ASYMPTOTIC DISTRIBUTION OF EIGENVALUES AND EIGEN-FUNCTIONS OF A GENERAL CLASS OF ELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS[1]

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The asymptotic distribution of eigenvalues and eigenfunctions of a class of elliptic pseudo-differential operators considered recently by Eskin and Visik [2], was studied by the writer in [6]. The purpose of this paper is to extend those results to the more general class of elliptic pseudo-differential operators A of positive order α on a bounded open set Ω of R^n.

More specifically, let A be an elliptic operator of positive order α on Ω with symbol $\tilde{A}(x, \xi)$ and let $\tilde{A}_j(x^j, \xi)$ be the symbol of the principal part of A in a local coordinates system. Suppose that

$$\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi) \quad \text{for } x_n^j = 0$$

where $\tilde{A}_j^+$ is homogeneous of order k in ξ, $k \geq 0$ and independent of x^j, analytic in $\operatorname{Im} \xi_n > 0$; $\tilde{A}_j^-$ is homogeneous of order $\alpha - k$ in ξ with an analytic continuation in $\operatorname{Im} \xi_n \leq 0$.

Let A_2 be the realization of A as an operator in $L^2(\Omega)$ under null "regular" boundary conditions. If A_2 is self-adjoint, it is shown that

$$\text{(i)} \qquad N(t) = \sum_{\lambda_j \leqq t} 1 = (2\pi)^{-n} t^{n/\alpha} \int_\Omega \int_{\tilde{A}(x,\xi)<1} d\xi \, dx + o(t^{n/\alpha})$$

$$\text{(ii)} \qquad e(x, x, t) = (2\pi)^{-n} t^{n/\alpha} \int_{\tilde{A}(x,\xi)<1} d\xi + o(t^{n/\alpha}); \qquad x \text{ in } \Omega.$$

$$e(x, y, t) = \textstyle\sum_{\lambda_j \leqq t} \varphi_j(x)\overline{\varphi_j(y)} = o(t^{n/\alpha}); \quad x \neq y.$$

λ_j, φ_j are respectively the eigenvalues and eigenfunctions of A_2.

We shall use the method of Garding [3] as extended by Browder in [1]. The notations and the definitions are essentially those of Eskin and Visik [2], they are given in Section 1. The asymptotic behavior of the kernel of $(A_2 + tI)^{-2m}$ where m is the smallest positive integer such that $m\alpha > n/2$. is studied in Section 2. The results are obtained by an application of the Hardy-Littlewood Tauberian theorem.

Section 1

Let Ω be a bounded open set of R^n with a smooth boundary $\partial\Omega$. $H^{s,2}(\Omega)$, $s \geq 0$, which shall be written as $H^s(\Omega)$ for short, denotes the usual Sobolev

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space and $H_+^s(\Omega)$ is the space of generalized functions f defined on all of R^n, equal to 0 on $R^n/\text{cl}\,\Omega$ and coinciding with functions in $H^s(\Omega)$ on $\text{cl}\,\Omega$.

DEFINITION 1. *$\tilde{A}_+(\xi)$ is in C_k^+ iff:*

(i) *$\tilde{A}_+(\xi)$ is homogeneous of order k in ξ, continuous for $\xi \neq 0$ and has an analytic continuation in* $\text{Im}\,\xi_n > 0$ *for each fixed* $\xi' = (\xi_1, \cdots, \xi_{n-1})$.

(ii) *$\tilde{A}_+(\xi) \neq 0$ for $\xi \neq 0$ and for any positive integer p, there is an expansion*

$$\tilde{A}_+(\xi) = \textstyle\sum_{s=0}^{p} c_s(\xi')\xi_+^{-ks} + R_{k,p+1-k}(\xi); \qquad \xi_+ = \xi_n + i|\xi'|$$

where all the terms are homogeneous of orders k in ξ *with analytic continuation in* $\text{Im}\,\xi_n > 0$ *and*

$$|R_{k,p+1-k}(\xi)| \leq C|\xi'|^{p+1}(|\xi'| + |\xi_n|)^{k-p-1}.$$

DEFINITION 2. *$\tilde{A}(x, \xi)$ is in $\hat{D}_{\alpha,1}^1$ iff:*

(i) *$\tilde{A}(x, \xi)$ is infinitely differentiable in X and in ξ for $\xi \neq 0$.*

(ii) *$\tilde{A}(x, \xi)$ is homogeneous of order α* in ξ.

(iii) $|D_x^p \tilde{A}(x, \xi)| \leq C_p(+|\xi|)^{\alpha}; \quad 0 \leq |p| = \sum_{j=1}^{n} p_j < \infty.$

(iv) *For any x in R^n and for any $s \geq -\alpha$, there is a decomposition*

$$(\xi_- - i)^s \tilde{A}(x, \xi) = \tilde{A}_-(x, \xi) + R(x, \xi); \qquad \xi_- = \xi_n - i|\xi'|,$$

