AN ARC THEOREM FOR PLANE CONTINUA

BY
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If H is a bounded aposyndetic plane continuum which does not separate the plane, then H is locally connected. This follows from a result of Jones' [3, Th. 10] that if p is a point of a bounded plane continuum H and H is aposyndetic at p, then the union of H and all but finitely many of its complementary domains is connected im kleinen at p.[1] As a corollary of these results, each bounded aposyndetic nonseparating plane continuum is arc-wise connected. Closely related to the notion of an aposyndetic continuum is that of a semi-aposyndetic continuum, studied in [2]. A continuum M is *semi-aposyndetic* if for each pair of distinct points x and y of M, there exists a subcontinuum F of M such that the sets $M - F$ and the interior of F relative to M each contain a point of $\{x, y\}$. Note that a bounded semi-aposyndetic nonseparating plane continuum may fail to be locally connected. In this paper it is proved that every bounded semi-aposyndetic nonseparating plane continuum is arc-wise connected.

Throughout this paper S is the plane and d is the Euclidean metric for S.

DEFINITION. Let E be an arc-segment (open arc) in S with endpoints a and b, D be a disk in a continuum M in S, and ε be a positive real number. The arc-segment E is said to be *ε-spanned* by D in M if $\{a, b\}$ is a subset of D and for each point x in a bounded complementary domain of $D \cup E$, either $d(x, E) < \varepsilon$ or x belongs to M.

LEMMA 1. *If an arc-segment E in S of diameter less than ε with endpoints a and b is ε-spanned by a disk D in M (a subcontinuum of S), then there exists an arc-segment $M(E)$ in M with endpoints a and b such that for each point x of $M(E)$, $d(x, E) \leqq 2\varepsilon$.*

Proof. Let w be a point of the unbounded complementary domain of $D \cup E$. Let B denote an arc in D with endpoints a and b. For each positive real number r, let $C(r)$ denote the set consisting of all points x of S such that $d(x, \text{Cl } E) < r$ (Cl E is the closure of E). For each positive real number r, Cl $C(r)$ is a bounded locally connected continuum in S which does not contain a separating point. By a simple argument, one can show that if $r \geq \varepsilon$, Cl $C(r)$ does not separate S. Hence for each real number $r \geq \varepsilon$, Cl $C(r)$ is a disk [5, Th. 4, p. 512]. Since B is locally connected, the set Q consisting of all components of $B -$ Cl E which meet $Bd\ C(\varepsilon)$ (the boundary

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[1] A continuum H is *aposyndetic* at a point p of H if for each point q of $H - \{p\}$, there exist a continuum L and an open set G in H such that $p \in G \subset L \subset H - \{q\}$. A continuum is said to be *aposyndetic* if it is aposyndetic at each of its points (Jones).

of $C(\varepsilon)$) is finite. Define Q_1 to be the set of all elements X of Q such that if Y is an element of $Q - \{X\}$, then $Y \cup \text{Cl } E$ does not separate X from w in S. For $n = 2, 3, 4, \cdots$, define Q_n to be the set of all elements X of $Q - \bigcup_{i=1}^{n-1} Q_i$ such that if Y is an element of $Q - (\{X\} \cup \bigcup_{1=i}^{n-1} Q_i)$, then $Y \cup \text{Cl } E$ does not separate X from w in S. Since Q is finite and the sets $Q_1, Q_2, Q_3, \cdots$ are mutually exclusive, there exists an integer n such that $\bigcup_{i=1}^{n} Q_i = Q$.

For each element X of Q, define the arc-segment $M(X)$ as follows. Let c and e be the endpoints of X and let I denote the arc in Cl E from c to e. Let Z be the bounded complementary domain of the simple closed curve $X \cup I$. Let m be the integer ($1 \leqq m \leqq n$) such that X belongs to Q_m. If X is contained in Cl $C(\varepsilon + \varepsilon/m)$, define $M(X)$ to be X. Suppose that X is not contained in Cl $C(\varepsilon + \varepsilon/m)$. Since

