APPLICATIONS OF A COMPARISON THEOREM FOR ELLIPTIC EQUATIONS

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In [1] the author applied a comparison theorem of Swanson [2] to derive criteria for the positivity of the Green's function associated with second order elliptic operators. For the special case of self-adjoint operators, similar criteria were established for the positivity of the Robin's functions associated with mixed boundary conditions. The latter results were based on a comparison theorem of the author [3].

The purpose of this paper is to extend the comparison theorem of [3] to cover a class of non self-adjoint equations and to use this comparison theorem to improve substantially on Theorem 2 of [1]. It will also be shown that a variational principle and the strong maximum principle for elliptic equations can be derived from this comparison theorem.

Let L be an elliptic operator with real coefficients defined by

$$Lu \equiv - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu \tag{1}$$

in a bounded domain $D \subset R^n$. Our comparison theorem will deal with functions $u(x)$ and $v(x)$ which are, respectively, solutions of the boundary value problems

$$Lu = 0 \text{ in } D, \qquad \partial u/\partial \nu + \sigma u = 0 \text{ on } \partial D, \tag{2}$$

and

$$Lv + pv = 0 \text{ in } D, \qquad \partial v/\partial \nu + \tau v = 0 \text{ on } \partial D. \tag{3}$$

In (2) we allow $-\infty < \sigma(x) \leq +\infty$, where $\sigma(x_0) = +\infty$ is used to denote the boundary condition $u(x_0) = 0$. Similar notation will be adhered to for (3). It is assumed that the boundary problems (2) and (3) are sufficiently regular so that certain resolvents for L and $L + pI$ can be represented as integral operators. Specifically, in the case of (2), we assume the existence of a constant K such that for $\gamma \geq K$ the boundary value problem

$$Lu + \gamma u = f \text{ in } D, \qquad \partial u/\partial \nu + \sigma u = 0 \text{ on } \partial D, \tag{4}$$

can be solved in the form

$$u(x) = \int_D G_\sigma(x, \xi; \gamma) f(\xi)\, d\xi.$$

Here $G_\sigma(x, \xi; \gamma)$ is the Robin's function for (4), having the following charac-

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teristic properties: for each $\xi_0 \in D$, G satisfies

$$LG + \gamma G = 0 \text{ in } D \ (x \neq \xi_0), \qquad \partial G/\partial \nu + \sigma G = 0 \text{ on } \partial D;$$

and for each fixed $x_0 \in D$, G satisfies

$$L^*G + \gamma G = 0 \text{ in } D \ (\xi \neq x_0), \qquad \partial G/\partial \nu + (\sigma + \sum b_i \, \partial \nu/\partial x_i)G = 0 \text{ on } \partial D,$$

where $\partial \nu/\partial x_i$ denotes the cosine of the angle between the exterior normal ν and the positive x_i-axis. The Robin's function associated with

$$(5) \qquad Lv + (p + \gamma)v = f \text{ in } D, \qquad \partial v/\partial \nu + \tau v = 0 \text{ on } D$$

has analogous properties and will be denoted by $H_\tau(x, \xi; \gamma)$. Our comparison theorem is as follows.

THEOREM 1. *Suppose $u(x)$ and $v(x)$ are solutions of* (2) *and* (3) *respectively and that $u(x)$ is positive in D. If $p(x) \leq 0$ in D and $-\infty < \tau(x) \leq \sigma(x) \leq +\infty$ on ∂D, then either $v(x)$ changes sign in D or else $v(x)$ is a constant multiple of $u(x)$.*

Before proceeding with the proof, it will be useful to recall some results of [4] where Theorem 1 is proven in the special case $\sigma(x) \equiv +\infty$. Given (2), it is possible to choose K sufficiently large so that $G_\sigma(x, \xi; \gamma)$ will be positive in $D \times D$ for all $\gamma \geq K$ and, even more important, that the resolvent $(L + \gamma I)^{-1}$ defined on $\mathfrak{L}^2(D)$ by

$$(L + \gamma I)^{-1} f = \int_D G_\sigma(x, \xi) f(\xi) \, d\xi$$

is u_0-positive in the sense of Krasnoselskii [5]. From this latter property it follows that $(L + \gamma I)^{-1}$ has exactly one normalized positive eigenfunction and that the corresponding eigenvalue is positive and larger than the absolute value of any other eigenvalue of $(L + \gamma I)^{-1}$. According to Theorem 2.6 of [4], the conclusions of Theorem 1 follow if one can show that for some $\gamma \geq K$,

$$(6) \qquad H_\tau(x, \xi; \gamma) \geq G_\sigma(x, \xi; \gamma).$$

Proof of Theorem 1. From the results of [4] as described above, it is sufficient to establish (6). To that end we consider the identity

$$\int_{D_r} (vLu - uL^*v) \, dy = \int_{\partial D_r} \left[u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} + \sum b_i \frac{\partial \nu}{\partial x_i} uv \right] ds$$

where D_r is the domain D with spheres of radius r deleted about two points $y = x$ and $y = \xi$. Setting $v(y) = G_\sigma(x, y; \gamma)$ and $u(y) = H_\tau(y, \xi; \gamma)$ and letting $r \to 0$ we get

$$\int_D [G_\sigma L H_\tau - H_\tau L^* G_\sigma] \, dy$$
$$= \int_{\partial D} \left[H_\tau \frac{\partial G_\sigma}{\partial \nu} - G_\sigma \frac{\partial H_\tau}{\partial \nu} + \sum b_i \frac{\partial \nu}{\partial x_i} H_\tau G_\sigma \right] ds$$
$$+ H_\tau(x, \xi; \gamma) - G_\sigma(x, \xi; \gamma).$$

Using the characteristic properties of G_σ and H_τ, this reduces to

$$\int_D - pH_\tau G_\sigma \, dy = \int_{\partial D} (\tau - \sigma) H_\tau G_\sigma \, ds + H_\tau - G_\sigma,$$

from which (6) follows readily.