$\tilde{A}_-(x, \xi)$ and $R(x, \xi)$ are infinitely differentiable with respect to x. Moreover $\tilde{A}_-(x, \xi)$ has an analytic continuation in $\text{Im}\,\xi_n \leq 0$ *and*

$$|D_x^p \tilde{A}_-(x, \xi)| \leq C_p(1 + |\xi|)^{s+\alpha}, \quad |D_x^p D_\xi \tilde{A}_-(x, \xi)| \leq c_p(1 + |\xi|)^{s+\alpha-1}$$

$$|D_x^p R(x, \xi)| \leq C_p(1 + |\xi'|)^{s+1+\alpha}(1 + |\xi|)^{-1},$$

$$|D_x^p D_\xi R(x, \xi)| \leq c_p(1 + |\xi'|)^{s+\alpha}(1 + |\xi|)^{-1}.$$

Let $\tilde{A}(\xi)$ be homogeneous of positive order α in ξ and $\tilde{A}(\xi) \neq 0$ for $\xi \neq 0$. Let $u \in H^s(R_+^n)$ with $u(x) = 0$ for $x_n \leq 0$. Then $Au = F^{-1}\{\tilde{A}(\xi)\tilde{u}(\xi)\}$ where the inverse Fourier transform F^{-1} is taken in the sense of the theory of distributions is well-defined. Here $\tilde{u}(\xi)$ denotes the Fourier transform of $u(x)$.

Suppose $\tilde{A}(x, \xi)$ for x in $\text{cl}\,\Omega$ is infinitely differentiable with respect to x and ξ, homogeneous of order α in ξ and $\tilde{A}(x, \xi) \neq 0$ for $\xi \neq 0$. We extend $\tilde{A}(x, \xi)$ with respect to x to all of R^n with preservation of homogeneity with respect to ξ. $\tilde{A}(x, \xi)$ may be expanded in Fourier series

$$\tilde{A}(x, \xi) = \textstyle\sum_{k=-\infty}^{\infty} \psi(x) \exp(-i\pi kx/p)\tilde{L}_k(\xi), \qquad k = (k_1, \cdots, k_n)$$

and

$$\tilde{L}_k(\xi) = (2p)^{-n} \int_{-p}^{p} \exp(-i\pi kx/p)\tilde{A}(x, \xi)\, dx,$$

$\psi(x) \in C_c^\infty(R^n); \psi(x) = 1$ for $|x| \leq p - \varepsilon; \psi(x) = 0$ for $|x| \geq p$.

Let P^+ be the restriction operator of functions from R^n to Ω. For $u \in H_+^\alpha(\Omega)$, define

$$P^+Au = P^+(\textstyle\sum_{k=-\infty}^{\infty} \psi(x) \exp(-i\pi kx/p)L_k\, u).$$

Let $\{\varphi_j\}$ be a finite partition of unity corresponding to a finite open covering

$\{N_j\}$ of cl Ω and let $\{\psi_j\}$ be the infinitely differentiable functions with compact supports in N_j and such that $\varphi_j \psi_j = \varphi_j$.

Throughout the paper, we consider elliptic pseudo-differential operators

$$P^+Au = \sum_j P^+\varphi_j A\psi_j + \sum_j P^+\varphi_j A(1 - \psi_j)$$

of positive order α on Ω with the following properties:

(i) If $\varphi_j A_j \psi_j$ is the principal part of $\varphi_j A\psi_j$ in a local coordinates system then $\tilde{A}_j(x^j, \xi)$ is homogeneous of order α in ξ and for $x_n^j = 0$, admits a factorization

$$\tilde{A}_j(x^j, \xi) = \tilde{A}_j^+(x^j, \xi)\tilde{A}_j^-(x^j, \xi)$$

where $\tilde{A}_j^+ \in C_k^+$, $\tilde{A}_j^-$ is homogeneous of order $\alpha - k$ in ξ and has an analytic continuation in $\operatorname{Im} \xi_n \leq 0$.

(ii) $\tilde{A}_j^+(x^j, \xi) \in \hat{D}^1_{\alpha,1}$ for $x \in N_j \cap \partial\Omega \neq 0$.

If $k > 0$, we consider

$$P^+B_r = \sum_j P^+\varphi_j B_r \psi_j + \sum_j P^+\varphi_j B_r(1 - \psi_j); \qquad r = 1, \cdots, k.$$

B_r are pseudo-differential operators of orders α_r with $0 \leq \alpha_r < \alpha$. Let $\varphi_j B_{rj} \psi_j$ be the principal part of $\varphi_j B_r \psi_j$ in a local coordinates system; then $\tilde{B}_{rj}(x^j, \xi)$ are assumed to be in $\hat{D}^1_{\alpha_i,1}$.

Set

$$\mathfrak{A} = \sum\nolimits'_{j,s} \varphi_j A\varphi_s$$

where the summation is taken over all j, s with supp $\varphi_j \cap$ supp $\varphi_s \neq 0$

Define the operator A_2 on $L^2(\Omega)$ as follows:

$$D(A_2) = \{u : u \in H_+^\alpha(\Omega); \gamma P^+B_r u = 0; r = 1, \cdots, k\}$$

and $A_2 u = P^+\mathfrak{A}u$ if $u \in D(A_2)$. γ denotes the passage to the boundary.

If $k = 0$, no boundary conditions are required.

ASSUMPTION (I). *We assume throughout the paper that for* $t \geq t_0 > 0$, $(A_2 + tI)$ *is a 1-1 mapping of* $D(A_2)$ *onto* $L^2(\Omega)$. *Moreover there exist positive constants* C_1, C_2 *independent of t such that*

$$\| u \|_{s\alpha} + t^s \| u \| \leq C_1 \| (A_2 + t)^s u \| \leq C_2\{\| u \|_{s\alpha} + t^s \| u \|\}$$

for all u in $D(A_2 + t)^s$; $s \geq 1$.