$$I \cap \text{Bd } C(\varepsilon + \varepsilon/m) = \emptyset,$$

there exists a simple closed curve J containing I in Bd $Z \cup$ Bd $C(\varepsilon + \varepsilon/m)$ such that $Z \cap C(\varepsilon + \varepsilon/m)$ contains a complementary domain V of J [6, Th. 15, p. 149]. In this case define $M(X)$ to be the arc-segment $J - I$. Let x be a point of $M(X) - X$. Cl V contains x and is a subset of Cl Z. Since Bd $Z = I \cup X$, x is not in Bd Z. Thus x belongs to Z. Hence for each point x of $M(X)$, either x belongs to D, or $d(x, E) > \varepsilon$ and x is in Z and therefore belongs to a bounded complementary domain of $D \cup E$. It follows that $M(X)$ is contained in M. Note that for each point x of $M(X)$, $d(x, E) \leqq 2\varepsilon$. For each arc-segment X in B belonging to Q,

$$M(X) \cap (B - \textstyle\bigcup_{Y \in Q} Y) = \emptyset;$$

for if there exists a point x in $M(X) \cap (B - \bigcup_{Y \in Q} Y)$, then x would belong to both X (since $B - \bigcup_{Y \in Q} Y \subset C(\varepsilon)$) and $B - X$. If X and Y are distinct elements of Q, then the corresponding arc-segments $M(X)$ and $M(Y)$ are disjoint. To see this first suppose that X and Y both belong to Q_m for some integer m. Assume there exists a point x in $M(X) \cap M(Y)$. Since B is an arc, $X \cap Y = \emptyset$ and x must belong to either $M(X) - X$ or $M(Y) - Y$. Suppose that x is in $M(X) - X$. It follows that x is in the bounded complementary domain of $X \cup \text{Cl } E$. If x belongs to Y then $X \cup \text{Cl } E$ separates Y from w in S. This contradicts the assumption that X and Y are both elements of Q_m. Hence x belongs to $M(Y) - Y$ and is contained in the bounded complementary domain of $Y \cup \text{Cl } E$. It follows that either $X \cup \text{Cl } E$ separates Y from w or $Y \cup \text{Cl } E$ separates X from w in S. Again this is impossible, since X and Y belong to Q_m. By the same argument, one can show that assuming x is in $M(Y) - Y$ also involves a contradiction. Suppose there exist distinct integers k and m such that X and Y are elements of Q_k and Q_m respectively. Assume without loss of generality that $k < m$. Since

$$\text{Bd } C(\varepsilon + \varepsilon/k) \cap \text{Cl } C(\varepsilon + \varepsilon/m) = \emptyset,$$

then

$$M(Y) \cap (M(X) - X) = \emptyset.$$

Furthermore $M(Y) \cap X = \emptyset$; for otherwise, $Y \cup \text{Cl } E$ would separate X from w in S which is impossible since X belongs to Q_k, Y belongs to Q_m, and $k < m$. Hence $M(X) \cap M(Y) = \emptyset$.

The set $M(E) = \bigcup_{X \epsilon Q} M(X) \cup (B - (\{a, b\} \cup \bigcup_{X \epsilon Q} X))$ is an arc-segment in M with endpoints a and b such that for each point x of $M(E)$, $d(x, E) \leq 2\varepsilon$ [1, Th. 20.1.10, p. 157].

LEMMA 2. *Suppose that M is a bounded continuum in S, E is an arc-segment in S of diameter less that $\varepsilon/4$, and D is a disk in M which contains the endpoints of E. If E is not ε-spanned by D in M, then there exist points x and y in* Bd D *and an arc-segment Y in $E - D$ such that*

(1) *$d(\{x, y\}, E) \geq \varepsilon/4$,*

(2) *$\{x, y\}$ is not contained in the closure of a complementary domain of $D \cup Y$, and*

(3) *if $d(x, y) = r$, then D contains a circular region U of diameter $r/2$.*

Proof. There exists a point v of $S - M$ such that v is in a bounded complementary domain of $D \cup E$ and $d(v, \text{Cl } E) = s \geq \varepsilon$. Let z be a point of Cl E such that $d(v, z) = s$ and let T be the straight line segment from v to z in S. Define c to be the point of T such that $d(z, c) = \varepsilon/2$ and let L denote the straight line in S which contains c and is perpendicular to T. Define X to be the component of $S - L$ which contains v. Let w be a point of X which also belongs to the unbounded complementary domain of $M \cup \text{Cl } E$. There exists an arc-segment Y in $E - D$ such that $Y \cup D$ separates v from w in S [6, Th. 27, p. 177]. Let a and b be the endpoints of Y and let A and B be the components of Bd $D - \{a, b\}$. Let Z denote the θ-curve $A \cup B \cup \text{Cl } Y$. Note that the complementary domain Q of Z whose boundary contains A and B is the interior of D [7, Th. 1.7, p. 105].

