A COMMUTATIVITY THEOREM FOR PRESPECTRAL OPERATORS

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The main result of this paper is that a prespectral operator of class Γ has a unique resolution of the identity of class Γ, and a unique Jordan decomposition for resolutions of the identity of all classes. The proof of this proceeds by way of a commutativity theorem for prespectral operators. This last result is weaker in form than the commutativity theorem for spectral operators. We observe that, although Theorem 5 of [4; p. 329] is valid for spectral operators, it is not true in general for prespectral operators. (See §6.2 of [2; p. 309].) Consequently, the arguments of Theorem 6 of [4; p. 333–4] cannot be applied in the situation considered here.

Theorems 1 and 2 have recently been proved for scalar-type prespectral operators [3]. In [2], a weaker version of Theorem 2 has been shown to hold in the following special cases:

(a) prespectral operators with totally disconnected spectra
(b) adjoints of spectral operators
(c) prespectral operators whose adjoints are spectral operators.

Theorems 4 and 5 are also known in these cases [2].

The reader is referred to [2] for the definition and properties of prespectral operators. Throughout the paper, X is a complex Banach space with dual space X^*. We write $\langle x, y\rangle$ for the value of the functional y in X^* at the point x of X. For brevity, the term "operator" is used to mean "bounded linear operator". The spectrum and resolvent set of an operator T are denoted by $\sigma(T)$ and $\rho(T)$ respectively. The Banach algebra of operators on X is denoted by $L(X)$. The complex plane is denoted by $\mathbf{C}$, and Σ denotes the σ-algebra of Borel subsets of $\mathbf{C}$. If $\tau \subseteq \mathbf{C}$, and $z \in \mathbf{C}$, then $\chi(\tau, z)$ denotes the characteristic function of the set τ evaluated at z. Let K be a compact Hausdorff space. $C(K)$ denotes the Banach algebra of complex functions continuous on K under the supremum norm. $\mathbf{R}$ denotes the real line.

We require a preliminary result.

Lemma. *Let T be a prespectral operator on X with a resolution of the identity $E(\cdot)$. Let A, in $L(X)$, satisfy $AT = TA$.*

(i) *If $\delta \subseteq \mathbf{C}$ is closed, then $AE(\delta) = E(\delta)AE(\delta)$.*
(ii) *If $\tau \subseteq \mathbf{C}$ is open, then $E(\tau)A = E(\tau)AE(\tau)$.*
(iii) *If $\delta \subseteq \mathbf{C}$ is closed, $\tau \in \Sigma$ and $\bar{\tau} \cap \delta = \emptyset$, then $E(\delta)AE(\tau) = 0$.*

Proof. If δ is a closed set, then by Theorem 4 of [4; p. 328]

$$E(\delta)X = \{x \in X : \sigma(x) \subseteq \delta\}.$$

Received October 29, 1971.

(For a discussion of the single-valued extension property and the notation used in this proof, the reader is referred to §2.2 of [2; p. 292–3].) Now if $x \in X$

$$(\zeta I - T)Ax(\zeta) = A(\zeta I - T)x(\zeta) = Ax \quad (\zeta \in \rho(x)),$$

since $AT = TA$. Also the map $\xi \to Ax(\xi)$ is analytic in $\rho(x)$, and so we obtain successively

$$\rho(Ax) \supseteq \rho(x); \qquad \sigma(Ax) \subseteq \sigma(x).$$

Hence if $x \in E(\delta)X$, then also $Ax \in E(\delta)X$. Therefore $AE(\delta) = E(\delta)AE(\delta)$. This proves (i). If now τ is open, then $\mathbf{C}\backslash\tau$ is closed and

$$A(I - E(\tau)) = (I - E(\tau))A(I - E(\tau)).$$

Consequently $E(\tau)A = E(\tau)AE(\tau)$, proving (ii). Finally, to see (iii) observe that by (i) and hypothesis

$$E(\delta)AE(\bar{\tau}) = E(\delta)E(\bar{\tau})AE(\bar{\tau}) = 0.$$

Now, post-multiplying both sides of the equation $E(\delta)AE(\bar{\tau}) = 0$ by $E(\tau)$ gives the desired result.

