MULTIPLICATION ALTERATION BY TWO-COCYCLES

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0. Introduction

If U is an associative unitary algebra with a commutative subalgebra A and $\sigma = \sum a_i \otimes b_i \otimes c_i \in A \otimes A \otimes A$ is an Amitsur 2-cocycle, then we can define a new multiplication $*$ on U by setting

$$0.1 \qquad u * v = \sum a_i u b_i v c_i$$

for all $u, v \in U$. The Amitsur 2-cocycle condition guarantees that U is associative and unitary with the $*$ multiplication. If U was originally a central separable (simple) algebra then U is still central separable under the new multiplication. We show that the central separable algebra resulting from an Amitsur 2-cocycle σ is isomorphic to the Rosenberg Zelinsky central separable algebra coming from the 2-cocycle σ^{-1}.

If K is an intermediate field ($A \supset K \supset k$) we show how mapping 2-cocycles in $A \otimes_k A \otimes_k A$ into $A \otimes_K A \otimes_K A$ corresponds to taking the centralizer of K in central separable k algebras with maximal commutative subfield A. On the way to these results we prove that if A is a finite purely inseparable field extension of k and U is an algebra containing A then U is isomorphic to $A \otimes_k A$ as an A-bimodule if and only if U is a central separable k algebra of k-dimension n^2.

By being careful about what we mean by a 2-cocycle we are able to obtain an associative unitary algebra by means of 0.1 even when A is not a commutative subalgebra of U. We prove that if U is a central separable n^2-dimensional k algebra and $\tilde{U}$ is *any* n^2-dimensional k-algebra then there is a 2-cocycle in $U \otimes U \otimes U$ making U isomorphic to $\tilde{U}$ (via 0.1). Moreover we show that if U is a central separable k algebra with simple subalgebra L which has centralizer A then there is a 2-cocycle in $A \otimes A \otimes A$ making U isomorphic to $\tilde{U}$ if and only if $\tilde{U}$ contains a copy of L and is isomorphic to U as an L-bimodule. If A is commutative and σ is a 2-cocycle in $A \otimes A \otimes A$ then σ is an Amitsur 2-cocycle if σ is invertible.

We define when two 2-cocycles in $A \otimes A \otimes A$ are cohomologous and show that this is equivalent to the associated algebras being isomorphic by an isomorphism which is the identity on L. This gives a bijective correspondence between a 2-cohomology set (not group) and equivalence classes of algebras.

1. Linear Algebra

Throughout this paper k is at least a commutative unitary ring (and sometimes a field). All k algebras are unitary. A subalgebra has the same unit

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as the over algebra. Unadorned $\otimes$, Hom and End mean $\otimes_k$, Hom_k and End_k respectively. We speak of a central separable k algebra in the sense of [8, p. 330, footnote 9]. "Finite projective module" means "finitely generated projective module". $\otimes_A^n M$ means $M \otimes_A \cdots \otimes_A M$ (n-times).

Suppose U is a k algebra with subalgebras L and A, where A is in the centralizer of L in U. We consider U as a right L-module by

$$u \cdot l \equiv ul \quad \text{for } u \in U, l \in L.$$

$\mathrm{Hom}_{-L}(U, U)$ denotes the set of right L-module morphisms from U to U. Since A is in the centralizer of L in U there is a map

1.1 $$f : U \otimes A \to \mathrm{Hom}_{-L}(U, U), \qquad u \otimes a \to f_{u \otimes a}$$

where $f_{u\otimes a}(v) = uva$ for $u, v \in U$, $a \in A$.

We say that (U, L, A) satisfies H1 if

0. A is the full centralizer of L in U,

1.2 1. f is a bijection,

2. A is a finite projective k-module.

We shall show in 1.6 that if k is a field and U a central separable k algebra, then (U, k, U) satisfies H1.

1.3 LEMMA. (a) *If (U, k, U) and (U, L, A) satisfy* H1 *and $\otimes^n U$ has the right L-module structure given by*

$$(u_1 \otimes \cdots \otimes u_n) \cdot l \equiv u_1 \otimes \cdots \otimes u_{n-1} \otimes (u_n l)$$

then

$$\otimes^n U \otimes A \xrightarrow{f^n} \mathrm{Hom}_{-L}(\otimes^n U, U),$$

$$u_1 \otimes \cdots \otimes u_n \otimes a \to f^n_{u_1 \otimes \cdots \otimes u_n \otimes a}$$

where

$$f^n_{u_1 \otimes \cdots \otimes u_n \otimes a}(v_1 \otimes \cdots \otimes v_n) = u_1 v_1 u_2 v_2 \cdots u_n v_n a$$

is a bijection.

(b) *Since U is also a left L-module we can form $U \otimes_L U$ (where $ul \otimes v = u \otimes lv$) which is a right L-module by*

$$(u \otimes v) \cdot l \equiv u \otimes (vl).$$

If (U, L, A) satisfies H1 *then*

$$U \otimes \otimes^n A \xrightarrow{\bar{f}^n} \mathrm{Hom}_{-L}(\otimes_L^n U, U),$$

$$u \otimes a_1 \otimes \cdots \otimes a_n \to \bar{f}^n_{u \otimes a_1 \otimes \cdots \otimes a_n}$$

where

$$\bar{f}^n_{u \otimes a_1 \otimes \cdots \otimes a_n}(v_1 \otimes \cdots \otimes v_n) = u v_1 a_1 v_2 a_2 \cdots v_n a_n$$

is a bijection.

Proof. Since the proofs of (a) and (b) run parallel we work on both simultaneously and keep track in the margin.

Given rings X, Y and modules M_X, ${}_X N_Y$, O_Y (notation of [6]) there is a natural correspondence

$$\mathrm{Hom}_{-Y}(M \otimes_X N, O) \leftrightarrow \mathrm{Hom}_{-X}(M, \mathrm{Hom}_{-Y}(N, O)),$$

[6, p. 25, Prop. 5.2′]. This gives natural correspondences

(a) $\mathrm{Hom}_{-L}(\otimes^n U, U) \leftrightarrow \mathrm{Hom}(U, \mathrm{Hom}_{-L}(\otimes^{n-1} U, U))$,
(b) $\mathrm{Hom}_{-L}(\otimes_L^n U, U) \leftrightarrow \mathrm{Hom}_{-L}(U, \mathrm{Hom}_{-L}(\otimes_L^{n-1} U, U))$.
By induction (taking f^{n-1} and $\bar{f}^{n-1}$ as identifications) the right hand sides are equal to

(a) $\mathrm{Hom}(U, \otimes^{n-1} U \otimes A)$,
(b) $\mathrm{Hom}\_L(U, U \otimes \otimes^{n-1} A)$,

where $U \otimes \otimes^{n-1} A$ is a right L-module by

$$(u \otimes a_1 \otimes \cdots \otimes a_{n-1})\cdot l = (ul) \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

The hypothesis of (a) implies that $\otimes^{n-2} U \otimes A$ is a finite projective k-module and the hypothesis of (b) implies that $\otimes^{n-1} A$ is a finite projective k-module so that the above terms are naturally equivalent to

(a) $\mathrm{Hom}(U, U) \otimes \otimes^{n-2} U \otimes A$,
(b) $\mathrm{Hom}_{-L}(U, U) \otimes \otimes^{n-1} A$.

Under the hypothesis of (a) and (b) these are isomorphic to

(a) $(U \otimes U) \otimes \otimes^{n-2} U \otimes A$,
(b) $(U \otimes A) \otimes \otimes^{n-1} A$.

Checking all the correspondences shows that they give f^n and $\bar{f}^n$, Q.E.D.

With the right and left L-module structures on U, U is an L-bimodule. We let $\mathrm{Hom}_{L-L}(U, U)$ denote the set of simultaneously right and left L-module morphisms from U to U. Since A is contained in the centralizer of L in U we have the map

$$A \otimes A \xrightarrow{\ g\ } \mathrm{Hom}_{L-L}(U, U), \qquad a \otimes b \rightarrow g_{a\otimes b}$$

where $g_{a\otimes b}(u) = aub$.

We say that (U, L, A) satisfies H2 if

1.4
1. g is a bijection,
2. (U, L, A) satisfies H1.