Concrete hypotheses on $\tilde{A}_j(x^j, \xi)$; $\tilde{B}_{rj}(x^j, \xi)$ may be given so that Assumption (I) is verified (cf. [5]).

Section 2

In this section, we shall first study the asymptotic behavior of the kernel $\mathcal{G}(x, y, t)$ of $(A_2 + tI)^{-2m}$ as $t \to +\infty$ where m is the smallest integer such that $2m\alpha > n$. Then we show that

$$\lim_{t\to+\infty} t^{2m-n/\alpha}\{\mathcal{G}(x, y, t) - G(x, y, t)\} = 0$$

where $G(x, y, t)$ is the kernel of $(A_2 + tI + T)^{-2m}$. T is such that T^j is A_2^{j}-bounded with zero A_2^{j}-bound; $1 \leq j \leq m$.

THEOREM 1: *Let A_2 be as in Section 1. Suppose further that*
(i) *Assumption* (I) *is satisfied,*
(ii) $C_c^\infty(\Omega) \subset D(A_2)$,
(iii) A_2 *is self-adjoint.*
Then for $t \geq t_0 > 0$,

$$(A_2 + tI)^{-2m} f(x) = \int_\Omega \mathcal{G}(x, y, t)\overline{f(y)}\, dy$$

for f in $L^2(\Omega)$. m is the smallest positive integer such that $2m\alpha > n$. Moreover

$$| \mathcal{G}(x, y, t) | \leq Ct^{-2m+n/\alpha}$$

for all x, y in Ω;

$$\| (A_2 + tI)^m \mathcal{G}(x, \cdot , t) \| \leq Ct^{-m+n/2\alpha}.$$

Let L be an extension of $\mathcal{G}(x, \cdot , t)$ from Ω to R^n such that

$$\| L\mathcal{G}(x, \cdot , t) \|_{H^{m\alpha}(R^n)} \leq C \| \mathcal{G}(x, \cdot , t) \|_{H^{m\alpha}(\Omega)} .$$

Then $L\mathcal{G}(x, \cdot , t) \in D(A_2 + tI)^m$. The different constants C are all independent of x, t.

Proof. The proof is essentially the same as that of Lemma 1.7 of Browder [1]. Cf. also [6]. We shall not reproduce it.

PROPOSITION 1. *Let $\varphi \in C_c^\infty(\Omega)$; then $\mathcal{A}\varphi \in C_c^\infty(\Omega)$.*

Proof. Since $\varphi \in C_c^\infty(\Omega)$ and $\tilde{A}_j(x^j, \xi) \in \hat{D}^1_{\alpha,1}$, it follows from a result of Eskin and Visik [2] that $\mathcal{A}\varphi \in C^\infty(\Omega)$. It is trivial to check that supp $(\mathcal{A}\varphi) \subset \Omega$.

PROPOSITION 2. $\mathcal{A}^s u = A^s u + T_s u$ *for all u in $H^{s\alpha}(R^n)$ where s is a positive integer and T_s is a bounded linear mapping of $H^{s\alpha+k}(R^n)$ into $H^{k+1}(R^n)$; $k \geq 0$.*

Proof. By hypothesis, we have

$$\mathcal{A}u = \textstyle\sum'_{j,s} \varphi_j A\varphi_s u,$$

$$\mathcal{A}^2 u = \mathcal{A}(\mathcal{A}u) = \textstyle\sum'_{r,k} \varphi_r A\varphi_k (\sum'_{j,s} \varphi_j A\varphi_s u) = \sum'_{r,k} \sum'_{j,s} \varphi_r A (\varphi_k \varphi_j A\varphi_s u)$$

By Lemma 3.D.1 of [2, p. 144], one may write

$$\varphi_r A (\varphi_k \varphi_j A\varphi_s u) = A (\varphi_r \varphi_k \varphi_j A\varphi_s u) + T^{(1)} (\varphi_k \varphi_j A\varphi_s u)$$

where $T^{(1)}$ is a "smoothing" operator with respect to A in the sense of Eskin-Visik; i.e. $\| T^{(1)}v \|_m \leq C \| v \|_{\alpha+m-1}$ for any positive integer m. So

$$\mathcal{A}^2 u = \textstyle\sum'_{j,s} A (\varphi_j A\varphi_s u) + T^{(1)} (\sum'_{j,s} \varphi_j A\varphi_s u).$$

Applying the same lemma again, one gets

$$\begin{aligned}\mathcal{A}^2 u &= A^2 u + T^{(2)} (Au) + T^{(1)} (\textstyle\sum'_{j,s} \varphi_j A\varphi_s u) \\ &= A^2 u + T^{(3)} u\end{aligned}$$

where $\| T^{(3)}u \|_m \leq C \| u \|_{2\alpha+m-1}$.