THEOREM 1. *Let T be a prespectral operator on X, with a resolution of the identity $E(\cdot)$ of class Γ. Let A, in $L(X)$, satisfy $AT = TA$. Define*

$$R = \int_{\sigma(T)} \operatorname{Re} \lambda E(d\lambda).$$

Then $AR = RA$.

Proof. By Theorem 3.10 of [2; p. 298], T^* is a prespectral operator on X^* with a resolution of the identity $F(\cdot)$ of class X such that

$$\left(\int_{\sigma(T)} f(\lambda)E(d\lambda)\right)^* = \int_{\sigma(T)} f(\lambda)F(d\lambda) \quad (f \in C(\sigma(T))).$$

Using this in conjunction with Theorem 3.1 of [2; p. 294], we see that R^* is a scalar-type prespectral operator on X^* with a resolution of the identity $G(\cdot)$ of class X such that

$$R^* = \int_{\sigma(R)} \lambda G(d\lambda), \qquad G(\mathbf{C}\backslash\mathbf{R}) = 0$$

and for every real number ξ,

$$G(\{\xi\}) = F(L_\xi), \tag{1}$$

where L_ξ is the line parallel to the imaginary axis through the point ξ. Let $x \in X$, $y \in X^*$. Define

$$g(\lambda) = \langle Ax, G((-\infty, \lambda])y\rangle \quad (\lambda \in \mathbf{R}),$$

$$h(\lambda) = \langle x, G((-\infty, \lambda])A^*y\rangle \quad (\lambda \in \mathbf{R}).$$

Now $\langle Ax, G(\cdot)y\rangle$ and $\langle x, G(\cdot)A^*y\rangle$ may be regarded as complex Borel measures on $\mathbf{R}$. Hence g and h are right-continuous complex functions of bounded variation on $\mathbf{R}$. Therefore the set D of points of $\mathbf{R}$ at which either g or h is discontinuous is countable. If $\xi \in \mathbf{R}\backslash D$ we have

$$\langle Ax, G(\{\xi\})y\rangle = \langle x, G(\{\xi\})A^*y\rangle = 0.$$

Hence, using (1) we obtain

$$\langle x, A^*F(L_\xi)y\rangle = \langle x, F(L_\xi)A^*y\rangle = 0 \quad (\xi \in \mathbf{R}\backslash D). \tag{2}$$

Now, $\sigma(T)$ is compact, and so there is a positive real number K such that

$$\sigma(T) \subseteq \{z \in \mathbf{C} : -K < \operatorname{Re} z < +K\}. \tag{3}$$

Let Ω denote the set on the right-hand side of (3). Observe that

$$F(\mathbf{C}\backslash\Omega) = F(\mathbf{C}\backslash\bar{\Omega}) = 0. \tag{4}$$

Next, we construct a suitable sequence of functions converging uniformly to $\operatorname{Re} z$ on Ω. Let n be a positive integer. Since D is countable, $\mathbf{R}\backslash D$ is dense in $\mathbf{R}$ and so we may choose points $\{\xi_m : m = 0, 1, \cdots, 2n+1\}$ in $\mathbf{R}\backslash D$ such that the following two conditions hold:

$$-K = \xi_0 < \xi_1 < \cdots < \xi_{2n+1} = +K; \tag{5}$$

$$|\xi_{m+1} - \xi_m - 2K/(2n+1)| < 2K/(2n+1)^2 \quad (m = 0, 1, \cdots, 2n). \tag{6}$$

