EUCLIDEAN AND NON-EUCLIDEAN NORMS IN A PLANE

BY
D. A. SENECHALLE

Introduction. Let L denote a 2-dimensional linear space. If f is any norm on L, $K(f)$ denotes the smallest number $r > 0$ such that for some Euclidean norm g dominated by f, the norm $r \cdot g$ dominates f. Note that $K(f) = 1$ or $K(f) > 1$ according as f is Euclidean or not. The following are the main results in this paper: (1) that

$$K(f) = \sup \{[(f^2(ax + y/a) + f^2(bx - y/b))/(a^2 + 1/a^2 + b^2 + 1/b^2)]^{1/2}: a, b > 0, f(x) = f(y) = 1\};$$

(2) a theorem which shows how to construct all norms f with $K(f)$ fixed; (3) some improvements on known conditions for inner product spaces with the change that they are required to hold only locally or in the limit.

Notation and preliminaries. For any linearly independent x and y in L, $C(x, y)$ denotes the set $\{ax + by: a, b \geqq 0\}$ and $W(x, y)$ denotes the set $\{ax + by: ab \geqq 0\}$. We call a quadruple of points (x, y, x', y') interlocking if the points are pairwise linearly independent, $C(y', y) \supset C(x, y) \supset C(x', y)$, and the unit sphere of some norm contains them. If f is any functional on L define $S(f)$ and $U(f)$ to be $f^{-1}(1)$ and $f^{-1}([0, 1])$ respectively. Define a subnorm to be any restriction f of a norm on L such that dom f is closed, $R \cdot \text{dom } f = \text{dom } f$, and there exists an interlocking quadruple of points of $S(f)$. Call a functional f on L a Euclidean pre-norm if either f is a Euclidean norm or $f = |g|$ for some $g \neq 0$ in $L^{\#}$. If f is any subnorm, E_f (E^f) denotes the set of all Euclidean pre-norms dominating (dominated by) f over dom f, and if g is in E_f of E^f, $d(f, g)$ denotes

$$\sup \{g(x), 1/g(x) : x \in S(f)\}.$$

If N is E_f or E^f, $d(f, N)$ denotes $\inf_{g \in N} d(f, g)$. Note that the definition of $K(f)$ can be extended to subnorms by an obvious modification.

If $w = (x, y, x', y')$ is any interlocking quadruple, define

$$k(w) = [(ab + cd)/(cd(a^2 + b^2) + ab(c^2 + d^2))]^{1/2},$$

where $x' = ax + by$ and $y' = cx - dy$. Thus $(a, b, c, d > 0)$. We list without proof the following four properties of any interlocking quadruple $w = (x, y, x', y')$:

(P_1) There exists only one ordered pair (r, C) such that $r > 0$, $C = S(f)$ for some Euclidean pre-norm f, C contains x' and y', and $r \cdot C$ contains x and y.

(P_2) $r = k(w)$.

Received December 23, 1968.

(P_3) If f is a subnorm, $x, y, x', y' \in S(f)$,

$$x' = (ax + y/a)/f(ax + y/a), \qquad y' = (bx - y/b)/f(bx - y/b),$$

then

$$k(w) = [(f^2(ax + y/a) + f^2(bx - y/b))/(a^2 + 1/a^2 + b^2 + 1/b_2)]^{1/2}.$$

(P_4) The quadruple $w' = (x', y', x, y)$ is also interlocking and $k(w') = 1/k(w)$.

THEOREM 1. *If f is a subnorm, then*

(a) *E_f and E^f have unique nearest elements g and h respectively to f.*

(b) *$d(f, E^f) = d(f, E_f) = K(f)$ and $g = K(f)h$.*

(c) *Each of the sets $S(f) \cap S(g)$ and $S(f) \cap S(h)$ contains two linearly independent points and if $W(x, y)$ contains one of these sets, it intersects the other.*

(d) *There is an interlocking quadruple $w = (x, y, x', y')$ such that $x, y \in S(f) \cap S(h)$, $x', y' \in S(f) \cap S(g)$.*

(e) *$K(f) = k(w) = \sup_{v \in V} k(v)$, where V is the set of all interlocking quadruples of points of $S(f)$.*

(f) $K(f) = \sup\{[(f^2(ax + y/a) + f^2(bx - y/b))/(a^2 + 1/a^2 + b^2 + 1/b^2)]^{1/2} : a, b > 0, f(x) = f(y) = 1, ax + y/a, bx - y/b \in \operatorname{dom} f\}$.