1.5 LEMMA. *If (U, L, A) satisfies* H2 *and $\otimes_L^n U$ has the L-bimodule structure given by*

$$l\cdot(u_1 \otimes \cdots \otimes u_n) \equiv (lu_1) \otimes u_2 \otimes \cdots \otimes u_n,$$

$$(u_1 \otimes \cdots \otimes u_n)\cdot l \equiv u_1 \otimes \cdots \otimes u_{n-1} \otimes (u_n l),$$

Then

$$\otimes^{n-1} A \xrightarrow{g^n} \mathrm{Hom}_{L-L}(\otimes_L^n U, U),$$

$$a_0 \otimes \cdots \otimes a_n \to g^n_{a_0} \otimes \cdots \otimes a_n$$

where

$$g^n_{a_0} \otimes \cdots \otimes a_n\,(u_1 \otimes \cdots \otimes u_n) = a_0 u_1 a_1 u_2 \cdots a_{n-1} u_n a_n$$

is a bijection.

Proof. Given rings X, Y, Z and modules ${}_XM_Y$, ${}_YN_Z$, ${}_YO_Z$ [notation of 6] there is a natural correspondence

$$\mathrm{Hom}_{X-Z}(M \otimes_Y N, O) \leftrightarrow \mathrm{Hom}_{X-Y}(M, \mathrm{Hom}_{-Z}(N, O))$$

induced by the correspondence in [6, p. 28, Prop. 5.2′]. This gives the natural correspondence

$$\mathrm{Hom}_{L-L}(\otimes_L^n U, U) \leftrightarrow \mathrm{Hom}_{L-L}(U, \mathrm{Hom}_{-L}(\otimes_L^{n-1} U, U)).$$

By the previous lemma and taking f^{n-1} as an identification we have that

$$\mathrm{Hom}_{-L}(\otimes_L^{n-1} U, U) = U \otimes \otimes^{n-1} A.$$

Plugging this in above gives

$$* \qquad \mathrm{Hom}_{L-L}(U, U \otimes \otimes^{n-1} A),$$

where $U \otimes \otimes^{n-1} A$ has the L-bimodule structure given by

$$l \cdot (u \otimes a_1 \otimes \cdots \otimes a_{n-1}) \cdot m \equiv (lum) \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

Since $\otimes^{n-1} A$ is a finite projective k-module $(*)$ is naturally isomorphic to

$$\mathrm{Hom}_{L-L}(U, U) \otimes \otimes^{n-1} A.$$

Since (U, L, A) satisfies condition 1 of H2 the above is isomorphic to

$$(A \otimes A) \otimes \otimes^{n-1} A.$$

Checking through the correspondences shows that they give g^n, Q.E.D.

1.6 PROPOSITION. *If k is a field, U a finite-dimensional central separable k algebra and L a simple subalgebra of U with A the centralizer of L in U then (U, L, A) satisfies* H1 *and* H2.

Proof. By [1, p. 53, Theorem 13],

1.7 L is the centralizer of A in U and A is simple,

1.8 $(\dim_k A)(\dim_k L) = \dim_k U$,

1.9 A and L have common center F which is a field and

$$A \otimes_F L \to U, \qquad a \otimes L \to al$$

induces an algebra isomorphism between $A \otimes_F L$ and the centralizer of F in U.

It can easily be shown using [1, p. 42, Theorem 14]

$$U \otimes U^{\mathrm{op}} \xrightarrow{M} \mathrm{End}\, U, \qquad u \otimes v^{\mathrm{op}} \to f_{u\otimes v}$$

where $f_{u\otimes v}(w) = uwv$ is an algebra isomorphism. (U^{op} is the opposite algebra to U, i.e., for $u, v \in U$ $u^{\mathrm{op}}v^{\mathrm{op}} = (vu)^{\mathrm{op}}$.)

Consider $k \otimes L^{\mathrm{op}} \subset U \otimes U^{\mathrm{op}}$ in the natural way. Then—via M—$\mathrm{Hom}_{-L}(U, U)$ corresponds to the centralizer of $k \otimes L^{\mathrm{op}}$ in $U \otimes U^{\mathrm{op}}$. Now $k \otimes L^{\mathrm{op}}$ is a simple subalgebra of the central separable k algebra $U \otimes U^{\mathrm{op}}$ and $U \otimes A^{\mathrm{op}}$ lies in the centralizer. Counting dimensions and applying 1.8 shows that $U \otimes A^{\mathrm{op}}$ is the full centralizer of $k \otimes L^{\mathrm{op}}$. In view of the correspondence M we have that condition 1 of H1 is satisfied. Condition 0 is satisfied by hypothesis and condition 2 is satisfied since we assume U is finite dimensional over k.

Consider $L \otimes L^{\mathrm{op}} \subset U \otimes I^{\mathrm{op}}$ in the natural way. Then—via $M \subset$ $\mathrm{Hom}_{L-L}(U, U)$ corresponds to the centralizer of $L \otimes L^{\mathrm{op}}$ in $U \otimes k^{\mathrm{op}}$. As shown above $U \otimes A^{\mathrm{op}}$ is the centralizer of $k \otimes L^{\mathrm{op}}$ and similarly $A \oplus U^{\mathrm{op}}$ is the centralizer of $L \otimes k^{\mathrm{op}} = L \otimes k$ in $U \otimes U^{\mathrm{op}}$. Thus the centralizer of $L \otimes L^{\mathrm{op}}$ is $(U \otimes A^{\mathrm{op}}) \cap (A \otimes U^{\mathrm{op}})$ which is equal to $A \otimes A^{\mathrm{op}}$. In view of the correspondence M we have that the first condition of H2 is satisfied. We have already shown that the second condition is satisfied, Q.E.D.

1.10 PROPOSITION. *Suppose k is a field with extension field L, U is a k algebra containing L and (U, L, A) satisfies* H1. *Then L is a finite field extension of k, U is a finite-dimensional central separable k algebra, A is a simple k algebra with center L and (U, L, A) satisfies* H2.

Proof. L is commutative implies that L lies in its centralizer A. Since A is a 'finite projective' k-module, L must be finite dimensional.

The composite

1.11 $$U \otimes A^{\mathrm{op}} \to U \otimes A \xrightarrow{f} \mathrm{Hom}_{-L}(U, U), \quad u \otimes a^{\mathrm{op}} \to u \otimes a$$

is an algebra homomorphism. It is an algebra isomorphism since the left map is bijective and the right map is bijective because (U, L, A) satisfies H1. If U has infinite dimension over k then the cardinality of $\dim_k U \otimes A^{\mathrm{op}}$ equals the cardinality of $\dim_k U$ since A has finite k dimension. U must also have infinite dimension over L so that the cardinality of $\dim_k \mathrm{Hom}_{-L}(U, U)$ is greater than the cardinality of $\dim_k U$. This contradicts the fact that 1.11 is an isomorphism. Thus U is a finite-dimensional k algebra and $\mathrm{Hom}_{-L}(U, U)$ is a finite-dimensional central separable L algebra. Thus the isomorphism 1.11 implies that U is a central separable k algebra and A^{op} (hence A) is a simple k algebra with center L, since L lies in the center of A (hence A^{op}). (U, L, A) satisfies H2 by 1.6, Q.E.D.

1.12 LEMMA. *Suppose A is a commutative k algebra. If (U, L, A) satisfies* H1 (H2) *then so does $(U \otimes A, L \otimes A, A \otimes A)$. If A is a faithful finite* *projective k-module and* $(U \otimes A, L \otimes A, A \otimes A)$ *satisfies* H1 (H2) *then so does* (U, L, A).

Proof. If B is a k algebra and M and N are left B-modules then

1.13
$$\operatorname{Hom}_B(M, N) \otimes_k A \to \operatorname{Hom}_B(M, N \otimes_k A), \qquad f \otimes a \to (m \to f(m) \otimes a),$$

is an isomorphism when A is a finite projective k-module.