We prove by induction. Suppose that

$$\mathfrak{A}^{s-1}u = A^{s-1}u + T_{s-1}\,u \quad \text{with } \| T_{s-1}\,u \|_m \leq C \, \| u \|_{(s-1)\alpha+m-1}.$$

We show that it is true for s.

$$\mathfrak{A}^{s}u = \mathfrak{A}(\mathfrak{A}^{s-1}u) = \textstyle\sum'_{j,k} \varphi_j \, A(\varphi_k \, \mathfrak{A}^{s-1}u)$$
$$= \textstyle\sum'_{j\,k} \varphi_j \, A(\varphi_k \, A^{s-1}u + \varphi_k \, T_{s-1}\,u)$$

Applying the same lemma again, we obtain

$$\mathfrak{A}^{s}u = A^{s}u + T'(A^{s-1}u) + \textstyle\sum'_{j\,k} \varphi_j \, A(\varphi_k \, T_{s-1}\,u) = A^{s}u + T_s\,u.$$

By a trivial computation, we get $\| T_s\,u \|_m \leq C \, \| u \|_{s\alpha+m-1}$.

PROPOSITION 3. *Let A be as in Section 1 and A_{x_0} be the pseudo differential operator A with symbol evaluated at x_0. Then*

$$\| (A^{s}_{x_0} A - AA^{s}_{x_0})u \|_k \leq C \, \| u \|_{s\alpha+\alpha+k-1} \quad \textit{for all } u \in H^{(s+1)\alpha+k}(R^n)$$

where k is any positive integer.

Proof. By definition, we have

$$A\varphi = \textstyle\sum_{m=-\infty}^{\infty} \psi(y) \exp\,(-i\pi ym/1) L_m \, \varphi$$

with $| \tilde{L}_m(\xi) | \leq C(N) \; |\xi|\alpha(1 + |m|)^{-N}$. N is a large positive number. Consider

$$A^{s}_{x_0} \, A\varphi = A^{s}_{x_0}(\textstyle\sum_{m=-\infty}^{\infty} \psi(y) \exp\,(-iym/1) L_m \, \varphi)$$
$$= A^{s}_{x_0}(\textstyle\sum_{m=-\infty}^{\infty} \phi_m \, L_m \, \varphi) \quad \text{with } \phi_m = \psi(y) \exp\,(-i\pi ym/1).$$

Let $g \in C^{\infty}_{c}(R^n)$. By the Parseval formula, we have

$$(A^{s}_{x_0} \, A\varphi, g) = (A^{s}_{x_0}\{ \textstyle\sum_{m=-\infty}^{\infty} \phi_m \, L_m \, \varphi\}, g) = (F\{ \textstyle\sum_{m=-\infty}^{\infty} \phi_m \, L_m \, \varphi\}, F(A^{s}_{x_0} \, g)).$$

From Lemma 1.D.1 of [2, p. 140], we get

$$\phi_m \, L_m \, \varphi = L_m \, \phi_m \, \varphi + T_m \, \varphi$$

with

$$\| T_m \, \varphi \|_k \leq C \, | m |^{n+3+k+}\alpha(1 + |m|)^{-N} \, \| \varphi \|_{k+\alpha-1}.$$

C is independent of m.

Let $T = \sum_{m=-\infty}^{\infty} T_m$. Taking N large enough, we obtain

$$\| T\varphi \|_k \leq C \, \| \varphi \|_{k+\alpha-1}.$$

So

$$(A^{s}_{x_0} \, A\varphi, g) = (F\{ \textstyle\sum_{m=-\infty}^{\infty} L_m(\phi_m \, \varphi)\}, F(A^{s}_{x_0} \, g)) + (A^{s}_{x_0} \, T\varphi, g).$$

It is easy to check that

$$\begin{aligned}(A_{x_0}^s A\varphi, g) &= \sum_{m=-\infty}^{\infty} (FL_m(\phi_m \varphi), F(A_{x_0}^s g)) + (A_{x_0}^s T\varphi, g)\\ &= \sum_{m=-\infty}^{\infty} (A_{x_0}^s L_m(\phi_m \varphi), g) + (A_{x_0}^s T\varphi, g)\\ &= \sum_{m=-\infty}^{\infty} (L_m(A_{x_0}^s(\phi_m \varphi)), g) + (A_{x_0}^s T\varphi, g).\end{aligned}$$

Again by applying Lemma 1.D.1 of [2], we get

$$A_{x_0}^s(\phi_m \varphi) = \phi_m A_{x_0}^s \varphi + S_m \varphi$$

with

$$\| S_m \varphi \|_k \leq C \,| m |^{n+3+k+s} \| \varphi \|_{s\alpha+k-1}.$$

Hence

$$(A_{x_0}^s A\varphi, g) = \sum_{m=-\infty}^{\infty} (L_m \phi_m A_{x_0}^s \varphi, g) + (\mathfrak{L}\varphi, g) + (A_{x_0}^s Y\varphi, g)$$

with

$$\mathfrak{L} = \sum_{m=-\infty}^{\infty} L_m S_m.$$

Moreover

$$\| \mathfrak{L}\varphi \|_k \leq C \sum_{m=-\infty}^{\infty} | m |^{n+3+k+s}(1 + | m |)^{-N} \| \varphi \|_{(s+1)\alpha+k-1} \leq C \| \varphi \|_{(s+1)\alpha+k-1}$$

by taking N large enough.