Proof. There is some Euclidean pre-norm h which is the pointwise limit of a sequence of Euclidean norms in E^f whose distances from f converge to $d(f, E^f)$, and thus $d(f, E^f) = d(f, h)$. If h is not a norm, then $S(h) = \alpha \cup -\alpha$ for some line α not containing 0. Suppose that either $S(f) \cap S(h)$ does not contain two linearly independent points, or that for some x, y, $W(x, y)$ contains $S(f) \cap S(h)$ but contains no point z of $S(f)$ such that $d(f, E^f) = 1/h(z)$. In either case, there exist some two points x and y of $S(h)$ such that $S(f) \cap S(h)$ is interior to $W(x, y)$ and such that

$$\sup\{1/h(p) : p \in S(f) \cap W(x, y)\} < d(f, E^f).$$

There is some Euclidean norm k such that $k(x) = k(y) = 1$, $k(x + y) < h(x + y)$, and which is close enough to h to insure that

$$\sup\{1/k(p) : p \in S(f) \cap W(x, y)\} < d(f, E^f) \quad \text{and} \quad U(k) \supset U(f).$$

Thus $k \in E^f$ and $d(f, k) < \sup\{1/h(p) : p \in S(f)\} = d(f, E^f)$, a contradiction. Therefore $S(f) \cap S(h)$ contains two linearly independent points and if $W(x, y)$ contains $S(f) \cap S(h)$, then it contains a point z of $S(f)$ such that $d(f, E^f) = 1/h(z)$.

Suppose that in E^f there is a Euclidean pre-norm $k \neq h$ such that $d(f, E^f) = d(f, k)$. Let $m = (h^2 + k^2)^{1/2}$. Note that $m \in E^f$, $d(f, m) = d(f, E^f)$, and m is a norm even though h and/or k may not be. There is some point x of $S(m) \cap \operatorname{dom} f$ such that $d(f, m) = f(x)$. We have that $h(x) \geqq 1$ because $f(x)/h(x) \leqq d(f, h) = d(f, m) = f(x)$. A similar argument shows that $k(x) \geqq 1$. But $m(x) = 1 = (h^2(x) + k^2(x))^{1/2}$, so $h(x) = k(x) = 1$. There is some point y where $S(m)$ touches $S(f)$, but $y \neq \pm x$ because $d(f, m) = f(x) \neq 1$. It follows that $S(h) \cap S(k) = \{\pm x, \pm y\}$. There is some Euclidean norm n such that $n(y) = m(y) = 1$, $n(x + y) = m(x + y)$, $n(x) > m(x)$, and which is close enough to m to insure that $n \in E^f$ and $d(f, n) < d(f, h)$. Since this is a contradiction, h is the unique nearest member of E^f to f. Now define $g = d(f, h)h$. It is easily checked that $g \in E_f$, that $d(f, g) = d(f, h)$, and that g is the unique nearest member of E_f to f. This completes parts (a) and (b).

With slight modification, the argument used to show $S(f) \cap S(h)$ has two linearly independent points will also work for $S(f) \cap S(g)$. It has been shown that if $W(x, y)$ contains $S(f) \cap S(h)$, then it contains a point z of $S(f)$ such that $d(f, h) = 1/h(z)$, and this implies that $z \in S(g)$. A similar argument shows that if $W(x, y)$ contains $S(f) \cap S(g)$, then it intersects $S(f) \cap S(h)$. This completes (c).