Suppose B, C, D and E are subalgebras of U and U is a left $B \otimes C^{\mathrm{op}}$-module by

$$(b \otimes c^{\mathrm{op}}) \cdot u \equiv buc \quad \text{for } b \in B,\ c \in C,\ u \in U.$$

If B centralizes D and C centralizes E then we have the map

1.14
$$D \otimes_k E \xrightarrow{h} \operatorname{Hom}_{B \otimes_k C^{\mathrm{op}}}(U, U), \quad d \otimes e \to (u \to due).$$

If we "base extend" by A the map in 1.14 becomes

1.15
$$(D \otimes_k A) \otimes_A (E \otimes_k A) \xrightarrow{h'} \operatorname{Hom}_{(B \otimes_k A) \otimes_A (C \otimes_k A)^{\mathrm{op}}}(U \otimes_k A, U \otimes_k A).$$

The left hand side in 1.15 is naturally isomorphic to $D \otimes_k E \otimes_k A$. For the right hand side we have the sequence of natural isomorphisms

$$\operatorname{Hom}_{(B \otimes_k A) \otimes_A (C \otimes_k A)^{\mathrm{op}}}(U \otimes_k A, U \otimes_k A)$$
$$\cong \operatorname{Hom}_{B \otimes_k C^{\mathrm{op}} \otimes_k A}(U \otimes_k A, U \otimes_k A) \cong \operatorname{Hom}_{B \otimes_k C^{\mathrm{op}}}(U, U \otimes_k A).$$

The map h' in 1.15 corresponds to

$$D \otimes_k E \otimes_k A \to \operatorname{Hom}_{B \otimes_k C^{\mathrm{op}}}(U, U \otimes_k A),$$
$$d \otimes e \otimes a \to (u \to due \otimes a),$$

which factors

1.16
$$D \otimes_k E \otimes_k A \xrightarrow{h \otimes I} \operatorname{Hom}_{B \otimes_k C^{\mathrm{op}}}(U, U) \otimes_k A \to \operatorname{Hom}_{B \otimes_k C^{\mathrm{op}}}(U, U \otimes_k A).$$

(The right hand map in 1.16 is the map in 1.13 and is an isomorphism when A is a finite projective k-module.)

Thus we have that h' is bijective if h is bijective and A is a finite projective k-module. Also, if A is a finite projective k-module and h' is bijective then $h \otimes I$ (in 1.16) is bijective. If A is also a faithful k-module then h must be bijective.

The three interesting cases are

(I) $B = L, C = U, D = A, E = k$,
(II) $B = k, C = L, D = U, E = A$,
(III) $B = L, C = L, D = A, E = A$.

For Case I bijectivity of 1.14 is equivalent to (U, L, A) satisfying condition O of H1 and bijectivity of 1.15 is equivalent to $(U \otimes A, L \otimes A, A \otimes A)$ satisfying condition 0 of H1. Similarly Case II covers condition 1 of H1 and Case III covers condition 1 of H2.

Finally $A \otimes_k A$ is a finite projective A-module when A is a finite projective k-module, Q.E.D.

2. Constructions

Suppose U is a k algebra.

2.1 Definition. $\sigma = \sum u_i \otimes v_i \otimes w_i \in U \otimes U \otimes U$ is called a 2-cocycle if

2.2 $$\sum_{i,j} u_i u_j \otimes v_j \otimes w_j v_i \otimes w_i = \sum_{i,j} u_i \otimes v_i u_j \otimes v_j \otimes w_j w_i,$$

and there is an element $e_\sigma \in U$ where

2.3 $$\sum_i u_i e_\sigma v_i \otimes w_i = 1 \otimes 1 = \sum_i u_i \otimes v_i e_\sigma w_i .$$

If σ is a 2-cocycle and both $e_\sigma, f_\sigma \in U$ satisfy 2.3 then considering

$$\sum u_i e_\sigma v_i f_\sigma w_i$$

shows that $e_\sigma = f_\sigma$.

Suppose U is commutative. If σ is an Amitsur 2-cocycle in $U \otimes U \otimes U$ [8, p. 327] where $\sigma^{-1} = \sum \bar{u}_i \otimes \bar{v}_i \otimes \bar{w}_i \in U \otimes U \otimes U$ then one easily checks that σ is a 2-cocycle in the above sense with $e_\sigma = \sum \bar{u}_i \bar{v}_i \bar{w}_i$. Clearly, if σ is a 2-cocycle—in the above sense—which is invertible then σ is an Amitsur 2-cocycle.

If U is a flat k-module and A is a subalgebra of U which is a flat k-module then the natural maps

$$A \otimes A \otimes A \to U \otimes A \otimes A \to U \otimes U \otimes A \to U \otimes U \otimes U$$

are injective and we take them for identifications.

2.4 Definition. We say that σ is a 2-cocycle in $U \otimes U \otimes A$ (respectively, $U \otimes A \otimes A$, $A \otimes A \otimes A$) if A is a subalgebra of U, both U and A are flat k-modules, σ is a 2-cocycle in $U \otimes U \otimes U$ and σ lies in $U \otimes U \otimes A$, (respectively, $U \otimes A \otimes A$, $A \otimes A \otimes A$).

If σ is a 2-cocycle we can define a new k-algebra U^σ. As a set U^σ is equal to U. For an element $u \in U$ we write u^σ to indicate that we are considering it as an element in U^σ. The multiplication in U^σ is given by

2.5 $$u^\sigma v^\sigma \equiv \left(\sum u_i u v_i v w_i\right)^\sigma$$

where $u, v \in U$, $\sigma = \sum u_i \otimes v_i \otimes w_i$ and the multiplication on the right hand side takes place in U. Associativity follows from 2.2. The unit of U^σ is e_σ^σ by 2.3.

Suppose σ is a 2-cocycle in $U \otimes U \otimes A$. We define

2.6 $$U^\sigma \xrightarrow{N} U \otimes A^{\text{op}}, \qquad U^\sigma \to \sum u_i u v_i \otimes w_i^{\text{op}}.$$

One easily checks that N is an algebra homomorphism. N is injective since the k-module morphism

$$U \otimes A^{\text{op}} \xrightarrow{m} U^\sigma, \qquad u \otimes a^{\text{op}} \to (ue_\sigma a)^\sigma$$

has the property $mN = I$.

Suppose $\sigma = \sum u_i \otimes v_i \otimes w_i$, $\tau = \sum r_i \otimes s_i \otimes t_i \in U \otimes U \otimes U$ are two 2-cocycles and $\varphi = \sum x_i \otimes y_i \in U \otimes U$.

2.7 Definition. $\sigma \sim^\varphi \tau$—read "σ is cohomologous to τ via φ"—if

2.8 $$\sum_{i,j} x_i u_j \otimes v_j \otimes w_j y_i = \sum_{i,jl} r_i x_j \otimes y_j s_i x_l \otimes y_l t_i$$

and

2.9 $$e_\tau = \sum_i x_i e_\sigma y_i .$$

This relation $\sim$ is not reflexive without further assumptions.

If $\sigma \sim^\varphi \tau$ we have the algebra homomorphism

2.10 $$U^\sigma \to U^\tau, \qquad u^\sigma \to (\textstyle\sum_i x_i u y_i)^\tau.$$

2.11 Definition. A 2-cocycle $\sigma = \sum u_i \otimes v_i \otimes w_i \in U \otimes U \otimes U$ is *vertible* with *verse* $\bar\sigma = \sum \bar u_i \otimes \bar v_i \otimes \bar w_i \in U \otimes U \otimes U$ if

2.12 $$\sum_{i,j} \bar u_i u_j \otimes v_j \bar v_i \otimes \bar w_i w_j = 1 \otimes 1 \otimes 1 = \sum_{i,j} u_i \bar u_j \otimes \bar v_j v_i \otimes w_i \bar w_j .$$

If σ is a vertible 2-cocycle in $U \otimes U \otimes A$ with verse $\bar\sigma \in U \otimes U \otimes A$ and A is commutative then

2.13 $$\tilde N : U^\sigma \otimes A \to U \otimes A, \qquad u^\sigma \otimes a \to \sum_i u_i u v_i \otimes w_i a$$

is an algebra isomorphism with inverse

$$\tilde N^{-1} : U \otimes A \to U^\sigma \otimes A, \qquad u \otimes a \to \sum_i (\bar u_i u \bar v_i)^\sigma \otimes \bar w_i a.$$

2.14 *Example.* Suppose A is a commutative k algebra which is a finite projective k-module. Suppose V is a finite projective and faithful A-module. We can consider V as a k-module and have the injective algebra representation $\pi : A \to \text{End } V$. Identify A with its image under π. If σ is a vertible 2-cocycle in End $V \otimes$ End $V \otimes A$ with verse in End $V \otimes$ End $V \otimes A$ then as algebras

$$(\text{End } V)^\sigma \otimes A \cong \text{End } V \otimes A$$

by 2.13. If the unit mapping $k \to A$, $\lambda \to \lambda \cdot 1$ is a split monomorphism then $(\text{End } V)^\sigma$ is a central separable k algebra by [8, p. 330, Lemma 3.1].

2.15 *Remark.* Suppose p is a prime, k has characteristic p and A is a k

algebra which is purely inseparable over k in the sense that for any $a \in A$ there is an n with $a^{p^n} \in K$. Then for any 2-cocycle $\sigma \in A \otimes A \otimes A$ there is a high enough m with $\sigma^{p^m} \in k \otimes k \otimes k$. One easily checks that $e_\sigma^{p^m} \otimes 1 \otimes 1$ is an inverse to σ^{p^m} so that σ is invertible, i.e. σ is an Amitsur 2-cocycle.

