$

Hence, for $k > k_0$,

$$\log M(r_k/\rho) \leq (1 + \xi_k)(1/\rho)^{\mu+\varepsilon_k} \log M(r_k). \tag{8}$$

If we set $\rho r = r_k$ for $k > k_0$, we find that (7) and (8) imply

$$\log M(\rho r) + \log \Lambda(\rho, \varepsilon) \leq \tfrac{1}{2} \log M(r) + \tfrac{1}{2} \log L,$$

and

$$(1 + \xi_k)^{-1}\rho^{\mu+\varepsilon_k} \log M(r) + \log \Lambda(\rho, \varepsilon) \leq \tfrac{1}{2} \log M(r) + \tfrac{1}{2} \log L. \tag{9}$$

But since ε_k and ξ_k both approach zero as k approaches infinity and $2\varepsilon\mu < \log 4/3$, there exists an integer k_1 for which

$$(1 + \xi_k)^{-1}\rho^{\mu+\varepsilon_k} = (1 + \xi_k)^{-1}e^{-2\varepsilon(\mu+\varepsilon_k)} > 3/4 \tag{10}$$

when $k > k_1$. Thus, using (9) and (10), we see, for $k > \max(k_0, k_1)$,

$$\tfrac{1}{4} \log M(\rho^{-1}r_k) \leq \tfrac{1}{2} \log L - \log \Lambda(\rho, \varepsilon),$$

in contradiction to the fact that $\log M(\rho^{-1}r_k)$ approaches infinity as k approaches infinity.

We remark that Theorem 1 shows that there is no path tending to infinity along which $|f(z)|$ is bounded. Hence, Theorem 2 can be obtained by applying Theorem 1 to

$$g(z) = (f(z) - \textstyle\sum_{n_k<n} c_k z^{n_k})/z^n$$

(The author would like to thank Professor W. H. J. Fuchs for this latter simplification.)

4. Proof of Theorem 3

We split the proof into two parts according to whether f has regular or irregular growth (i.e., whether the order of f and the lower order of f are equal or distinct).

If f has regular growth, for each η, $0 < \eta < 1$, we choose $\varepsilon > 0$ so that $e^{-2\varepsilon\mu} > 1 - \eta/2$. Proceeding as in the proof of Theorem 1, we find (7) replaced by

$$M(\rho r)\cdot\Lambda(\rho, \varepsilon) \leq M(r)^{1/2} e^{r^{\alpha}/2}\cdot K',$$

for $0 < \rho < e^{-\varepsilon}$ where K' is a positive constant, and (9) replaced by

$$(1 + \xi_k)^{-1}\rho^{\mu+\varepsilon_k} \log M(r) + \log \Lambda(\rho, \varepsilon) \leq \tfrac{1}{2} \log M(r) + \log K' + r^{\alpha}/2 \tag{11}$$

where $\rho = e^{-2\varepsilon}$ and $\rho r = r_k$ for $k > k_0$.

Since ε_k and ξ_k both approach zero as k approaches infinity and $e^{-2\varepsilon\mu} > 1 - \eta/2$, there exists an integer k_1 such that for $k > k_1$ the inequality

(11) implies

$$(1 - \eta/2 - 1/2) \log M(r) \le (\log K'' - \log \Lambda(\rho, \varepsilon)) + r^{\alpha}/2,$$

and

$$(12) \qquad (1 - \eta) \log M(r) \le 2(\log K'' - \log \Lambda(\rho, \varepsilon)) + r^{\alpha}.$$

But (12) easily implies that $\mu \le \alpha$.

We now turn to the case when the lower order of f is strictly less than the order of f. Let τ be a real number for which $\tau > \mu$ and

$$\limsup_{t \to \infty} (\log M(t))/t^{\tau} = +\infty.$$

Choose σ to satisfy the conditions of Lemma 3 with $G(t) = \log M(t)$. Choose ε so that $\frac{1}{2}e^{2\varepsilon\tau} < \eta < 1$. Proceeding as in the proof of Theorem 1, we obtain (in place of (7))

$$(13) \qquad M(\rho r) \cdot \Lambda(\rho, \varepsilon) \le M(r)^{1/2} \cdot K'' e^{r^{\alpha}/2}$$

for $0 < \rho < e^{-\varepsilon}$ and K'' a positive constant.