Part (d) is obvious if f is Euclidean, so suppose f is not. There exist linearly independent points x and y^* of $S(f) \cap S(h)$ such that $W(x, y^*) \supset S(f) \cap S(h)$. By (c) and the symmetry of $S(f) \cap S(h)$, $C(x, y^*)$ intersects $S(f) \cap S(g)$. There is some point y of $S(f) \cap S(h) \cap C(x, y^*)$ such that $C(x, y)$ contains some point x' of $S(f) \cap S(g)$ but such that if z is any point of $S(f) \cap S(h)$ interior to $C(x, y)$, then $C(x, z)$ contains no point of $S(f) \cap S(g)$. Suppose that $W(x, y) \supset S(f) \cap S(g)$. Then $C(x, y)$ contains some two linearly independent points z_1 and z_2 of $S(f) \cap S(g)$ such that $W(z_1, z_2) \supset S(f) \cap S(g)$. There is a point z of $S(f) \cap S(h)$ in $C(z_1, z_2)$. Thus z is interior to $C(x, y)$ and $C(x, z)$ contains either z_1 or z_2, so it intersects $S(f) \cap S(g)$, a contradiction. Therefore, $W(x, y)$ does not contain $S(f) \cap S(g)$, and this implies that there is some point y' of $S(f) \cap S(g)$ not in $C(x, y)$ and such that $C(y', y) \supset C(x, y)$. The points x, y, x', y' have the required properties. This completes (d).

According to property (P_1) of interlocking quadruples, there exists only one pair (r, C) such that $r > 0$, $C = S(f)$ for some Euclidean pre-norm f, $x', y' \in C$, and $x, y \in r \cdot C$. By property (P_2), $r = k(w)$, where $w = (x, y, x', y')$. The pair $(K(f), S(f))$ has these properties of the pair (r, C), so $k(w) = K(f)$. Suppose the quadruple $u = (p, q, p', q')$ is in V. Let m be the subnorm such that $S(m) = \{\pm p, \pm q, \pm p', \pm q'\}$. Since m is a restriction of f, $K(m) \leqq K(f)$. Just as it has been shown that $K(f) = k(w)$, it may be proved that $K(m) = k(u)$. Thus $k(u) = K(m) \leqq K(f) = k(w)$ and $k(w) = \sup_{v \in V} k(v)$. This completes (e). Part (f) follows from (d) and property (P_3).

The following corollary shows how to construct all the subnorms f with a fixed $K(f)$. (For any f, $1 \leqq K(f) \leqq 2^{1/2}$, as may be checked by finding the maximum of the expression in part (f) of Theorem 1.) This corollary is stated without proof since it is a straightforward consequence of Theorem 1 and the four properties of interlocking quadruples.

COROLLARY. *Suppose* $1 < r \leqq 2^{1/2}$. *Let* C *be some ellipse in* L *with center* 0. *Let* W *be the set of all subnorms* f *such that*: (a) $S(f) \subset [1, r]C$ *and* (b) *there exists* *an interlocking quadruple* (x, y, x', y') *of points of* $S(f)$ *such that* $x, y \in rC$ *and* $x', y' \in C$. *Finally, let* W' *denote the set of all subnorms* f' *such that* $S(f') = T(S(f))$ *for some reversible linear* T *and some* f *in* W. *Then* W' *is the set of all subnorms* f *such that* $K(f) = r$.

Suppose that $\sim$ is one of the relations $\leqq$ and $\geqq$. Say that a subnorm f has property $(D, \sim)$ provided that if (x, y, x', y') is any interlocking quadruple of points of $S(f)$, then there exists an interlocking quadruple $w = (x, y, x'', y'')$ of points of $S(f)$ such that $k(w) \sim 1$. M. M. Day proves in [2] that every norm with property $(D, \sim)$ is Euclidean. Calling a subnorm f Euclidean whenever $K(f) = 1$, we prove:

THEOREM 2. *Every subnorm with property* $(D, \sim)$ *is Euclidean.*

Proof. Let f be a subnorm with property $(D, \sim)$. Suppose f is not Euclidean. Let g and h denote the nearest members to f of E_f and E^f respectively. Using part (d) of Theorem 1 and that $S(f)$, $S(g)$, and $S(h)$ are closed, there exists an interlocking quadruple (p, q, p', q') such that

$$C(p, q) \cap S(f) \cap S(h) = \{p, q\} \quad \text{and} \quad C(p', q') \cap S(f) \cap S(g) = \{p', q'\}.$$

If $w = (p, q, x, y)$ is an interlocking quadruple of points of $S(f)$, then $k(w) > 1$, and if $w' = (p', q', x, y)$ is an interlocking quadruple of points of $S(f)$, then $k(w') < 1$. Since this yields the contradiction that f does not have property $(D, \sim)$, it follows that f is Euclidean.

In what follows, $\|\cdot\|$ denotes a norm on L and S its unit sphere. We are concerned with conditions which make $\|\cdot\|$ Euclidean. Brief surveys of the results of this type may be found in [1] and [3] and a more extensive survey in [5].

THEOREM 3. *Let* $\sim$ *denote one of the relations* $\leqq$ *and* $\geqq$. *Suppose that there exists some* $\varepsilon > 0$ *such that if* $\|x\| = \|y\| = 1$ *and* $\|x - y\| < \varepsilon$, *then there exist* $a, b > 0$ *such that*

$$\|ax + by\|^2 + ab\|x - y\|^2 \sim (a + b)^2.$$

Then $\|\cdot\|$ *is Euclidean.*

Proof. Suppose that $\sim$ is $\geqq$ and that $\|\cdot\|$ is not Euclidean. Let g and h be the nearest members to $\|\cdot\|$ of $E_{\|\cdot\|}$ and $E^{\|\cdot\|}$ respectively. By Theorem 1, there exists an interlocking quadruple (p, q, p', q') such that $p, q \in S(f) \cap S(h)$ and $p', q' \in S(f) \cap S(g)$. If $g(p' - q') \leqq 2^{1/2}$, let $u = p'$ and $v = q'$, and if $g(p' - q') > 2^{1/2}$, let $u = p'$ and $v = -q'$. In either case, $u, v \in S \cap S(g)$, $g(u - v) \leqq 2^{1/2}$, and $S \cap S(g) \cap C(u, v) \neq S \cap C(u, v)$ because $S \cap C(u, v)$ contains either p or q.

For every $r \geqq 1$, let g_r be the Euclidean norm such that $g_r(u) = g_r(v) = 1$ and $g_r(u + v) = g(u + v)/r$. Let

$$M = \sup\{r \geqq 1 : g_r(z) > 1 \text{ for some } z \text{ in } S \cap C(u, v)\}.$$

For every $r \geqq 1$, let

$$S_r = \{z \in S \cap C(u, v) : g_r(z) \geqq 1\}.$$

If $1 \leqq r \leqq M$, let

$$l(r) = \sup \{\| x - y \| : x \text{ and } y \text{ are in the same component of } S_r\},$$

and if $1 \leqq r < M$, let u_r and v_r be points such that $\| u_r - v_r \| = l(r)$ and the arc $C(u_r, v_r) \cap S$ is a component of S_r. Note that each of the points u_r and v_r belongs to $S(g_r)$ and that the function l is nonincreasing over $[1, M]$. One of the following two statements is true: (a) $l(M) > 0$; (b) $\lim_{r \to M} l(r) = 0$. If (a) is true, S_M contains an arc of $S(g_M)$. If (b) is true, there is some r, $1 < r < M$, such that $\| u_r - v_r \| < \varepsilon$. In either case, there is some r, $1 < r \leqq M$, and some two points x and y in $C(u, v) \cap S \cap S(g_r)$ such that $\| x - y \| < \varepsilon$ and such that if $a, b > 0$, then $\| ax + by \| \leqq g_r(ax + by)$. But since $g(u - v) \leqq 2^{1/2}$ and $u, v \in S \cap S(g)$, $x - y$ is interior to $W(u, -v)$ implying that $\| x - y \| < g_r(x - y)$. Therefore, if $a, b > 0$, then

$$\| ax + by \|^2 + ab \| x - y \|^2 < g_r^2(ax + by) + abg_r^2(x - y) = (a + b)^2.$$

This is a contradiction. Therefore $\| \cdot \|$ is Euclidean if $\sim$ is $\geqq$.