Suppose σ is a 2-cocycle in $U \otimes U \otimes A$ and L is a subalgebra of U which centralizes A. The map

2.16 $$L \xrightarrow{H} U^\sigma, \qquad l \to (e_\sigma l)^\sigma$$

is easily checked to be an algebra homomorphism. H is injective since

$$\sum_i u_i \otimes v_i e_\sigma l w_i = 1 \otimes l$$

if $\sigma = \sum u_i \otimes v_i \otimes w_i$, $l \in L$. The algebra homomorphism H gives U^σ a right, left and *bi* L-module structure in the obvious fashion.

2.17 Lemma. (a) *If σ is in $U \otimes U \otimes A$ (as above) then*

$$U \to U^\sigma, \qquad u \to u^\sigma$$

is a right L-module isomorphism.

(b) *If σ is in $U \otimes A \otimes A$ then*

$$U \to U^\sigma, \qquad u \to u^\sigma$$

is an L-bimodule isomorphism.

Proof. The map is bijective since U^σ equals U as a set. If $u \in U$, $l \in L$,

$$u^\sigma (e_\sigma l)^\sigma = (\textstyle\sum_i u_i u v_i e_\sigma l w_i)^\sigma$$

which is equal to

$$(\textstyle\sum_i (u_i u v_i e_\sigma w_i) l)^\sigma$$

since L centralizes A. By 2.3 the above equals

$$(ul)^\sigma.$$

A similar calculation shows that the map is also a left L-module homomorphism if $\sigma \in U \otimes A \otimes A$, Q.E.D.

If A centralizes L and $A \subset L$ then A is commutative and we have a copy of A in U^σ via

$$A \to L \xrightarrow{H} U^\sigma.$$

2.18 Lemma. *Suppose A and L are subalgebras of U where A centralizes L and $A \subset L$. Furthermore, suppose A is a faithful k-module and σ is a vertible 2-cocycle in $U \otimes A \otimes A$. Then (U, L, A) satisfies* H1 (H2) *if and only if $(U^\sigma, H(L), H(A))$ does.*

Proof. If either (U, L, A) or $(U^\sigma, H(L), H(A))$ satisfies H1 or H2 then A (or $H(A)$) is a finite projective k-module. Thus we may assume that A is a faithful finite projective k-module. By 1.12, (U, L, A) satisfies H1 (H2) if and only if $(U \otimes A, L \otimes A, A \otimes A)$ does. An easy calculation shows that the isomorphism $\tilde{N}$ in 2.13 gives rise to the commutative diagram

$$\begin{array}{ccc} U^{\sigma} \otimes A & \xrightarrow{\quad \tilde{N} \quad} & U \otimes A \\ & {\scriptstyle H \otimes I} \nwarrow \qquad \nearrow {\scriptstyle \iota \otimes I} & \\ & L \otimes A. & \end{array}$$

Thus $(U \otimes A, L \otimes A, A \otimes A)$ satisfies H1 (H2) if and only if

$$(U^{\sigma} \otimes A, H(L) \otimes A, H(A) \otimes A)$$

does. By 1.12 this is equivalent to $(U^{\sigma}, H(L), H(A))$ satisfying H1 (H2), Q.E.D.

3. Characterizations of U^{σ}

In the sequel U and A are always assumed to be flat as k-modules.

We define algebra homomorphisms

$$e_1 : U \otimes A^{\mathrm{op}} \to U \otimes A^{\mathrm{op}} \otimes A^{\mathrm{op}}, \qquad u \otimes a^{\mathrm{op}} \to u \otimes 1 \otimes a^{\mathrm{op}}$$

$$e_2 : U \otimes A^{\mathrm{op}} \to U \otimes A^{\mathrm{op}} \otimes A^{\mathrm{op}}, \qquad u \otimes a^{\mathrm{op}} \to u \otimes a^{\mathrm{op}} \otimes 1$$

3.1 Proposition. *Suppose σ is a 2-cocycle in $U \otimes A \otimes A$. If*

$$\sigma = \sum u_i \otimes v_i \otimes w_i$$

let σ^0 denote $\sum u_i \otimes v_i^{\mathrm{op}} \otimes w_i^{\mathrm{op}}$ in $U \otimes A^{\mathrm{op}} \otimes A^{\mathrm{op}}$. Then N (see 2.6) induces an isomorphism between U^{σ} and

$$V = \{x \in U \otimes A^{\mathrm{op}} \mid \sigma^0 e_2(x) = e_1(x)\sigma^0\}$$

Proof. We have shown at 2.6 that N is injective so it remains to prove that the image is precisely V. Suppose $x = \sum x_i \otimes a_i^{\mathrm{op}} \in V$. We shall show that

3.2 $$x = N(\textstyle\sum x_i e_{\sigma} a_i)^{\sigma}.$$

Since $x \in V$,

$$\textstyle\sum_{i,j} u_i x_j \otimes (a_j v_i)^{\mathrm{op}} \otimes w_i^{\mathrm{op}} = \sigma^0 e_2(x) = e_1(x)\sigma^0 = \sum_{i,j} x_i u_j \otimes v_j^{\mathrm{op}} \otimes (w_j a_i)^{\mathrm{op}}.$$

Thus

3.3 $$\begin{aligned} N(\textstyle\sum_i x_i e_{\sigma} a_i)^{\sigma} &= \textstyle\sum_{i,j} u_i x_j e_{\sigma} a_j v_i \otimes w_i \\ &= \textstyle\sum_{i,j} x_i u_j e_{\sigma} v_j \otimes (w_j a_i)^{\mathrm{op}} = \sum_i x_i \otimes a_i^{\mathrm{op}} = x. \end{aligned}$$

We have shown that $V \subset \mathrm{Im}\, N$. The first cocycle condition 2.2 implies that $V \supset \mathrm{Im}\, N$, Q.E.D.

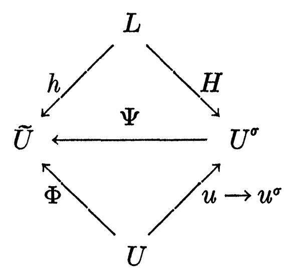
In [8, p. 339, Theorem 2] Rosenberg and Zelinsky give a correspondence from Amitsur 2-cocycles to central separable algebras. If A is a commutative algebra over k which is a finite projective k-module and where the unit mapping

$k \to A$ is a split monomorphism then by example 2.14, $(\text{End } A)^\sigma$ is a central separable k algebra, if σ is an Amitsur 2-cocycle in $A \otimes A \otimes A$. Since A is commutative $A = A^{\text{op}}$ and Proposition 3.1 shows that $(\text{End } A)^\sigma$ is isomorphic to a subalgebra V of $(\text{End } A) \otimes A$. Since A is a finite projective k-module the two maps

3.4
$$\begin{gathered}(\text{End } A) \otimes A \to \text{End}_{1 \otimes A}(A \otimes A),\\ f \otimes a \to (b \otimes c \to f(b) \otimes ac)\\ (\text{End } A) \otimes A \otimes A \to \text{End}_{1 \otimes A \otimes A}(A \otimes A \otimes A),\\ f \otimes a \otimes b \to (c \otimes d \otimes e \to f(c) \otimes ad \otimes be)\end{gathered}$$

are A and $A \otimes A$ algebra isomorphisms respectively. (See [8] for the notation $\text{End}_{1 \otimes A}$ and $\text{End}_{1 \otimes A \otimes A}$.) Moreover with these isomorphisms η_2 and η_3 [8, p. 339, Theorem 2] correspond to our e_1 and e_2 respectively. Thus if σ is an Amatsur 2-cocycle the subalgebra V of $(\text{End } A) \otimes A$ is isomorphic to $A(\sigma^{-1})$ [8, p. 339, Theorem 2] with the isomorphism induced by 3.4.