Since Y is in $S - D$, both $A \cup \text{Cl } Y$ and $B \cup \text{Cl } Y$ separate v from w in S. Furthermore, since $\text{Cl } X \cap \text{Cl } Y = \emptyset$ and $\{v, w\}$ is a subset of X, both A and B meet X. There exist a positive real number r and points x and y in $A \cap \text{Cl } X$ and $B \cap \text{Cl } X$ respectively such that

$$d(A \cap \text{Cl } X, B \cap \text{Cl } X) = d(x, y) = r.$$

Let g be the midpoint of the straight line segment in Cl X from x to y. Let G be the circular region in S which is centered on g such that $\{x, y\}$ is contained in Bd G. Since

$$(G \cap \text{Cl } X) \cap (A \cup B) = \emptyset$$

and Cl $(G \cap X)$ meets both A and B, $G \cap X$ is a subset of Q [6, Th. 116, p. 247]. The set $G \cap X$ contains a circular region U of diameter $r/2$. Since $G \cap X$ is a subset of D, U is contained in D.

The circular disk J of radius $\varepsilon/4$ centered on z contains E. Note that $d(J, X) = \varepsilon/4$. It follows that $d(\{x, y\}, E) \geq \varepsilon/4$. Since $\{x, y\}$ is contained in Cl Q and

$$\{x, y\} \cap \{a, b\} = \emptyset,$$

$\{x, y\}$ is not contained in the closure of a complementary domain of $D \cup Y$ [6, Th. 116, p. 247].

LEMMA 3. *Suppose that E is an arc-segment in S, N is a disk in M (a subcontinuum of S which does not separate S), and N contains the endpoints of E. For each positive integer n, there exists a disk D in M containing N such that if*

(1) *W is a complementary domain of $D \cup E$,*
(2) *x is a point of* Cl $W \cap$ Bd D, *and*
(3) *$d(x, E) > 1/n$,*

then there exists a point t of $W - M$ such that $d(x, t) < 1/2n$.

Proof. There exists a 1-complex K (a finite collection of arcs no two of which interesect in an interior point of either) in Cl $(S - N)$ such that (1) Bd N is contained in K, (2) each vertex of K has order 3 in K, and (3) if L is a component of $S - (K \cup N)$ and Cl $L \cap M \neq \emptyset$ then the diameter of L is less than $1/2n$. Define H to be the finite set consisting of all components of $S - K$ which are subsets of M, and let D be the component of $\bigcup_{X \in H}$ Cl X which contains N. Since M does not separate S, D is a disk.

Let W be a complementary domain of $D \cup E$. Suppose there exists a point x of Bd $D \cap$ Cl W such that $d(x, E) > 1/n$. Note that W is the only complementary domain of $D \cup E$ which has x as a limit point. The point x belongs to K. There exist a component L of $S - (K \cup D)$ and a point t of $S - M$ such that x belongs to Cl L and t belongs to L; for otherwise, x would belong to the interior of D. Since the diameter of L is less than $1/2n$, $d(x, t) < 1/2n$. L is a connected set in $S - (D \cup E)$. It follows that t is a point of $W - M$.

DEFINITION. A point y of a continuum M *cuts* x from z in M if x, y and z are distinct points of M and y belongs to each subcontinuum of M which contains $\{x, z\}$.

LEMMA 4. *If M is a compact semi-aposyndetic metric continuum and x, y and z are points of M such that y cuts x from z in M, then z does not cut x from y in M.*

Proof. Suppose y cuts x from z and z cuts x from y in M. For each positive integer i, let G_i be the set of all points v of M such that $\rho(v, z) < 1/i$ (ρ is a metric for M) and let L_i be the x-component of $M - G_i$. The limit superior L of $L_1, L_2, L_3, \cdots$ is a continuum in M which contains $\{x, z\}$. Since y cuts x from z in M, y is in L. Note that for each positive integer i, y does not belong to L_i.