If $\sim$ is $\leqq$, an argument similar to the preceding one may be used. The main differences are that u and v are picked from $S \cap S(h)$ instead of $S \cap S(g)$, the norms g_r are replaced by Euclidean norms h_r defined by $h_r(u) = h_r(v) = 1$ and $h_r(u + v) = rh(u + v)$,

$$M = \sup \{r \geqq 1 : h_r(z) < 1 \text{ for some } z \text{ in } S \cap C(u, v)\},$$

and $S_r = \{z \in S \cap C(u, v) : h_r(z) \leqq 1\}$.

THEOREM 4. *Suppose that if $\| x \| = 1$ there exist $a, b > 0$ (depending on x) and a sequence (y_i) in $S \backslash x$ converging to x such that*

$$\lim_{i \to \infty} ((a + b)^2 - \| ax + by_i \|^2)/(ab \| x - y_i \|^2) \geqq 1.$$

(This limit may be ∞.) Then $\| \cdot \|$ is Euclidean.

Proof. Suppose that $\| \cdot \|$ is not Euclidean. Let g and h be the nearest members to $\| \cdot \|$ of $E_{\|\cdot\|}$ and $E^{\|\cdot\|}$ respectively. By an affine transformation, we may assume that L is the plane, $S(g)$ and $S(h)$ are circles, $S(g)$ with radius 1 and $S(h)$ with radius $R > 1$, $(1, 0) \in S \cap S(g)$, and that $C((1, 0), (1, 1))$ contains a point of $S \cap S(h)$. We use the following notation: r and p are the functions such that for every α, $(r(\alpha) \cos \alpha, r(\alpha) \sin \alpha) = p(\alpha) \in S$; $\theta = -\operatorname{atan}(r'/r)$, and $\theta_- = -\operatorname{atan}(r'_-/r)$, where r'_- denotes the left-hand derivative of r.

Let k denote $1 - 1/R^2$. $0 < k \leqq \frac{1}{2}$. Suppose that $(\theta_-)'(\alpha) \geqq -k$ for every α in dom $(\theta_-)'$. The function $\theta_-(\alpha) + \alpha + \pi/2$ gives the direction of the left-hand tangent to S at $p(\alpha)$, so it is nondecreasing. Thus $\theta_-(\alpha) + \alpha$ is nondecreasing and if $\alpha \in$ dom $(\theta_-)'$, then $(\theta_-)'(\alpha) + 1 \geqq 1 - k$. Since $(1, 0) \in S \cap S(g)$, $\theta_-(0) = 0$. These conditions imply that for every α,

$\theta_-(\alpha) \geqq (1-k)\alpha$, so $\theta_-(\alpha) \geqq -k\alpha$ and we get the inequalities

$$\text{atan } (r'_-(\alpha)/r(\alpha)) \leqq k\alpha, \quad (\ln r)'_-(\alpha) \leqq \tan (k\alpha) \text{ and}$$
$$r(\alpha) \leqq \exp \left(\int_0^\alpha \tan (kt)\, dt\right) = \cos (k\alpha)^{-1/k}.$$

But there is an α in $[0, \pi/4]$ such that $r(\alpha) = R$. Since $R = (1-k)^{-1/2}$, $(1-k)^{-1/2} \leqq \cos (k\alpha)^{-1/k}$, $(1-k)^{k/2} \geqq \cos (k\alpha) \geqq \cos (k\pi/4)$, and $f(k) \geqq 0$, where for every t in $[0, \frac{1}{2}]$,

$$f(t) = (\ln (1-t))t/2 - \ln (\cos (t\pi/4)).$$

To get a contradiction, we observe that $f(0) = f'(0) = 0$ and that $f''(t) < 0$ if $t \in [0, \frac{1}{2}]$. These conditions imply that $f(t) < 0$ if $t \in (0, \frac{1}{2}]$, and, in particular, that $f(k) < 0$. Thus there is some α such that $(\theta_-)'(\alpha) < -k$. By a rotation, we may assume that $\alpha = 0$.