3.5 THEOREM. (a) *Assume* (U, k, U) *and* (U, L, A) *satisfy* H1. *Let* $\tilde{U}$ *be a* k *algebra and* $h : L \to \tilde{U}$ *an algebra homomorphism, (giving* $\tilde{U}$ *a right* L*-module structure.) If* $\Phi : U \to \tilde{U}$ *is a right* L*-module isomorphism then there is a unique* 2*-cocycle* $\sigma \in U \otimes U \otimes A$ *and algebra isomorphism* $\Psi : U^\sigma \to \tilde{U}$ where the following diagram is commutative:

3.6


(b) *Assume* (U, L, A) *satisfies* H2. *Let* $\tilde{U}$ *be a* k *algebra and* $h : L \to \tilde{U}$ *an algebra homomorphism (giving* $\tilde{U}$ *and* L*-bimodule structure.) If* $\Phi : U \to \tilde{U}$ *is an* L*-bimodule isomorphism then there is a unique* 2*-cocycle* $\sigma \in A \otimes A \otimes A$ *(with* $e_\sigma \in A$*) and algebra isomorphism* $\Psi : U^\sigma \to \tilde{U}$ *such that* 3.6 *is commutative. In particular* h *is injective.*

(c) *Assume* (U, L, A) *satisfies* H2, $A \subset L$ *(so that* A *is commutative) and* $h : L \to \tilde{U}$ *is an injective algebra homomorphism giving* $\tilde{U}$ *an* L*-bimodule structure. If* $\Phi : U \to \tilde{U}$ *is an* L*-bimodule isomorphism and* $(\tilde{U}, h(L), h(A))$ *satisfies* H2 *then there is a unique invertible* 2*-cocycle* $\sigma \in A \otimes A \otimes A$ *(with* $e_\sigma \in A$*) and an algebra isomorphism* $\Psi : U^\sigma \to \tilde{U}$ *such that* 3.6 *is commutative.*

Proof. Since $\Phi : U \to \tilde{U}$ is a right L-module isomorphism it is a k-module isomorphism and we have defined

$$\Phi^n \equiv \otimes^n \Phi : \otimes^n U \to \otimes^n \tilde{U}.$$

Similarly for $\Phi^{-1} : \tilde{U} \to U$, the right L-module isomorphism inverse to Φ, we have defined

$$\Phi^{-n} \equiv \otimes^n \Phi^{-1} : \otimes^n \tilde{U} \to \otimes^n U.$$

If $\otimes^n U$ and $\otimes^n \tilde{U}$ have the right L-module structure given by

$$(u_1 \otimes \cdots \otimes u_n)\cdot l = u_1 \otimes \cdots \otimes u_{n-1} \otimes (u_n l)$$

$$(\tilde{u}_1 \otimes \cdots \otimes \tilde{u}_n)\cdot l = \tilde{u}_1 \otimes \cdots \otimes \tilde{u}_{n-1} \otimes (\tilde{u}_n \cdot l)$$

then Φ^n is a right L-module isomorphism with inverse Φ^{-n}.

We use Φ to give U a new algebra structure by "pulling back" the algebra structure of $\tilde{U}$. Let C be the composite

3.7 $$U \otimes U \xrightarrow{\Phi^2} \tilde{U} \otimes \tilde{U} \xrightarrow{\text{multiplication}} \tilde{U} \xrightarrow{\Phi^{-1}} U$$

and let

3.8 $$e = \Phi^{-1}(1_{\tilde{U}}).$$

Then C is an associative multiplication with unit e and $\Phi : U \to \tilde{U}$ is an algebra isomorphism between U with this new algebra structure and $\tilde{U}$.

All the maps in the composite 3.7 are right L-module morphisms so that C is. Thus by 1.3 there is a unique element $\sigma = \sum_i u_i \otimes v_i \otimes w_i \in U \otimes U \otimes A$ where

$$C(u \otimes v) = \sum_i u_i u v_i v w_i$$

for all $u \otimes v \in U \otimes U$. The map $C(C \otimes I): U \otimes U \otimes U \to U$ is a right L-module homomorphism. However,

$$U \otimes U \otimes U \to U, \qquad x \otimes y \otimes z \to \sum_{i,j} u_i u_j x v_j y w_j v_i z w_i$$

is precisely $C(C \otimes I)$, so that $\sum_{i,j} u_i u_j \otimes v_j \otimes w_j v_i \otimes w_i$ is the unique element in $U \otimes U \otimes U \otimes A$ corresponding to $C(C \otimes I)$, per 1.3. Similarly

$$\sum_{i,j} u_i \otimes v_i u_j \otimes v_j \otimes w_j w_i$$

is the unique element corresponding to $C(I \otimes C)$. By associativity of C it follows that σ satisfies 2.2.

Since e is the unit for C the map

$$U \to U, \qquad u \to C(u \otimes e)$$

is the identity. This map is precisely

$$U \to U, \qquad x \to \sum_i u_i x v_i e w_i.$$

Thus $\sum u_i \otimes v_i e w_i$ is the unique element in $U \otimes U$ corresponding to the identity, per 1.3. (Here we are using that (U, k, U) satisfies H1 and using (U, k, U) for (U, L, A) in 1.3). But $1 \otimes 1$ corresponds to the identity so that $\sum u_i \otimes v_i e w_i = 1 \otimes 1$. Similarly using that $C(e \otimes u) = u$ we have that $\sum u_i e v_i \otimes w_i = 1 \otimes 1$. Thus σ satisfies 2.3 and σ is a 2-cocycle.

If

$$U^\sigma \xrightarrow{\Psi} \tilde{U}, \qquad u^\sigma \to \Phi(u)$$

then

$$\begin{array}{ccc} \tilde{U} & \xleftarrow{\quad\Psi\quad} & U^\sigma \\ & {\scriptstyle\Phi}\nwarrow \quad \nearrow & \quad u^\sigma \\ & & \quad\nearrow \\ & U & u \end{array}$$

is commutative and Ψ is the unique map $U^\sigma \to \tilde{U}$ making the diagram commutative. Ψ is an algebra isomorphism by the definition of C and σ. Since σ is uniquely determined by C it is the unique 2-cocycle making Ψ an algebra homomorphism.

By definition of the right L-module structure on $\tilde{U}$ we have that

$$h(l) = 1_{\tilde{U}} \cdot l.$$

Thus

$$\Phi^{-1}h(l) = \Phi^{-1}(1_{\tilde{U}} \cdot l) = \Phi^{-1}(1_{\tilde{U}})l = el.$$

This gives commutativity of

$$\begin{array}{ccc} & L & \\ {\scriptstyle h}\swarrow & & \searrow{\scriptstyle H} \\ \tilde{U} & \xleftarrow{\quad\Psi\quad} & U^\sigma \end{array}$$

and we have proved (a).

Now we prove (b). We define

$$\Phi^n = \otimes_L^n \Phi : \otimes_L^n U \to \otimes_L^n \tilde{U}.$$

Note we are tensoring over L. Φ^n is well defined since Φ is an L-bimodule isomorphism. We have

$$\Phi^{-n} = \otimes_L^n \Phi^{-1} : \otimes_L^n \tilde{U} \to \otimes_L^n U.$$

If $\otimes_L^n U$ and $\otimes_L^n \tilde{U}$ have the L-bimodule structure given by

$$l \cdot (u_1 \otimes \cdots \otimes u_n) \cdot m = (lu_1) \otimes u_2 \otimes \cdots \otimes u_{n-1} \otimes (u_n m)$$

$$l \cdot (\tilde{u}_1 \otimes \cdots \otimes \tilde{u}_n) \cdot m = (l \cdot \tilde{u}_1) \otimes \tilde{u}_2 \otimes \cdots \otimes \tilde{u}_{n-1} \otimes (\tilde{u}_n \cdot m),$$

then Φ^n is an L-bimodule isomorphism with inverse Φ^{-n}.

As in the proof of part (a) we give U a new algebra structure by letting C be the composite

$$U \otimes_L U \xrightarrow{\Phi^2} \tilde{U} \otimes_L \tilde{U} \xrightarrow{\text{multiplication}} \tilde{U} \xrightarrow{\Phi^{-1}} U$$

and let

$$e = \Phi^{-1}(1_{\tilde{U}})$$

Clearly, $l1_{\tilde{U}} = 1_{\tilde{U}}\, l$ for all $l \in L$. Since Φ is an L-bimodule isomorphism we have that $le = el$ for all $l \in L$. This implies that e is in A, the centralizer of L.

The rest of the proof of part (b) is analogous to the proof of part (a) except that we rely on Lemma 1.5 instead of Lemma 1.3.

Now we prove part (c). By part (b) we only need to show that σ is invertible. The correspondence

3.9 $$\otimes^{n+1} A \to \mathrm{Hom}_{L-L}\,(\otimes_L^n \tilde{U}, \tilde{U}), \qquad a_0 \otimes \cdots \otimes a_n \to f_{a_0 \otimes \cdots \otimes a_n}$$

where

$$f_{a_0 \otimes \cdots \otimes a_n}(\tilde{u}_1 \otimes \cdots \otimes \tilde{u}_n) = h(a_0)\tilde{u}_1\, h(a_1)\tilde{u}_2 \cdots \tilde{u}_n\, h(a_n)$$

is bijective by 1.5 since $(\tilde{U}, h(L), h(A))$ satisfies H2 and h is injective by part (b).