M is not aposyndetic at y with respect to z. That is, the point z belongs to each subcontinuum of M which contains y in its interior (relative to M). To see this assume there exist a continuum H and open sets U and V in M such that $z \in V$ and $y \in U \subset H \subset M - V$. There exists an integer i such that G_i is contained in V. Since y does not belong to an element of $L_1, L_2, L_3, \cdots$, for each integer j $(j > i)$, $L_j \cap U = \emptyset$. This contradicts the fact that y is in L.

By the same argument, M is not aposyndetic at z with respect to y. Since M is semi-aposyndetic, this is a contradiction. Hence z does not cut x from y in M.

THEOREM. *If M is a semi-aposyndetic bounded subcontinuum of the plane S which does not separate S, then M is arc-wise connected.*

Proof. Let p and q be distinct points of M. According to a theorem by Jones, if no point cuts p from q in M, then p and q belong to a simple closed curve in M and are therefore the extremities of an arc lying in M [4]. Suppose that there exists a point which cuts p from q in M. Let K be the closed subset of M consisting of p, q and all points x such that x cuts p from q in M. Define the binary relation R on K as follows. For distinct points x and y of K, $x\, R\, y$ if x cuts p from y in M or $x = p$.

If x and y are distinct points of K, either $x\, R\, y$ or $y\, R\, x$. To see this first suppose that $\{x, y\} \cap \{p, q\} = \emptyset$. Either x does not cut y from q or y does not cut x from q in M (Lemma 4). Assume that x does not cut y from q in M. There exists a continuum H in $M - \{x\}$ containing $\{y, q\}$. The point x cuts p from y in M; for otherwise, there would exist a continuum F such that $\{p, y\} \subset F \subset M - \{x\}$ and $\{p, q\}$ would be a subset of the continuum $H \cup F$ in $M - \{x\}$ which is impossible since x belongs to K. Hence $x\, R\, y$. By the same argument, if y does not cut x from q, then $y\, R\, x$. If $\{x, y\} \cap \{p, q\} \neq \emptyset$, the conclusion follows immediately.

The binary relation R is anti-symmetric. For if x and y belong to K and $x\, R\, y$, then by Lemma 4, $y \not R\, x$ ($y\, R\, x$ does not hold). R is also transitive. To see this suppose there exist points x, y and z of K such that $x\, R\, y$, $y\, R\, z$ and $x \not R\, z$. There exists a continuum H in $M - \{x\}$ containing $\{p, z\}$. Since $y\, R\, z$, y must belong to H. This contradicts the assumption that $x\, R\, y$.

For each point x of K, define $P(x)$ to be the set of all points z of K such that $z\, R\, x$ and define $F(x)$ to be the set of all points z of K such that $x\, R\, z$. Note that $P(p) = F(q) = \emptyset$. Let x be a point of $K - \{p, q\}$ and let z be a point of $F(x)$. Since R is anti-symmetric, $z \not R\, x$. Hence there exists a continuum J such that $\{p, x\} \subset J \subset M - \{z\}$. $P(x)$ is a subset of J and since J is closed in M, z is not in $\mathrm{Cl}\, P(x)$. It follows that for each point x of K, $\mathrm{Cl}\, P(x) \cap F(x) = \emptyset$. Suppose that x is a point of $K - \{p, q\}$ and z is a point of $P(x)$. Since $z\, R\, x$, $x \not R\, z$. Consequently there exists a continuum P in $M - \{x\}$ containing $\{p, z\}$. The point x cuts z from q in M; for otherwise, there would exist a continuum L such that $\{z, q\} \subset L \subset M - \{x\}$ and $\{p, q\}$ would be a subset of the continuum $L \cup P$ in $M - \{x\}$ which contradicts the assumption that x belongs to K. By Lemma 4, the point z does not cut x from q in M. Therefore there exists a continuum T such that

$$\{x, q\} \subset T \subset M - \{z\}.$$

Let y be a point of $F(x)$. If y is not in T, there exists a continuum H such that $\{p, q\} \subset H \subset M - \{y\}$. This contradicts the assumption that y is in K. It follows that $F(x)$ is contained in the closed set T and z is not in Cl $F(x)$. Hence for each point x in K, $P(x) \cap \text{Cl } F(x) = \emptyset$.