By the hypothesis, there exist $a, b > 0$ and a sequence of numbers (α_i) converging to 0 and all different from 0 such that

$$\lim_{i\to\infty} ((a+b)^2 - \| ap(0) + bp(\alpha_i) \|^2)/(ab \| p(0) - p(\alpha_i) \|^2) \geqq 1.$$

By replacing a and b by $a/(a+b)$ and $b/(a+b)$ respectively in the above expression, we may assume that $a + b = 1$.

Let $|\cdot|$ be the norm for the plane; i.e., $|(c, d)| = (c^2 + d^2)^{1/2}$. For any point $y \neq 0$, denote $y/\|y\|$ by sgn (y). If $y \neq 0$, $\|y\| = |y|/|\text{sgn } (y)|$. Also, if $|\alpha| < \pi/2$, then

$$|\text{sgn } (ap(0) + bp(\alpha))| = r(\text{atan } (br(\alpha) \sin \alpha/(ar(0) + br(\alpha) \cos \alpha))).$$

$$\lim_{i\to\infty} (1 - \| ap(0) + bp(\alpha_i) \|^2)/(ab \| p(0) - p(\alpha_i) \|^2)$$
$$= \lim_{i\to\infty} (|\text{sgn } (p(0) - p(\alpha_i))|^2/|\text{sgn } (ap(0) + bp(\alpha_i))|^2) F(\alpha_i)/G(\alpha_i),$$

where for every α in $[-\pi/2, \pi/2]$,

$$F(\alpha) = r^2(\text{atan } (br(\alpha) \sin \alpha/(ar(0) + br(\alpha) \cos \alpha)) - (a^2r^2(0) + 2abr(0)r(\alpha) \cos \alpha + b^2r^2(\alpha))$$

and

$$G(\alpha) = ab(r^2(0) - 2r(0)r(\alpha) \cos \alpha + r^2(\alpha)).$$

For every α, $1 \leqq r(\alpha) \leqq R$, so $\lim_{i\to\infty} F(\alpha_i)/G(\alpha_i) \geqq 1/R^2$. The function G is both left- and right-differentiable because it is the sum of three functions of this type. The same is true about the second half of the expression for F. Since the function

$$\text{atan } (br(\alpha) \sin \alpha/(ar(0) + br(\alpha) \cos \alpha))$$

is increasing over an open subinterval s of $(-\pi/2, \pi/2)$ containing 0, the second half of the expression for F inherits from r left- and right-differentiability over s.

Let F'' and G'' denote the derivatives of F'_- and G'_- respectively. We note that F and G are differentiable at 0 and that since θ_- is differentiable at 0, so are r'_-, F'_-, and G'_-. A computation shows that $F(0) = G(0) = F'(0) = G'(0) = 0 \neq G''(0)$. Thus we have the following three properties:

(a) $s \subset \operatorname{dom} F \cap \operatorname{dom} F'_- \cap \operatorname{dom} F'_+ \cap \operatorname{dom} G \cap \operatorname{dom} G'_- \cap \operatorname{dom} G'_+$;

(b) $0 \in \operatorname{dom} F' \cap \operatorname{dom} F'' \cap \operatorname{dom} G' \cap \operatorname{dom} G''$; and

(c) $F(0) = G(0) = F'(0) = G'(0) = 0 \neq G''(0)$.

These three conditions on any real functions F and G imply that

$$\lim_{\alpha \to 0} F(\alpha)/G(\alpha) = F''(0)/G''(0).$$

Therefore, $\lim_{i\to\infty} F(\alpha_i)/G(\alpha_i) = F''(0)/G''(0) \geqq 1/R^2$. The computation referred to above also shows that $F''(0)/G''(0) = 1 + \theta'(0)$. But $\theta' = (\theta_-)'$, so $(\theta_-)'(0) \geqq 1/R^2 - 1$, which is contrary to what we have shown above. Therefore $\| \cdot \|$ is Euclidean.