The composite D

$$\tilde{U} \otimes_L \tilde{U} \xrightarrow{\Phi^{-2}} U \otimes U \xrightarrow{\text{multiplication}} U \xrightarrow{\Phi} \tilde{U}$$

is an L-bimodule morphism and so there is a unique element

$$\sigma' = \textstyle\sum_i u_i' \otimes v_i' \otimes w_i'$$

in $A \otimes A \otimes A$ where

$$D(\tilde{u} \otimes \tilde{v}) = \textstyle\sum_i h(u_i')\tilde{u}h(v_i')\tilde{v}h(w_i')$$

for all $\tilde{u} \otimes \tilde{v} \in \tilde{U} \otimes_L \tilde{U}$.

In view of the fact that Ψ is an algebra isomorphism and 3.6 is commutative we have that for $u, v \in U$

$$\begin{aligned} \Psi((uv)^\sigma) = \Phi(uv) &= D(\Phi(u) \otimes_L \Phi(v)) \\ &= \textstyle\sum_i h(u_i')\Phi(u)h(v_i')\Phi(v)h(w_i') \\ &= \Psi(\textstyle\sum_i H(u_i')u^\sigma H(v_i')V^\sigma(w_i')) \end{aligned}$$

so that

$$(uv)^\sigma = \textstyle\sum_i H(u_i')u^\sigma H(v_i')v^\sigma H(w_i')$$

where the indicated multiplications on the right hand side taken place in U^σ. Since $\sigma \in A \otimes A \otimes A$ we can use (2.17, b) to simplify the right hand side and obtain (recall $A \subset L$)

$$(uv)^\sigma = \textstyle\sum_i (u_i'\, uv_i')^\sigma (vw_i')^\sigma.$$

Applying the multiplication formula 2.5 we obtain

$$(uv)^\sigma = \textstyle\sum_{i,j} (u_j\, u_i'\, uv_i'\, v_j\, vw_i'\, w_j)^\sigma$$

for all $u, v \in V$. Then by 1.5 we have that

$$\textstyle\sum_{i,j} u_j\, u_i' \otimes v_i'\, v_j \otimes w_i'\, w_j = 1 \otimes 1 \otimes 1 \in A \otimes A \otimes A$$

and $\sigma' = \sigma^{-1}$, Q.E.D.

Note that in the proof of part (b) we showed that $e \in A$ by considering e as

$\Phi^{-1}(1_{\tilde{U}})$. Actually if σ is a 2-cocycle in $U \otimes A \otimes U$ then e lies in A since for all $l \in L$,

$$el = \sum_i u_i \, elv_i \, ew_i = \sum_i u_i \, ev_i \, lew_i = le$$

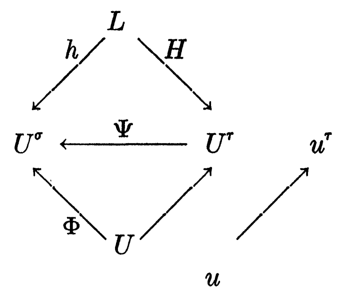
where the first and third equalities follow from (2.3) and the second from the fact that A is the centralizer of L. Thus e centralizes L so lies in A.

3.10 Corollary. *Assume (U, L, A) satisfies* H2. *If $\sigma \in U \otimes A \otimes A$ is a 2-cocycle then actually $\sigma \in A \otimes A \otimes A$ and $e_\sigma \in A$.*

Proof. We let

$$h : L \to U^\sigma, \quad l \to (el)^\sigma; \qquad \Phi : U \to U^\sigma, \quad u \to u^\sigma$$

By 2.17(b), this is an L-bimodule isomorphism. By the theorem, part (b), there is a unique 2-cocycle $\tau \in A \otimes A \otimes A$ with $e_\tau \in A$ and algebra isomorphism $\Psi : U^\tau \to U^\sigma$ where



is commutative. The commutativity of the bottom triangle implies that Ψ is the map

$$U^\tau \to U^\sigma, \qquad u^\tau \to u^\sigma.$$

Since Ψ is an algebra isomorphism it follows from 1.3(b) that $\sigma = \tau$, Q.E.D.

3.11 Corollary. *Suppose A and L are subalgebras of U where A centralizes L, $A \subset L$, A is a faithful k-module and σ is a* 2-*cocycle in $A \otimes A \otimes A$. Then if any two of the following conditions hold so does the third:*

(a) (U, L, A) *satisfies* H2,

(b) $(U^\sigma, H(L), H(A))$ *satisfies* H2,

(c) *σ is invertible, i.e. σ is an Amitsur* 2-cocycle.

Proof. This is just a combination of 2.18 and 3.5(c), Q.E.D.

4. Coboundaries

Recall the definition 2.7 for two 2-cocycles to be cohomologous. If σ, τ are 2-cocycles in $U \otimes U \otimes A$ we have the algebra homomorphisms

$$H^\sigma : L \to U^\sigma, \quad l \to (e_\sigma l)^\sigma; \qquad H^\tau : L \to U^\tau, \quad l \to (e_\tau l)^\sigma$$

as at 2.16. Suppose $\sigma \sim^\varphi \tau$ and $\varphi \in U \otimes A$. There is the algebra homomorphism $R^\varphi : U^\sigma \to U^\tau$ as defined at 2.10. One easily checks that with the additional assumption that $\varphi \in U \otimes A$ we have commutativity of

4.1
$$\begin{array}{ccc} & L & \\ {}^{H^\tau}\swarrow & & \searrow^{H^\sigma} \\ U^\tau & \xleftarrow{\quad R^\varphi \quad} & U^\sigma \end{array}$$

We call $\varphi = \sum_i x_i \otimes y_i \in U \otimes U$ vertible if there is

$$\bar{\varphi} = \sum \bar{x}_i \otimes \bar{y}_i \in U \otimes U$$

where

4.2
$$\textstyle\sum_{i,j} \bar{x}_i x_j \otimes y_j \bar{y}_i = 1 \otimes 1 = \sum_{i,j} x_i \bar{x}_j \otimes \bar{y}_j y_i .$$

(Of course this is equivalent to $\sum \bar{x}_i \otimes \bar{y}^{\text{op}}$ being the inverse to $\sum x_i \otimes y_i^{\text{op}}$.) If φ is vertible then $R^\varphi : U^\sigma \to U^\tau$ is an algebra isomorphism with inverse

$$R^{-\varphi} : U^\tau \to U^\sigma, \qquad u^\tau \to (\textstyle\sum_i \bar{x}_i u \bar{y}_i)^\sigma.$$

4.3 Theorem. (a) *Assume* (U, L, A) *and* (U, k, U) *satisfy* H1 *and* σ, τ *are two* 2-*cocycles in* $U \otimes U \otimes A$. *If* $r : U^\sigma \to U^\tau$ *is an algebra homomorphism where*

4.4
$$\begin{array}{ccc} & L & \\ {}^{H^\tau}\swarrow & & \searrow^{H^\sigma} \\ U^\tau & \xleftarrow{\quad r \quad} & U^\sigma \end{array}$$

is commutative then there is a unique element $\varphi \in U \otimes A$ *where* $\sigma \sim^{\varphi} \tau$ *and* $r = R^\varphi$. *Also, r is an algebra isomorphism if and only if* φ *is vertible. In this case if* $\bar{\varphi}$ *is the verse, then* $\tau \sim^{\bar{\varphi}} \sigma$.

(b) *Assume* (U, L, A) *satisfies* H2 *and* σ, τ *are two* 2-*cocycles in* $A \otimes A \otimes A$. *If* $r : U^\sigma \to U^\tau$ *is an algebra homomorphism where* 4.4 *is commutative then there is a unique element* $\varphi \in A \otimes A$ *where* $\sigma \sim^{\varphi} \tau$ *and* $r = R^\varphi$. *Also, r is an algebra isomorphism if and only if* φ *is vertible. In this case if* $\bar{\varphi}$ *is the verse then* $\tau \sim^{\bar{\varphi}} \sigma$.

(c) *Assume* (U, L, A) *satisfies* H2, $A \subset L$ *and* σ, τ *are two invertible* 2-*cocycles in* $A \otimes A \otimes A$. *If* $r : U^\sigma \to U^\tau$ *is an algebra homomorphism where* 4.4 *is commutative then there is a unique element* $\varphi \in A \otimes A$ *where* $\sigma \sim^{\varphi} \tau$ *and* $r = R^\varphi$. *Moreover, r is an algebra isomorphism and* φ *is vertible, which is the usual notion of invertibility since A is commutative. If* $\bar{\varphi}$ *is the inverse then* $\tau \sim^{\bar{\varphi}} \sigma$.