The binary relation R is a natural ordering of K [7, p. 41]. Hence there exists an arc A (not necessarily in S) containing K such that p and q are endpoints of A and R is the order induced on K from A [7, Th. 6.4, p. 56]. If a and b are points of K such that $a\,R\,b$ and a point x cuts a from b in M, then x belongs to K, $a\,R\,x$, and $x\,R\,b$. To see this first note that since x cuts a from b in M and $a\,R\,b$, x is not p (Lemma 4). Since a belongs to every subcontinuum of M which contains $\{p, b\}$, x cuts p from b in M. It follows that x belongs to K and $x\,R\,b$. Suppose that x cuts p from a in M. By Lemma 4, there exists a continuum H such that $\{p, b\} \subset H \subset M - \{a\}$. This contradicts the assumption that $a\,R\,b$. Hence $x \not\!R\, a$ and $a\,R\,x$. Let E be a component of $A - K$ with endpoints a and b and assume $a\,R\,b$. Suppose there exists a point x such that x cuts a from b in M. The point x belongs to K. Furthermore since $a\,R\,x$ and $x\,R\,b$, x must belong to E. This contradicts the assumption that E is a subset of $A - K$. Hence no point cuts a from b in M. Let C denote the set of components of $A - K$. It follows from Jones' theorem that for each E belonging to C, there exists a simple closed curve $J(E)$ in M which contains the endpoints of E [4]. Since M does not separate S, there exists a disk $N(E)$ in M such that the endpoints of E are in $N(E)$. Note that if C is finite, one can easily define an arc in M with endpoints p and q.

Assume that C is infinite. For each element E of C define E' to be the straight line segment in S which has the endpoints of E as endpoints. Suppose that for some positive real number ε, there exists an infinite subset I of C such that for each element of E of I, E' is not ε-spanned by a disk in M. There exist a point z in K and a sequence $E_1, E_2, E_3, \cdots$ of elements of I such that (1) $E_1, E_2, E_3, \cdots$ converges to z and (2) for each positive integer n, the diameter of E_n' is less than $\varepsilon/4$. By Lemma 3, for each positive integer n, there exists a disk D_n in M containing $N(E_n)$ such that if (1) W is a complementary domain of $D_n \cup E_n'$, (2) x is a point of Cl $W \cap \text{Bd } D_n$, and (3) $d(x, E_n') > 1/n$, then there exists a point t of $W - M$ such that $d(x, t) < 1/2n$. According to Lemma 2, for each positive integer n, there exist points x_n and y_n in Bd D_n, an arc-segment Y_n in $E_n' - D_n$, a positive real number r_n, and a circular region U_n in S such that (1) $d(\{x_n, y_n\}, E_n') \geq \varepsilon/4$, (2) $\{x_n, y_n\}$ is not contained in the closure of a complementary domain of $D_n \cup Y_n$, (3) $d(x_n, y_n) = r_n$, and (4) U_n has diameter $r_n/2$ and is contained in D_n. If i and j are distinct positive integers, then $U_i \cap U_j = \emptyset$; for otherwise,

$$(K - \{p, q\}) \cap \text{Cl } (E_i \cup E_j)$$

would contain a point which does not cut p from q in M. Since M is bounded and the regions $U_1, U_2, U_3, \cdots$ are mutually exclusive, the sequence $r_1, r_2, r_3, \cdots$ has limit 0. There exists a point x of $M - \{z\}$ such that x is a cluster point of $x_1, x_2, x_3, \cdots$. Suppose that there exists a continuum F in $M - \{z\}$ such that x belongs to the interior of F (relative to M). There exist a region G containing z in $S - F$ and distinct integers i and j such that (1) D_i and D_j both meet F and (2) Cl $(E_i' \cup E_j')$ is a subset of G. It follows that $(K - \{p, q\}) \cap \text{Cl}\,(E_i \cup E_j)$ contains a point which does not cut p from q in M. This is a contradiction. Hence each subcontinuum of M which contains x in its interior (relative to M) must also contain z (that is, M is not aposyndetic at x with respect to z).