Remarks. 1. If L is self-adjoint, Theorem 1 reduces to a special case of the comparison theorem of [2].

2. It is not strictly necessary to assume that $u(x)$ is positive in D. If $u(x)$ changes sign then the above argument can be applied on any nodal domain of $u(x)$.

3. Whereas Swanson's comparison theorem [2] deals with two equations which differ in all the coefficients, the equations $Lu = 0$ and $Lu + pu = 0$ differ only in the last coefficient. This fact allows us to establish Theorem 1 under conditions which are in most instances substantially weaker than Swanson's.

As an immediate application of Theorem 1 we are able to strengthen Theorem 2 of [1] as follows.

THEOREM 2. *Suppose the Robin's function $G_\sigma(x, \xi; \gamma)$ associated with* (4) *exists. If there exists a solution of $Lv + \gamma v = 0$ which is positive in D and satisfies $\partial v/\partial \nu + \tau v = 0$ on ∂D with $\tau \leq \sigma$, then $G_\sigma(x, \xi; \gamma)$ is non-negative in $D \times D$.*

Proof. Suppose to the contrary that $G(x_0, \xi_0) < 0$ for some $(x_0, \xi_0) \in D \times D$. Since $\lim_{x \to \xi_0} G(x, \xi_0; \gamma) = +\infty$, there exists a proper sub-domain $D_0 \subset D$ (not containing ξ_0) such that $G(x, \xi_0; \gamma) < 0$ for $x \in D_0$, $G(x, \xi_0; \gamma) = 0$ for $x \in \partial D_0 \cap D$ and $\partial G/\partial \nu + \sigma G = 0$ for $x \in \partial D_0 \cap \partial D$. Applying Theorem 1 in D_0 with $p(x) \equiv 0$ leads to the conclusion that $v(x)$ changes sign in D_0. This contradiction shows that $G(x, \xi; \gamma)$ is non-negative in $D \times D$.

Remarks. 1. Setting $\sigma(x) \equiv +\infty$, one obtains a substantially stronger result than Theorem 2 of [1].

2. As in Theorem 2 of [1], one can formulate a similar theorem in case $L^* v = 0$ has a solution which is positive in D.

As a second application of Theorem 1, we consider the eigenvalue problem

$$Lv = \lambda v \text{ in } D, \qquad v = 0 \text{ on } \partial D, \tag{7}$$

under the additional assumption that the real part of the spectrum of (7) is bounded below. Applying the spectral mapping theorem to (7) and to the related u_0-positive operator $(L + \gamma I)^{-1}$ for a sufficiently large γ, it follows that (7) has a real eigenvalue which is simple, is smaller than all other real eigenvalues, and corresponds to an eigenfunction which is positive in D. In case L is self-adjoint, it is well known [6, p. 409] that λ_1 is a strictly decreasing function of the domain D. The following theorem asserts a similar result for the more general case of (7).

THEOREM 3. *Let λ_1 be the smallest real eigenvalue of* (7) *and λ_1' be the smallest real eigenvalue of*

$$(7') \qquad Lu = \lambda u \text{ in } D', \qquad u = 0 \text{ on } \partial D'.$$

If D' is a proper subdomain of D, then $\lambda_1' > \lambda_1$.

Proof. Suppose to the contrary that $\lambda_1' \leq \lambda_1$. Then we can apply Theorem 1 with $\sigma(x) \equiv +\infty$ to the eigenfunctions u_1 and v_1 satisfying

$$Lu_1 - \lambda_1' u_i = 0 \quad \text{and} \quad Lv_1 - \lambda_1 v_1 = 0$$

in D', respectively. By Theorem 1 we conclude that v_1 changes sign in D' or is a constant multiple of u_1. Since both of these conclusions contradict the known properties of v_1, it follows that $\lambda_1' > \lambda_1$.

As a final application of Theorem 1 we give a simple proof of a strong maximum principle for elliptic equations.

THEOREM 4. *Let $u(x)$ be a solution of*

$$(8) \qquad -\sum \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum b_i \frac{\partial u}{\partial x_i} + cu = 0$$

in a closed domain $\bar{D}$ in which $c(x) \geq 0$. If $u(x)$ attains its positive maximum at some $x_0 \in D$, then $u(x)$ is constant in some neighborhood of x_0.

Proof. If $u(x)$ attains its positive maximum at x_0, then there exists a neighborhood D' of x_0, $D' \subset D$, in which $u(x)$ is positive and for which we have $\partial u/\partial \nu + \sigma u = 0$ on $\partial D'$ with $\sigma(x) \geq 0$ on $\partial D'$. We consider also the boundary value problem

$$(9) \qquad -\sum \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial v}{\partial x_i}\right) + \sum b_i \frac{\partial v}{\partial x_i} = 0 \text{ in } D', \qquad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial D'$$

which has the obvious non-trivial solution $v =$ constant. Applying Theorem 1 to $u(x)$ and $v(x) =$ constant and noting that v does not change sign in D', we conclude that u is a constant multiple of v in D'.

Remarks. 1. By the unique continuation principle it is possible to conclude that $u(x) \equiv$ constant in D.

2. Using comparison theorems for sub-solutions and super-solutions such as those of Swanson [7], one can also obtain a maximum principle for solutions of differential inequalities.

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