Set $\rho = e^{-2\varepsilon}$. By Lemma 3 (with $\rho r = r'$) there exists a sequence S of radii r approaching infinity for which

$$\log M(r) \le (r^{\tau}/(\rho r)^{\tau}) \log M(\rho r).$$

Hence, for r in S, (13) implies

$$\begin{aligned} \log M(\rho r) + \log \Lambda(\rho, \varepsilon) &\le (1/2\rho^{\tau}) \log M(\rho r) + \log K'' + r^{\alpha}/2, \\ &\le \eta \log M(\rho r) + \log K'' + r^{\alpha}/2. \end{aligned}$$

Thus, for r in S,

$$(1 - \eta) \log M(\rho r) + \log (\Lambda(\rho, \varepsilon)(K'')^{-1}) \le r^{\alpha}/2.$$

However, (ii) of Lemma 3 gives

$$(14) \qquad \rho^{\tau} r^{\tau}(1 - \eta) + \log (\Lambda(\rho, \varepsilon)(K'')^{-1}) \le r^{\alpha}/2$$

for r in S, and since the radii in the sequence S approach infinity, (14) yields $\tau \le \alpha$. It follows from our choice of τ that the order of f is not greater than α.

5. Proof of Lemma 1

Consider the function

$$G(z) = (s_0/(z + 1)^2) \prod_{k=0}^{\infty} (m_k + 1 - z)/(m_k + 1 + z),$$

where the m_i are the midpoints of the segments (n_k, n_{k+1}) for which s_k and s_{k+1} are distinct. By (2) we see that $\sum 1/m_k < \infty$. Hence $G(z)$ defines an analytic function in $\operatorname{Re} z > -1$.

A Laplace inversion theorem (see Churchill [2, p. 178]) implies that $G(z)$ is the Laplace transform of the function

$$g(e^{-s}) = (1/2\pi i) \int_{c_0 - i\infty}^{c_0 + i\infty} e^{zs} G(z)\, dz,$$

where c_0 is any real number greater than -1. Therefore the function

$$g(t) = (1/2\pi) \int_{-\infty}^{+\infty} e^{-yi\log t} G(iy)\, dy$$

satisfies (i), with $B_k = | G(n_k + 1) |$.

(ii) follows easily, because

$$\int_0^1 | g(t) | \, dt \leq \frac{1}{2\pi} \int_{-\infty}^{+\infty} | G(iy) | \, dy$$
$$= (1/2\pi) \int_{-\infty}^{+\infty} \left(\frac{1}{1 + y^2}\right) \prod_{k=0}^{\infty} \left| \frac{m_k + 1 - iy}{m_k + 1 + iy} \right| dy \leq \frac{1}{2}.$$

To obtain (iii), we proceed to estimate $| G(n_q + 1) |$ for fixed $q > 1$. We note that for positive z

$$| G(z) | \geq \frac{1}{| z + 1 |^2} \prod_{k=0}^{\infty} \left| \frac{\frac{1}{2}(n_k + n_{k+1}) + 1 - z}{\frac{1}{2}(n_k + n_{k+1}) + 1 + z} \right|,$$

so that

$$\frac{1}{| G(n_q + 1) |} \leq (n_q + 2)^2 \prod_{k=0}^{\infty} \left| \frac{n_k + n_{k+1} + 4 + 2n_q}{n_k + n_{k+1} - 2n_q} \right|. \tag{15}$$

Setting $\mu_k = n_k + n_{k+1}$, we estimate separately the terms of the products Π_1, Π_2, and Π_3 with $\mu_k < 2n_q$, $2n_q < \mu_k \leq 4n_q$, and $\mu_k > 4n_q$, respectively.