Again by the same lemma, we have

$$L_m \phi_m A_{x_0}^s \varphi = \phi_m L_m(A_{x_0}^s \varphi) + R_m(A_{x_0}^s \varphi)$$

where

$$\| R_m(A_{x_0}^s \varphi) \|_k \leq C \,| m |^{n+3+k+\alpha}(1 + | m |)^{-N} \| A_{x_0}^s \varphi \|_{k+\alpha-1}$$

and C is independent of m. Therefore

$$\begin{aligned}&(A_{x_0}^s A\varphi, g)\\ &\quad= \sum_{m=-\infty}^{\infty} (\phi_m L_m(A_{x_0}^s \varphi), g) + (\mathfrak{Z}\varphi, g) \quad \text{with } \| \mathfrak{Z}\varphi \|_k \leq C \| \varphi \|_{(s+1)\alpha+k}\end{aligned}$$

By an easy argument, we obtain

$$(A_{x_0}^s A\varphi, g) = (AA_{x_0}^s \varphi, g) + (\mathfrak{Z}\varphi, g) \quad \text{for all } g \text{ in } C_c^\infty(R^n).$$

Hence $(A_{x_0}^s A - AA_{x_0}^s)\varphi = \mathfrak{Z}\varphi$, Q.E.D.

PROPOSITION 4. *Suppose the hypotheses of Theorem* 1 *are satisfied. Then*

$$\phi(x) = ((\mathfrak{A} + t)^m L\mathfrak{G}(x, \cdot, t), (\mathfrak{A} + t)^m \phi) \quad \textit{for all } \phi \in C_c^\infty(R^n).$$

Proof. From Theorem 1, we have

$$\phi(x) = ((A_2 + t)^m L\mathfrak{G}(x, \cdot, t), (A_2 + t)^m \phi) \quad \text{for all } \phi \in D(A_2 + t)^m.$$

Let $f \in D(A_2 + t)^{2m-1}$; then since A_2 is self-adjoint,

$$\begin{aligned}f(x) &= ((A_2 + t)L\mathfrak{G}(x, \cdot, t), (A_2 + t)^{2m-1}f)\\ &= ((\mathfrak{A} + t)L\mathfrak{G}(x, \cdot, t), (A_2 + t)^{2m-1}f).\end{aligned}$$

So

$$|((\mathcal{A} + t)L\mathcal{G}(x, \cdot, t), (A_2 + t)^{2m-1}f)|$$
$$= |f(x)| \leq \max_{x \in \bar{\Omega}} |f(x)| \leq M \|f\|_{2m-1} \leq C \|(A_2 + t)^{2m-2}f\|$$

by using the Sobolev imbedding theorem and Theorem 1.

Let $v = (A_2 + t)^{2m-2}f$; then

$$((\mathcal{A} + t)L\mathcal{G}(x, \cdot, t), (A_2 + t)v)| \leq M \|v\|$$

for v in $D(A_2) \cap R(A_2 + t)^{2m-2}$. The inequality is true for all v in $D(A_2)$. Indeed, $R(A_2 + t)^{2m-2} = L^2(\Omega)$.

Therefore $L(v) = ((\mathcal{A} + t)L\mathcal{G}(x, \cdot, t), (A_2 + t)v)$ is a linear functional on $D(A_2)$ and since $D(A_2)$ is dense in $L^2(\Omega)$, we may extend $L(v)$ to all of $L^2(\Omega)$. Using the Riesz representation theorem, we get

$$L(v) = ((\mathcal{A} + t)L\mathcal{G}(x, \cdot, t), (A_2 + t)v) = (h, v)$$

for all v in $D(A_2)$. h is an element of $L^2(\Omega)$. Hence $L\mathcal{G}(x, \cdot, t) \in D(A_2)$ since $A_2 + t$ is self-adjoint.

Repeating the same argument $m - 2$ times, we get $(\mathcal{A} + t)^{m-1}L\mathcal{G}(x, \cdot)$ in $D(A_2)$. Therefore if $\phi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\phi(x) &= ((A_2 + t)^m L\mathcal{G}(x, \cdot, t), (A_2 + t)^m\phi)\\ &= ((\mathcal{A} + t)L\mathcal{G}(x, \cdot, t), (A_2 + t)^{2m-1}\phi)\\ &= ((\mathcal{A} + t)^2 L\mathcal{G}(x, \cdot, t), (A_2 + t)^{2m-2}\phi)\\ &= ((\mathcal{A} + t)^m L\mathcal{G}(x, \cdot, t), (A_2 + t)^m\phi)\\ &= ((\mathcal{A} + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m\phi)\end{aligned}$$

by taking into account Proposition 1.

THEOREM 2. *Suppose the hypotheses of Theorem* 1 *are satisfied. Then*

$$\mathcal{G}(x, x, t) = (2\pi)^{-n_{\xi}-2m+n/\alpha} \int_{R^n} (\tilde{A}(x, \xi) + 1)^{-2m}\, d\xi + o(t^{-2m+n/\alpha})$$

as $t \to +\infty$, *for* x *in* Ω.

Proof. Let $N_d(x) = \{y : |y - x| < d\}$ and d_0 be such that $N_{d_0}(x) \subset \Omega$. $N_d(x)$ is contained in Ω for $d < d_0$.