The following three corollaries are less general versions of Theorem 4.

COROLLARY 1. *Suppose that if* $\| x \| = 1$, *there exist* $a, b > 0$ *such that if* $\varepsilon > 0$, *there is a point* y *such that* $\| y \| = 1$, $0 < \| x - y \| < \varepsilon$, *and* $\| ax + by \|^2 + ab \| x - y \|^2 \leqq (a + b)^2$. *Then* $\| \cdot \|$ *is Euclidean.*

COROLLARY 2. *Suppose that if* $\| x \| = 1$, *there exist* $a, b > 0$ *such that*

$$\lim_{y \to x, y \in S} \left((a + b)^2 - \| ax + by \|^2 \right) / \left(ab \| x - y \|^2 \right)$$

exists and is $\geqq 1$. (*This limit may be* ∞.) *Then* $\| \cdot \|$ *is Euclidean.*

Let the modulus of convexity for $\| \cdot \|$ be the function δ defined by

$$\delta(\varepsilon) = \inf \{ 1 - \| (x + y)/2 \| : \| x \| = \| y \| = 1, \| x - y \| = \varepsilon \},$$

where $0 \leqq \varepsilon \leqq 2$. Nordlander [4] has proved that $\delta(\varepsilon) \leqq 1 - (1 - \varepsilon^2/4)^{1/2}$. Thus if (ε_i) is any sequence of positive numbers converging to 0, $\lim_{i\to\infty} \delta(\varepsilon_i)/\varepsilon_i^2 \leqq \frac{1}{8}$ if the limit exists.

COROLLARY 3. *If there exists a sequence* (ε_i) *of positive numbers converging to* 0 *such that* $\lim_{i\to\infty} \delta(\varepsilon_i)/\varepsilon_i^2 = \frac{1}{8}$, *then* $\| \cdot \|$ *is Euclidean.*

Next, using methods inspired by Nordlander's argument in [4], we obtain a stronger result.

THEOREM 5. *Let* $a, b, c, d > 0$, $| c - d | < \varepsilon < c + d$, *and let*

$$W = \{ (cd \| ax + by \|^2 + ab \| cx - dy \|^2)/(cd(a^2 + b^2) + ab(c^2 + d^2)) : \| x \| = \| y \| = 1, \| cx - dy \| = \varepsilon \}.$$

Then $1 \in W$ *and* W *contains a number* <1 *if and only if it contains a number* >1.

Proof. By an affine mapping, we may assume that L is the plane. Let r be the positive function such that for every θ, $(r(\theta) \cos \theta, r(\theta) \sin \theta) \in S$, and let $x = r \cdot \cos$, $y = r \cdot \sin$, and $p = (x, y)$. For each θ,

$$\| cp(\theta) - dp(\theta) \| = | c - d | \quad \text{and} \quad \| cp(\theta) - dp(\theta + \pi) \| = c + d,$$

so there is a least number $k(\theta)$ in $[\theta, \theta + \pi]$ such that $\| cp(\theta) - dp(k(\theta)) \| = \varepsilon$. Now define $u = x \circ k$, $v = y \circ k$, and $q = (u, v) = p \circ k$. Then p and q are continuous, both have range S, and $\| cp(\theta) - dq(\theta) \| = \varepsilon$ for all θ. For any α and β, let $A(\alpha, \beta)$ denote the area of the curve traced by $\alpha p + \beta q$. Then

$$\begin{aligned}(\alpha + \beta)^2 A(1, 0) - A(\alpha, \beta) &= (\alpha^2 + \alpha\beta) A(1, 0) \\ &\qquad + (\beta^2 + \alpha\beta) A(0, 1) - A(\alpha, \beta) \\ &= (\alpha^2 + \alpha\beta) \int_0^{2\pi} x\, dy + (\beta^2 + \alpha\beta) \int_0^{2\pi} u\, dv \\ &\qquad - \int_0^{2\pi} (\alpha x + \beta u)\, d(\alpha y + \beta v) \\ &= \alpha\beta \int_0^{2\pi} (x - u)\, d(y - v) = \alpha\beta A(1, -1).\end{aligned}$$