Since M is semi-aposyndetic, there exists a continuum F_z in $M - \{x\}$ such that z is contained in the interior of F_z (relative to M). There exist mutually exclusive circular regions U and V in S such that (1) $x \in U$ and $z \in V$, (2) Cl $U \cap F_z = \emptyset$, and (3) $M \cap V \subset F_z$. There exists a positive interger n such that (1) $1/n < \varepsilon/4$, (2) the set

$$\{u \in S \mid d(u, \{x_n, y_n\}) < 1/n\}$$

is contained in U, and (3) Cl E_n' is contained in V. Since

$$d(E_n', \{x_n, y_n\}) > 1/n,$$

there exist points t and u of $(S - M) \cap U$ such that $\{t, u\}$ is not contained in a complementary domain of $D_n \cup Y_n$. Let W and Z be the complementary domains of $D_n \cup Y_n$ (there are only two) which contain t and u respectively. Since M does not separate S, there exists an arc L in $S - M$ from t to u. Let k denote the first point of $L \cap \text{Bd}\, U \cap Z$ and let h be the last point of $L \cap \text{Bd}\, U$ which precedes k with respect to the order of L. Let H denote the subarc of L which has endpoints h and k. Note that h belongs to W and $H \cap \text{Cl}\, U = \{h, k\}$. $(D_n \cup Y_n) - U$ separates h from k in $S - U$. There exists a continuum N in $(D_n \cup Y_n) - U$ which separates h from k in $S - U$ [6, Th. 27, p. 177]. Let B_1 and B_2 be the mutually exclusive arc-segments in Bd U which have endpoints h and k. For $i = 1$ and 2, there exists a point c_i in $B_i \cap N$. The points c_1 and c_2 are contained in distinct components of $N - Y_n$ [6, Th. 28, p. 156]. For $i = 1$ and 2, let d_i be a point of Cl $Y_n \cap$ (c_i-component of $N - Y_n$). The set (θ-curve) $H \cup \text{Bd}\, U$ separates d_1 from d_2 in S [6, Th. 28, p. 156]. $H \cup \text{Bd}\, U$ is contained in $S - F_z$ and $\{d_1, d_2\}$ is a subset of F_z. Since F_z is connected, this is a contradiction. Hence for each positive real number ε, the set consisting of all elements E of C such that E' is not ε-spanned by a disk in M must be finite.

For each positive integer n, let C_n be the finite set consisting of all elements E of C such that either the diameter of E' is greater than or equal to $1/2n$, or E' is not $(1/2n)$-spanned by a disk in M. Let $H_1 = C_1$, and for $n = 2, 3, 4, \cdots$, let $H_n = C_n - C_{n-1}$. Note that the sets $H_1, H_2, H_3, \cdots$ are mutually exclusive and $C = \bigcup_{n=1}^{\infty} H_n$. For each element E of C, define the arc-segment $M(E)$ as follows. Assume that a and b are the endpoints of E. There exists an integer n such that E belongs to H_n. If $n = 1$, define $M(E)$ to be an arc-segment in $N(E)$ with endpoints a and b. According to

Lemma 1, if $n > 1$, there exists an arc-segment $M(E)$ in M with endpoints a and b such that for each point x of $M(E)$, $d(x, E') \leq 1/(n-1)$. For each positive real number ε, the set consisting of all elements E of C such that the diameter of $M(E)$ is greater than ε must be finite. Suppose that for some element X of C, the arc-segment $M(X)$ meets $K \cup \bigcup_{E \in C-\{X\}} M(E)$. It follows that $(K - \{p, q\}) \cap \mathrm{Cl}\, X$ contains a point which does not cut p from q in M. This is a contradiction. Hence for each element X of C,

$$(K \cup \bigcup_{E \in C-\{X\}} M(E)) \cap M(X) = \emptyset.$$

For each element E of C, let f_E be a homeomorphism from E onto $M(E)$. Define the function f from A to $K \cup \bigcup_{E \in C} M(E)$ as follows. For each point x of K, define $f(x) = x$. If x is a point of $A - K$, define $f(x) = f_E(x)$ $(x \in E)$. The function f is a homeomorphism. Hence $K \cup \bigcup_{E \in C} M(E)$ is an arc in M from p to q. It follows that M is arc-wise connected.

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