Let $\phi \in C_c^\infty(N_d(x))$, then from Theorem 1 we have

$$\begin{aligned}\phi(x) &= ((A_2 + t)^m L\mathcal{G}(x, \cdot, t), (A_2 + t)^m\phi)\\ &= ((\mathcal{A} + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m\phi)\end{aligned}$$

by taking into account Proposition 4.

We may write $(\mathcal{A} + t)^m = \sum_{k=0}^{m} t^k \mathcal{A}^{m-k}$. Taking into account Proposition 2 we get

$$(\mathcal{A} + t)^m L\mathcal{G}(x, \cdot, t) = (A + t)^m L\mathcal{G}(x, \cdot, t) + \sum_{k=0}^{m-1} t^k T_{m-k} L\mathcal{G}(x, \cdot, t)$$

where T_j is a "smoothing" operator with respect to A^j, i.e.

$$\| T_j u \|_k \leq M \| u \|_{j\alpha+k-1} .$$

Hence

$$\phi(x)$$
$$= ((A + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi) + \sum_{k=0}^{m-1} t^k (T_{m-k} L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)$$

Since $\phi \in C_c^\infty(\Omega)$, the first expression may be written as

$$((A + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)$$
$$= \int_{R^n} (A + t)^m L\mathcal{G}(x, y, t)\overline{(\mathcal{A} + t)^m \phi(y)}\, dy.$$
$$= ((A + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n}.$$

Let A_x be the operator A with symbol evaluated at the fixed point x. Then

$$((A + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n}$$
$$= ((A_x + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n}$$
$$+ (\{(A + t)^m - (A_x + t)^m\} L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n};$$

$$((A + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n}$$
$$= ((A_x + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n}$$
$$+ \sum_{k=0}^{m-1} t^k ((A^{m-k} - A_x^{m-k}) L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n}.$$

One can show easily that

$$A^s - A_x^s = \sum_{j=0}^{s-1} A_x^j (A - A_x) A^{s-j-1}$$

Hence

$$((A + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n}$$
$$= ((A_x + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n}$$
$$+ \sum_{k=0}^{m-1} \sum_{j=0}^{m-k-1} t^k (A_k^j (A - A_x) A^{m-k-j-1} L\mathcal{G}(x, \cdot . t), (\mathcal{A} + t)^m \phi).$$

Applying Proposition 2 to the first expression of the equation, one obtains

$$((A + t)^m L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n}$$
$$= ((A_x + t)^m L\mathcal{G}(x, \cdot, t), (A_x + t)^m \phi)_{R^n}$$
$$+ \sum_{k=0}^{m-1} \sum_{j=0}^{m-k-1} t^k (A_x^j (A - A_x) A^{m-k-j-1} L\mathcal{G}(x, \cdot, t), (\mathcal{A} + t)^m \phi)_{R^n}$$

$$+ \sum_{k=0}^{m-1} \sum_{j=0}^{m-k-1} t^k ((A_x + t)^m L\mathcal{G}(x, \cdot, t), A_x^j (A - A_x) A^{m-k-j-1} \phi)_{R^n}$$
$$+ \sum_{k=0}^{m-1} t^k ((A_x + t)^m L\mathcal{G}(x, \cdot, t), T_{m-k} \phi)_{R^n}.$$

Denote by R_1, R_2, R_3 the second, third, and fourth expressions on the right hand side of the equation respectively, then

$$| \phi(x) - ((A_x + t)^m L\mathcal{G}(x, \cdot, t), (A_x + t)^m \phi)_{R^n} | \leq | R_1 | + | R_2 | + | R_3 | + | R_4 |$$

where

$$R_4 = \sum_{k=0}^{m-1} t^k (T_{m-k} L\mathcal{G}(x, \cdot, t), (\mathfrak{A} + t)^m \phi)$$

We have

$$| R_3 | \leq \sum_{k=0}^{m-1} t^k \| (A_x + t)^m L\mathcal{G}(x, \cdot, t) \|_{L^2(R^n)} \| T_{m-k} \phi \|_{L^2(R^n)}$$
$$\leq \sum_{k=0}^{m-1} t^{k-m+n/2\alpha} \| \phi \|_{H^{(m-k)\alpha-1}(R^n)}.$$

by applying Theorem 1.

Using a well-known inequality of the theory of Sobolev spaces, we get

$$| R_3 | \leq t^{-m+n/2\alpha} \{ \sum_{k=0}^{m-1} t^k \varepsilon \| \phi \|_{(m-k)\alpha} + K(\varepsilon) t^{m-1} \| \phi \| \}$$
$$\leq t^{-m+n/2\alpha} \{ \varepsilon \| (A_2 + t)^m \phi \| + K(\varepsilon) t^{-1} \| (A_2 + t)^m \phi \| \}$$
$$\leq t^{-m+n/2\alpha} \{ \varepsilon + K(\varepsilon) t^{-1} \} \| (A_2 + t)^m \phi \|$$

by taking into account Assumption (I).