We have the inequalities

$$\begin{aligned} \Pi_1 &= \prod_{k=0}^{q-1} \left| \frac{n_k + n_{k+1} + 4 + 2n_q}{n_k + n_{k+1} - 2n_q} \right| \\ &\leq \left\{ \prod_{k=0}^{q-2} \left(\frac{4 + 4n_q}{q - k - 1} \right) \right\} (4 + 4n_q) \\ &\leq (2 + 2n_q)^q \cdot \frac{2}{(q-1)!} \\ &\leq \left(\frac{2e(1 + n_q)}{q - 1} \right)^q \cdot \frac{2(q-1)}{e}, \end{aligned}$$

since $n^n/n! \leq e^n$ for $n = 0, 1, 2, \cdots$. But then

$$\log \Pi_1 \leq n_q \cdot (q/n_q) \{\log (n_q/q) + C\} + \log (2(q-1)/e),$$

where C a constant. Since $k/n_k \to 0$ as $k \to \infty$, we have

$$\log \Pi_1 = o(n_q) \qquad (q \to \infty). \tag{16}$$

Assume Π_2 contains N factors (if $N = 0$, put $\Pi_2 = 1$). Then as above

$$\Pi_2 \leq ((4n_q + 4 + 2n_q)/2)^N \cdot 2/N! \leq ((3n_q + 2)e/N)^N \cdot 2,$$

and therefore,

$$\log \Pi_2 \leq n_q \cdot (N/n_q)\{\log (n_q/N) + C'\} + \log 2,$$

where C' is a constant. But $q + N = o(n_{q+N}) = o(n_q)$, so $N = o(n_q)$, and

$$\log \Pi_2 = o(n_q) \qquad (q \to \infty). \tag{17}$$

Finally,

$$\Pi_3 = \prod_{\mu_k > 4n_q} \left(1 + \frac{4 + 4n_q}{n_k + n_{k+1} - 2n_q}\right) \leq \prod_{\mu_k > 4n_q} \left(1 + \frac{4(1 + n_q)}{n_k}\right).$$

Thus,

$$\begin{aligned} \log \Pi_3 &\leq \textstyle\sum_{\mu_k > 4n_q} \log\,(1 + 4(1 + n_q)/n_k) \\ &< 4(1 + n_q) \textstyle\sum_{\mu_k > 4n_q} 1/n_k = o(n_q) \qquad (q \to \infty). \end{aligned} \tag{18}$$

Combining (15), (16), (17), and (18), we find

$$1/|\, G(n_q + 1) \,| \leq (n_q + 2)^2 e^{o(n_q)} \qquad (q \to \infty).$$

Hence, for a given ε in $(0, 1)$ we see for all n_q

$$1/|\, G(n_q + 1) \,| \leq K_1 (n_q)^2 e^{\varepsilon n_q},$$

where K_1 is a positive constant. So if $K_2 = (K_1)^{-1}$, for all n_q,

$$|\, G(n_q + 1) \,| \geq K_2\, n_q^{-2} e^{-\varepsilon n_q}.$$

For $0 < \rho < e^{-\varepsilon}$, let

$$h(u) = K_2\, u^{-2} e^{-\varepsilon u} (1/\rho^u) \qquad (u > 0);$$

then

$$h'(u)/h(u) = -2/u - \varepsilon + \log\,(1/\rho).$$

Therefore, inf $h(u)$ occurs when

$$\log\,(1/\rho) = 2/u + \varepsilon;$$

That is,

$$u = 2(\log\,(1/\rho) - \varepsilon)^{-1}.$$

Returning to $h(u)$, we have for $0 < \rho < e^{-\varepsilon}$,

$\inf_k B_k\, \rho^{-n_k} > K_2\, ((\log \rho^{-1} - \varepsilon)/2)^2 \exp\{-2\varepsilon\, (\log \rho^{-1} - \varepsilon)^{-1}\} \rho^{-2(\log \rho^{-1} - \varepsilon)^{-}}$

from which (iii) is clear.

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