Since $\| cp(\theta) - dq(\theta) \| = \varepsilon$ for all θ, $A(c, -d) = \varepsilon^2 A(1, 0)$. This fact and the above yield the equations

$$(a + b)^2 A(1, 0) - A(a, b) = abA(1, -1),$$
$$(c - d)^2 A(1, 0) - \varepsilon^2 A(1, 0) = -cdA(1, -1)$$

from which we conclude

$$cdA(a, b)/A(1, 0) + ab\varepsilon^2 = cd(a^2 + b^2) + ab(c^2 + d^2).$$

Let s be the positive function such that for each θ, $(s(\theta) \cos \theta, s(\theta) \sin \theta)$ is a point of the curve $ap + bq$. Then

$$cd\left(\int_0^{2\pi} s^2(\theta)\, d\theta\right) \Big/ \left(\int_0^{2\pi} r^2(\theta)\, d\theta\right) + ab\varepsilon^2 = cd(a^2 + b^2) + ab(c^2 + d^2).$$

Thus there is a θ such that

$$cds^2(\theta)/r^2(\theta) + ab\varepsilon^2 < cd(a^2 + b^2) + ab(c^2 + d^2)$$

if and only if there is a θ such that

$$cds^2(\theta)/r^2(\theta) + ab\varepsilon^2 > cd(a^2 + b^2) + ab(c^2 + d^2),$$

and there is a θ where equality holds. This is the desired result, because for every θ, $s^2(\theta)/r^2(\theta) = \| ap(\theta') + bq(\theta') \|^2$ for some θ'. This completes the proof.

One geometric interpretation of Theorem 5 is worth stating explicitly: Given, in the plane, any four points of the unit circle and any convex and symmetric about 0 simple closed curve C, there exists a linear mapping carrying all four points into C.

In [6] this author proves that $\| \cdot \|$ is Euclidean if there is a function F defined on $[0, 2]$ such that $F(\| x - y \|) = \| x + y \|$ whenever x and y are in S. The following stronger result is an easy consequence of Theorems 4 and 5.

THEOREM 6. *Suppose that there exist numbers* $a, b > 0$, *a subset* M *of* $[0, 2]$ *having* 0 *as an accumulation point, and a function* F *defined on* M *such that* $F(\| x - y \|) = \| ax + by \|$ *whenever* $x, y \in S$ *and* $\| x - y \| \in M$. *Then* $\| \cdot \|$ *is Euclidean.*

Proof. By Theorem 5, $F(\| x - y \|) = ((a + b)^2 - ab \| x - y \|^2)^{1/2}$ if $\| x - y \| \in M$. Theorem 4 then asserts $\| \cdot \|$ is Euclidean, since for any $x \in S$,

$$\lim_{y \to x, y \in S, \| x - y \| \in M} ((a + b)^2 - \| ax + by \|^2)/(ab \| x - y \|^2) = 1.$$

REFERENCES

1. M. M. DAY, *Some characterizations of inner product spaces*, Trans. Amer. Math. Soc., vol. 62 (1947), pp. 320–337.
2. ———, *On criteria of Kasahara and Blumenthal for inner product spaces*, Proc. Amer. Math. Soc., vol. 10 (1959), pp. 92–100.
3. ———, *Normed linear spaces*, Springer-Verlag, Berlin, 1958.
4. G. NORDLANDER, *The modulus of convexity in normed linear spaces*, Ark. Mat., vol. 4 (1960), pp. 15–17.
5. J. A. OMAN, *Characterizations of inner product spaces*, Ph.D. Thesis, Michigan State University, East Lansing, Mich., 1969.
6. D. A. SENECHALLE, *A characterization of inner product spaces*, Proc. Amer. Math. Soc., vol. 19 (1968), pp. 1306–1312.

UNIVERSITY OF GEORGIA
ATHENS, GEORGIA