Consider a typical term in R_2. We have

$$t^k (A_x^j (A - A_x) A^{m-k-j-1} L\mathcal{G}(x, \cdot, t), (\mathfrak{A} + t)^m \phi)_{R^n}.$$

From Proposition 4, we know that $A_x^j A - A A_x^j = T_{j+1}$ and T_{j+1} is a "smoothing" operator with respect to A^{j+1}. So

$$t^k (A_x^j (A - A_x) A^{m-k-j1} L\mathcal{G}(x, \cdot, t), (\mathfrak{A} + t)^m \phi)_{R^n}$$
$$= t^k ((A - A_x) A_x^j A^{m-k-j-1} L\mathcal{G}(x, \cdot, t), (\mathfrak{A} + t)^m \phi)_{R^n}$$
$$+ t^k (T_{j+1} A^{m-k-j-1} L\mathcal{G}(x, \cdot, t), (\mathfrak{A} + t)^m \phi)_{R^n}.$$

Since $\phi \in C_c^\infty(N_d(x))$, $(\mathfrak{A} + t)^m \phi \in C_c^\infty(N_d(x))$. Let $\varphi \in C_c^\infty(N_{2d}(x))$ with $\varphi = 1$ on $N_d(x)$ and 0 outside of $N_{d_1}(x)$, $d < d_1$. Using Lemma 2.7 of [2, p. 117], we have

$$| t^k (A_x^j (A - A_x) A^{m-k-j-1} L\mathcal{G}(x, \cdot, t), (\mathfrak{A} + t)^m \phi)_{R^n} |$$
$$= | t^k (\varphi (A - A_x) A_x^j A^{m-k-j-1} L\mathcal{G}(x, \cdot, t), (\mathfrak{A} + t)^m \phi)_{R^n}$$
$$+ t^k (T_{j+1} A^{m-k-j-1} L\mathcal{G}(x, \cdot, t), (\mathfrak{A} + t)^m \phi)_{R^n} |$$
$$\leq \{ C t^k d \| \mathcal{G}(x, \cdot, t) \|_{(m-k)\alpha} + t^k \| \mathcal{G}(x, \cdot, t) \|_{(m-k)\alpha-1} \} \| (\mathfrak{A} + t)^m \phi \|$$

where C is independent of t, d. Taking into account Theorem 1, we get

$$| R_2 | \leq C t^{-m+n/2\alpha} (d + \varepsilon + K(\varepsilon) t^{-1}) \| (\mathfrak{A} + t)^m \phi \|.$$

A similar argument gives

$$|R_1| \leq Ct^{-m+n/1\alpha}(d + \varepsilon + K(\varepsilon)t^{-1}) \, \| (\mathfrak{A} + t)^m \phi \|$$

and

$$|R_4| \leq Ct^{-m+n/2\alpha}(\varepsilon + K(\varepsilon)t^{-1}) \, \| (\mathfrak{A} + t)^m \phi \|.$$

Hence

$$|\phi(x) - ((A_x + t)^m L\mathcal{G}(x, \cdot, t), (A_x + t)^m \phi)_{R^n}| \leq Mt^{-m+n/2\alpha}\{\varepsilon + K(\varepsilon)t^{-1} + d\} \, \| (\mathfrak{A} + t)^m \phi \|.$$

A simple computation yields

$$\| (\mathfrak{A} + t)^m \phi \| \leq C\{\| \phi \|_{m\alpha} + t^m \| \phi \|\} \leq C_2 \| (A_x + t)^m \phi \| \leq C_3 t^{-m+n/2\alpha},$$

where $\phi \in C_c^\infty(N_d(x))$ with $d = t^{-1/\alpha}$ (cf. [1]).
Therefore

$$|\phi(x) - ((A_x + t)^m L\mathcal{G}(x, \cdot, t), (A_x + t)^m \phi)_{R^n}| \leq M(\varepsilon + K(\varepsilon)t^{-1} + t^{-1/\alpha})$$

Now we may take Fourier transform of the expressions on the left hand side of the inequality. A proof, almost identical (with only trivial changes) to that of Theorem 3 of [1] gives the wanted result.

THEOREM 3. *Under the hypotheses of Theorem* 1, *if* $x \neq y$, x, y *in* Ω, *then*

$$\lim_{t \to +\infty} t^{2m-n/\alpha}\mathcal{G}(x, y, t) = 0.$$

Proof. Same idea as in the proof of Theorem 2 with ϕ replaced by

$$\phi \in C_c^\infty(N_d(y)) \quad \text{and} \quad d < |x - y|.$$

We shall not reproduce it.

THEOREM 4. *Suppose the hypotheses of Theorem* 1 *are satisfied. Let* T *be a symmetric operator in* $L^2(\Omega)$. *Suppose further that* T^j *is* A_2^j*-bounded with zero* A_2^j*-bound for* $1 \leq j \leq m$, *where* m *is the smallest positive integer such that* $m\alpha > n/2$. *Then*

(i) $A_2 + tI + T$ *is a self-adjoint operator in* $L^2(\Omega)$;

(ii) $(A_2 + tI + T)^{-2m}f(x) = \int_\Omega G(x, y, t)f(y)\, dy$, f *in* $L^2(\Omega)$;

(iii) $|G(x, y, t)| \leq Ct^{-2m+n,\alpha}$, $\| (A_2 + t + T)^m G(x, \cdot, t) \| \leq Ct^{-m+n,2\alpha}$ *for* x, y *in* Ω, C *independent of* t, x.

Proof. Since $A_2 + tI$ is self-adjoint and T is symmetric with zero A_2-bound, it follows by a well-known result that $A_2 + tI + T$ is again a self-adjoint operator in $L^2(\Omega)$. All the other assertions of the theorem may be proved as in Theorem 1.