Proof. r induces $\tilde{r} : U \to U$ where

$$\begin{array}{cccccc} u^\tau & U^\tau & \xleftarrow{\quad r \quad} & U^\sigma & u^\sigma \\ \uparrow & \uparrow \cong & & \cong \uparrow & \uparrow \\ u & U & \xleftarrow{\quad \tilde{r} \quad} & U & u \end{array}$$

is commutative. By 3.4 (a) the vertical maps are right L-module isomorphisms and by the commutativity of 4.4, r is a right L-module morphism. Thus $\tilde{r} \in \mathrm{Hom}_{-L}(U, U)$ and since (U, L, A) satisfies H1 there is a unique element $\varphi = \sum x_i \otimes y_i \in U \otimes A$ where

$$\tilde{r}(u) = \sum_i x_i u y_i$$

for all $u \in U$. Thus

4.5 $$r(u^\sigma) = \left(\sum_i x_i u y_i\right)^\tau$$

for all $u^\sigma \in U^\sigma$.

Since r is an algebra homomorphism we have that for all $u, v \in U$,

4.6 $$\begin{aligned}\left(\sum_{i,j} x_i u_j uv_j vw_j y_i\right)^\tau = r(u^\sigma v^\sigma) &= r(u^\sigma) r(v^\sigma) \\ &= \left(\sum_{i,j,q} r_i (x_j u y_j) s_i (x_q v y_q) t_i\right)^\tau\end{aligned}$$

and

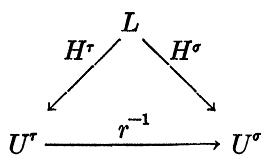
4.7 $$e_\tau = 1_{U^\tau} = \mathrm{r}(1_{U^\sigma}) = \sum_i x_i e_\sigma y_i,$$

where $\sigma = \sum_i u_i \otimes v_i \otimes w_i$ and $\tau = \sum r_i \otimes s_i \otimes t_i$.
By 4.6 and 1.3 (a),

$$\sum_{i,j} x_i u_j \otimes v_j \otimes w_j y_i = \sum_{i,j,q} r_i x_j \otimes y_j s_i x_q \otimes y_q t_i .$$

This and 4.7 show that $\sigma \sim^{\varphi} \tau$. Equation 4.5 shows that $r = R^{\varphi}$.

The remarks just before the theorem plus the fact $r = R^{\varphi}$ imply that r is an isomorphism if φ is vertible. If r is an algebra isomorphism with inverse r^{-1} then



is commutative and by what we have just shown there is a unique element

$$\bar{\varphi} = \sum \bar{x}_i \otimes \bar{y}_i \in U \otimes A$$

where $\tau \sim^{\bar{\varphi}} \sigma$ and $r^{-1} = R^{\bar{\varphi}}$. Then for all $u \in U$

$$\left(\sum_{i,j} \bar{x}_i x_j u y_j \bar{y}_j\right)^\sigma = r^{-1} r(u_\sigma) = u^\sigma$$

and

$$\left(\sum_{i,j} x_i \bar{x}_j u \bar{y}_j y_i\right)^\tau = r r^{-1}(u^\tau) = u^\tau .$$

Thus, since (U, L, A) satisfies H1 we have that

$$\sum_{i,j} \bar{x}_i x_j \otimes y_j \bar{y}_i = 1 \otimes 1 = \sum_{i,j} x_i \bar{x}_j \otimes \bar{y}_j y_i$$

and φ is vertible.

The proof of part (b) is analogous to the proof of part (a). We leave the details to the reader.

It follows from part (b) that under the hypotheses of part (c) there is an element $\varphi \in A \otimes A$ where $\sigma \sim^{\varphi} \tau$ and $r = R^{\varphi}$. If we show that φ has inverse $\bar{\varphi}$ then r is an algebra isomorphism with inverse $R^{\bar{\varphi}}$ and $\tau \sim_{\bar{\varphi}} \sigma$ by part (b).

Say $\sigma = \sum_i u_i \otimes v_i \otimes w_i$, $\tau = \sum_i r_i \otimes s_i \otimes t_i$ and $\varphi = \sum_i x_i \otimes y_i$. Since $\sigma \sim^{\varphi} \tau$ we have from 2.8 that

$$\sum_{i,j} x_i u_j \otimes v_j \otimes w_j y_i = \sum_{i,j,q} r_i x_j \otimes y_j s_i x_q \otimes y_q t_i .$$

This and the commutativity of A imply

4.8 $$\sum_{i,j} x_i y_i u_j w_j \otimes v_j = \sum_{i,j,q} x_j y_q r_i t_i \otimes y_j x_q s_i .$$

By hypothesis, σ and τ are invertible so that

$$a = \sum_j u_j w_j \otimes v_j , \quad b = \sum_i r_i t_i \otimes s_i , \quad e_\sigma \quad \text{and} \quad e_\tau$$

are all invertible. Also,

$$e_\tau e_\sigma^{-1} = \sum_i x_i y_i$$

by 2.9. Thus 4.8 implies that

$$(e_\tau^{-1} e_\sigma \otimes 1) b a^{-1} (\sum y_q \otimes x_q) \in A \otimes A$$

is the inverse to φ, Q.E.D.

5. Formalities

We can now apply our theorems to show what is classified by the 2-cohomology.

5.1 Case A. Assume (U, L, A) and (U, k, U) satisfy H1. Consider pairs $(\tilde{U}, h)$ where $\tilde{U}$ is a k algebra and $h : L \to \tilde{U}$ is an algebra homomorphism making $\tilde{U}$ isomorphic to U as a right L-module. Two pairs $(\tilde{U}, h)$ and $(\tilde{U}, h')$ are equivalent if there is an algebra isomorphism $r : \tilde{U} \to \tilde{U}'$ where $rh = h'$.

By 3.5(a) and 4.3(a) the equivalence classes of such pairs are in bijective correspondence with the equivalence classes of 2-cocycles in $U \otimes U \otimes A$ where two 2-cocycles are considered equivalent if they are cohomologous via a vertible element of $U \otimes A$ (with verse in $U \otimes A$).

5.2 Case B. Assume (U, L, A) satisfies H2. Consider pairs $(\tilde{U}, h)$ where $\tilde{U}$ is a k algebra and $h : L \to \tilde{U}$ is an algebra homomorphism making $\tilde{U}$ isomorphic to U as an L-bimodule. Two pairs $(\tilde{U}, h)$ and $(\tilde{U}', h')$ are equivalent if there is an algebra isomorphism $r : \tilde{U} \to \tilde{U}'$ where $rh = h'$.

By 3.5(b) and 4.3(b) the equivalence classes of such pairs are in bijective correspondence with the equivalence classes of 2-cocycles in $A \otimes A \otimes A$ where two 2-cocycles are considered equivalent if they are cohomologous via a vertible element in $A \otimes A$.

5.3 Case C. Assume (U, L, A) satisfies H2, $A \subset L$ (so that A is commutative) and A is a faithful k-module. Consider pairs $(\tilde{U}, h)$ where $\tilde{U}$ is a k algebra and $h : L \to \tilde{U}$ is an algebra homomorphism making $(\tilde{U}, h(L), h(A))$ satisfy H2 and making $\tilde{U}$ isomorphic to U as an L-bimodule. Two pairs

$(\tilde{U}, h)$ and $(\tilde{U}', h')$ are equivalent if there is an algebra isomorphism $r : \tilde{U} \to \tilde{U}'$ where $rh = h'$.

By 2.18, 3.5(c) and 4.3(c) the equivalence classes of such pairs are in bijective correspondence with the second Amitsur cohomology group of A over k.

6. Applications

6.1 THEOREM. *Assume k is a field and U is an n^2-dimensional ($n^2 < \infty$) central separable k algebra. If $\tilde{U}$ is any n^2-dimensional k algebra then there is a 2-cocycle $\sigma \in U \otimes U \otimes U$ where $e_\sigma = 1$ and $\tilde{U} \cong U^\sigma$ as an algebra.*

Proof. By 1.6 (U, k, U) satisfies H1. Let $L = k$ and $h : L \to \tilde{U}$ the "unit" map. Since U and $\tilde{U}$ have the same dimension they are isomorphic right L-modules and we can choose such an isomorphism $\Phi : U \to \tilde{U}$ which has the further property that $\Phi(1_U) = 1_{\tilde{U}}$. By Theorem 3.5(a) there is a 2-cocycle $\sigma \in U \otimes U \otimes U$ where $\tilde{U} \cong U^\sigma$ as an algebra. From 3.8 in the proof of 3.5 we see that $e_\sigma = 1$, Q.E.D.

6.2 THEOREM. *Assume k is a field and L a finite n-dimensional extension field of k. Let $L \otimes L$ have the L-bimodule structure given by*

$$l \cdot (m \otimes n) \cdot \bar{l} = (lm) \otimes (n\bar{l}).$$

Let $\tilde{U}$ be a k algebra with subalgebra L which gives $\tilde{U}$ and L-bimodule structure. If $\tilde{U}$ is an n^2-dimensional central separable k algebra *then $\tilde{U}$ is isomorphic to $L \otimes L$ as an L-bimodule. If k has positive characteristic and L is a purely inseparable extension of k, then $\tilde{U}$ is an n^2-dimensional central separable k algebra if $\tilde{U}$ is isomorphic to $L \otimes L$ as an L-bimodule.*

Proof. Suppose that $\tilde{U}$ is an n^2-dimensional central separable k algebra. $\tilde{U}$ is a projective left L-module (since L is a field) and $\tilde{U}^{\text{op}}$ is a projective left $L^{\text{op}} = L$-module so that $\tilde{U} \otimes_k \tilde{U}^{\text{op}}$ is a projective left $L \otimes L$-module.

$$\tilde{U} \otimes \tilde{U}^{\text{op}} \to \text{End}\, \tilde{U}, \qquad \tilde{u} \otimes \tilde{v}^{\text{op}} \to f_{\tilde{u} \otimes \tilde{v}}$$

where $f_{\tilde{u}\otimes\tilde{v}}(\tilde{w}) = \tilde{u}\tilde{w}\tilde{v}$ is an algebra isomorphism so that—by the induced action—$\tilde{U}$ is a projective faithful $\tilde{U} \otimes \tilde{U}^{\text{op}}$-module. Thus considering $\tilde{U}$ as an $L \otimes L$-module by

$$(l \otimes m) \cdot \tilde{w} = l\tilde{w}m$$

we have that $\tilde{U}$ is a projective faithful n^2-dimensional $L \otimes L$-module. (Here we have used "projective over projective is projective.")

$L \otimes L$ is a commutative Artinian algebra so that as an algebra $L \otimes L = \oplus_{i=1}^m R_i$ where each R_i is a primary hence local algebra. Following the decomposition of $L \otimes L$ we have that

$$\tilde{U} = \oplus_{i=1}^m \tilde{U}_i$$

where each $\tilde{U}_i$ is a projective R_i-module. Since R_i is local each $\tilde{U}_i$ is a free R_i-module and so $\dim \tilde{U}_i = n_i \dim R_i$. Since $\tilde{U}$ is a faithful $L \otimes L$-module no $\tilde{U}_i$ is equal to zero. Thus for each $i = 1, \cdots, m, n_i \geq 1$. With the equalities

$$\textstyle\sum_{i=1}^{m} n_i \dim R_i = \dim \tilde{U} = \dim L \otimes L = \sum_{i=1}^{m} \dim R_i$$

we have that each $n_i = 1$ and

$$\tilde{U} \cong \oplus_{i=1}^{m} R_i = L \otimes L$$

as an $L \otimes L$-module. Thus $\tilde{U}$ is isomorphic to $L \otimes L$ as an L-biomodule.

Conversely suppose $\tilde{U} \cong L \otimes L$ as an L-bimodule. As in 2.14 we consider L as a subalgebra of $U = \text{End } L$. Since U is an n^2-dimensional central separable k algebra we have that $U \cong L \otimes L$ as an L-bimodule by what we have already shown. Thus $U \cong \tilde{U}$ as an L-bimodule. By 1.6, 2.15 and 3.5(b) there is an invertible 2-cocycle σ in $L \otimes L \otimes L$ where $\tilde{U} \cong U^\sigma$ *as an algebra.* By 2.14, U^σ hence $\tilde{U}$ is an n^2-dimensional central separable k algebra, Q.E.D.

We first announced Theorem 6.2 without the hypothesis of L being purely inseparable over k; whereupon Chase gave a direct proof of the theorem *with* the hypothesis of pure inseparability and then Waterhouse gave an example showing that the hypothesis is needed. The example of Waterhouse shows that there is a 2-cocycle in $C \otimes_R C \otimes_R C$ which is not invertible. (C is the complexes and R the reals.)

6.3 LEMMA. *Assume that L is an n-dimensional field extension of the field k, $\tilde{U}$ and $\tilde{U}'$ are n^2-dimensional central separable k algebras and $h : L \to \tilde{U}$, $h' : L \to \tilde{U}'$ are algebra homomorphisms. Then $\tilde{U} \cong U'$ as algebras if and only if there is an algebra isomorphism $r : \tilde{U} \to \tilde{U}'$ where $rh = h'$.*

Proof. This follows immediately from [5, p. 110, Theorem of Skolem-Noether].

If K is an intermediate field $L \supset K \supset k$ there is a natural map

6.4 $$\chi : L \otimes L \otimes L \to L \otimes_K L \otimes_K L$$

which maps Amitsur 2-cocycles for L over k to Amitsur 2-cocycles for L over K. The collection of maps of the form

$$\otimes^n L \to \otimes_K^n L$$

induces a homomorphism from the Amitsur cohomology of L over k to the Amitsur cohomology of L over K.

If U is a central separable k algebra with maximal subfield L then by 1.7 and 1.9 the centralizer of K in U is a central separable K algebra. Let us denote the centralizer of K in U by $\xi(U)$. If $[L:K] = n_1$ and $[K:k] = n_2$ then $[L:k] = n_1 n_2$ and $\dim_k U = n_1^2 n_2^2$. By 1.8, $\dim_k \xi(U) = n_1^2 n_2$ so that $\dim_K \xi(U) = n_1^2$. This implies that L is a maximal subfield of $\xi(U)$.

We now are in a position to prove how χ and ξ correspond.

Consider two central separable k algebras with maximal subfield L equivalent if they are isomorphic as k algebras. Let $\mathcal{C}(L, k)$ be the equivalence classes of the central separable k algebras with maximal subfield L. Let U be End L so that (U, L, L) satisfies H2. By 1.6, 1.10, 6.3 and 6.2 the equivalence classes of pairs in 5.3 correspond to the "elements" of $\mathcal{C}(L, k)$. And by 5.3 we have a bijective correspondence with the second Amitsur cohomology group of L over k. Explicitly, this correspondence is

$$[\sigma] \leftrightarrow [U^\sigma]$$

where [ ] denotes "equivalence class" and σ is an Amitsur 2-cocycle.

Recall $H : L \to U^\sigma$ is given by $l \to (e_\sigma\, l)$ (3.3).

In $U = \text{End}\, L$ we have $\text{End}_K L$ which is in fact $\xi(U)$. For $x \in \text{End}_K L$, $\lambda \in K$,

$$H(\lambda)x^\sigma = (\lambda x)^\sigma = (x\lambda)^\sigma = xH(\lambda),$$

the first and third equality follow from 2.17(b). Thus

$$(\text{End}_K\, L)^\sigma \equiv \{x^\sigma \in (\text{End}\, L)^\sigma \mid x \in \text{End}_K\, L\}$$

is contained in $\xi((\text{End}\, L)^\sigma)$. Counting K dimension shows that $(\text{End}_K\, L)^\sigma$ is exactly $\xi((\text{End}\, L)^\sigma)$. Clearly the (sub) algebra structure induced on $(\text{End}_K\, L)^\sigma$ is the same as if we had taken

$$\chi(\sigma) \in L \otimes_K L \otimes_K L$$

and formed $(\text{End}_K\, L)^{\chi(\sigma)}$, by 2.5. Thus we have the commutative diagram

6.5
$$\begin{array}{ccccc}
H^2(L, k) & \longleftrightarrow & & \longleftrightarrow & \mathcal{C}(L, k) \\
\Big\downarrow & [\sigma] & \longleftrightarrow & [U^\sigma] & \Big\downarrow \\
& \Big\downarrow & & \Big\downarrow & \\
& [\chi(\sigma)] & \longleftrightarrow & [\xi(U^\sigma)] & \\
H^2(L, K) & \longleftrightarrow & & \longleftrightarrow & \mathcal{C}(L, K)
\end{array}$$

where $H^2(L, -)$ denotes the second Amitsur cohomology group of L over $-$, and σ is an Amitsur 2-cocycle in $L \otimes L \otimes L$.

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