THEOREM 5. *Under the hypotheses of Theorem* 4,

$$\lim_{t \to +\infty} t^{2m-n/\alpha}\mathcal{G}(x, y, t) = \lim_{t \to +\infty} t^{2m-n/\alpha}G(x, y, t); \quad x, y \text{ in } \Omega.$$

$\mathcal{G}(x, y, t)$, $G(x, y, t)$ *are defined respectively by Theorems* 1, 4.

Proof. For f in $D(A_2^m)$, we have

$$f(x) = ((A_2 + t)^m \mathcal{G}(x, \cdot, t), (A_2 + t)^m f)$$
$$= ((A_2 + t + T)^m G(x, \cdot, t), (A_2 + t + T)^m f).$$

Since $(A_2 + t + T)^m u = (A_2 + t)^m u + \sum_{k=0}^{m-1} (A_2 + t)^k T^{m-k} u$,

$$((A_2 + t + T)^m G(x, \cdot, t), (A_2 + t + T)^m f)$$
$$= ((A_2 + t)^m G(x, \cdot, t), (A_2 + t)^m f)$$
$$+ \sum_{k=0}^{m-1} ((A_2 + t)^m G(x, \cdot, t), (A_2 + t)^k T^{m-k} f)$$
$$+ \sum_{k=0}^{m-1} ((A_2 + t)^m T^{m-k} G(x, \cdot, t), (A_2 + t)^m f)$$
$$+ \sum_{k=0}^{m-1} \sum_{s=0}^{m-1} ((A_2 + t)^k G(x, \cdot, t), (A_2 + t)^s T^{m-s} f).$$

Denote by R_1, R_2, R_3 the last three expressions on the right hand side of the equation. Then

$$((A_2 + t)^m \{\mathcal{G}(x, \cdot, t) - G(x, \cdot, t)\}, (A_2 + t)^m f) = R_1 + R_2 + R_3.$$

Consider a typical term in the expression R_1. We have

$$|((A_2 + t)^m G(x, \cdot, t), (A_2 + t)^k T^{m-k} f)|$$
$$\leq Ct^{-m+n/2\alpha} \{\| T^{m-k} f \|_{k\alpha} + t^k \| T^{m-k} f \|\}$$

by taking into account Theorem 4. Hence

$$|R_1| \leq Ct^{-m+n/2\alpha} \{\varepsilon + K(\varepsilon) t^{-1}\} \| (A_2 + t)^m f \|$$

using the definition of T and Assumption (I).

Consider a typical term in the expression R_2:

$$|((A_2 + t)^k T^{m-k} G(x, \cdot, t), (A_2 + t)^m f)|$$
$$\leq Ct^{-m+n/2\alpha} \{\varepsilon + K(\varepsilon) t^{-1}\} \| (A_2 + t)^m f \|$$

where we have used Theorem 4. So

$$|R_2| \leq Ct^{-m+n/2\alpha} \{\varepsilon + K(\varepsilon) t^{-1}\} \| (A_2 + t)^m f \|.$$

We estimate R_3 in a similar fashion. Finally, we get

$$|((A_2 + t)^m \{\mathcal{G}(x, y, t) - G(x, \cdot, t)\}, (A_2 + t)^m f)|$$
$$\leq Ct^{-m+n/2\alpha} \{\varepsilon + K(\varepsilon) t^{-1}\} \| (A_2 + t)^m f \|.$$

Since $(A_2 + t)^m$ is onto $L^2(\Omega)$, we obtain

$$\| (A_2 + t)^m \{\mathcal{G}(x, \cdot, t) - G(x, \cdot, t)\} \| \leq Ct^{-m+n/2\alpha} \{\varepsilon + K(\varepsilon) t^{-1}\}.$$

But

$$|\mathcal{G}(x, y, t) - G(x, y, t)| \leq Mt^{-m+n/2\alpha} \| (A_2 + t)^m \{\mathcal{G}(x, \cdot, t) - G(x, \cdot, t)\} \|$$
$$\leq Mt^{-2m+n/\alpha} \{\varepsilon + K(\varepsilon) t^{-1}\}$$

(cf. [1]). Therefore $\lim_{t \to +\infty} t^{2m-n/\alpha} \{\mathcal{G}(x, y, t) - G(x, y, t)\} = 0.$

THEOREM 6. *Suppose the hypotheses of Theorem 5 are satisfied. Let λ_j, φ_j be respectively the eigenvalues and eigenfunctions of $A_2 + T$. Then*

$$N(t) = \sum_{\lambda_j \leqq t} 1 = (2\pi)^{-n} t^{n/\alpha} \int_\Omega \int_{\tilde{A}(x,\xi)<1} d\xi\, dx + o(t^{n/\alpha}),$$

$$e(x, x, t) = (2\pi)^{-n} t^{n/\alpha} \int_{\tilde{A}(x,\xi)<1} d\xi + o(t^{n/\alpha}), \quad x \text{ in } \Omega,$$

$$e(x, y, t) = \textstyle\sum_{\lambda_j \leqq t} \varphi_j(x)\overline{\varphi_j(y)} = o(t^{n/\alpha}), \quad x \neq y.$$

Proof. Applying the Tauberian theorem of Hardy-Littlewood and taking into account the results of Theorems 4, 5, 3, 2, we get the stated results.

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