We obtain immediately from (6)

$$\xi_{m+1} - \xi_m < K/n \quad (m = 0, 1, \cdots, 2n). \tag{7}$$

For $m = 0, 1, \cdots, 2n+1$, let L_m be the line parallel to the imaginary axis through the point ξ_m. Define

$$\tau_m = \{z \in \mathbf{C} : \xi_{m-1} < \operatorname{Re} z < \xi_m\} \quad (m = 1, \cdots, 2n+1); \tag{8}$$

$$\begin{aligned} &\delta_m = \{z \in \mathbf{C} : (\xi_{m-1} + \xi_m)/2 < \operatorname{Re} z < \xi_m\} \quad (m = 1, \cdots, 2n+1); \\ &f_n(z) = \textstyle\sum_{m=0}^{n} \xi_{2m+1}\chi(\tau_{2m+1}, z) + \sum_{m=1}^{n} \xi_{2m}\chi(\bar{\tau}_{2m}, z) \quad (z \in \bar{\Omega}). \end{aligned} \tag{9}$$

Observe that by (7), $f_n(z)$ converges to $\operatorname{Re} z$ uniformly on Ω and so as $n \to \infty$

$$\int_\Omega f_n(\lambda)F(d\lambda) \to \int_\Omega \operatorname{Re}\lambda F(d\lambda) = \int_{\sigma(T)} \operatorname{Re}\lambda F(d\lambda) = R^*. \tag{10}$$

(The first equality follows from (3).) This leads us to consider the expression η defined by

$$\begin{aligned} \eta = {} & \langle x, \textstyle\sum_{m=0}^{n} \xi_{2m+1}(A^*F(\tau_{2m+1}) - F(\tau_{2m+1})A^*)y\rangle \\ & + \langle x, \textstyle\sum_{m=1}^{n} \xi_{2m}(A^*F(\bar{\tau}_{2m}) - F(\bar{\tau}_{2m})A^*)y\rangle. \end{aligned} \tag{11}$$

Now, by (8),

$$\bar{\tau}_m = \tau_m \cup L_{m-1} \cup L_m \quad (m = 1, \cdots, 2n+1)$$

and the sets on the right-hand side of this equation are pairwise disjoint.

Therefore

$$(12)\quad F(\bar{\tau}_m) = F(\tau_m) + F(L_{m-1}) + F(L_m) \quad (m = 1, \cdots, 2n+1).$$

However by (2)

$$\langle x, A^*F(L_m)y\rangle = \langle x, F(L_m)A^*y\rangle = 0 \quad (m = 1, \cdots, 2n+1),$$

and so (11) becomes

$$(13)\qquad \eta = \langle x, \textstyle\sum_{m=1}^{2n+1} \xi_m(A^*F(\tau_m) - F(\tau_m)A^*)y\rangle.$$

Observe that $A^*T^* = T^*A^*$, and so by Lemma 1,

$$A^*F(\bar{\tau}_m) = F(\bar{\tau}_m)A^*F(\bar{\tau}_m) \quad (m = 1, \cdots, 2n+1).$$

Combining this with (12) gives for $m = 1, \cdots, 2n+1$,

$$A^*(F(\tau_m) + F(L_m) + F(L_{m-1}))$$
$$= (F(\tau_m) + F(L_m) + F(L_{m-1}))A^*(F(\tau_m) + F(L_m) + F(L_{m-1})).$$

This may be rewritten as

$$(14)\quad A^*F(\tau_m) - F(\tau_m)A^* = F(L_{m-1})A^*F(\tau_m) + F(L_m)A^*F(\tau_m)$$

by virtue of the equations

$$F(\tau_m)A^* = F(\tau_m)A^*F(\tau_m), \quad A^*F(L_m) = F(L_m)\,A^*F(L_m),$$
$$A^*F(L_{m-1}) = F(L_{m-1})A^*F(L_{m-1}),$$
$$F(\tau_m)A^*F(L_m) = F(\tau_m)F(L_m)A^*F(L_m) = 0,$$
$$F(\tau_m)A^*F(L_{m-1}) = F(\tau_m)F(L_{m-1})A^*F(L_{m-1}) = 0,$$
$$F(L_m)A^*F(L_{m-1}) = F(L_m)F(L_{m-1})A^*F(L_{m-1}) = 0,$$
$$F(L_{m-1})A^*F(L_m) = F(L_{m-1})F(L_m)A^*F(L_m) = 0,$$

all of which follow from the lemma. From (13) and (14) we obtain

$$(15)\qquad \eta = \langle x, \textstyle\sum_{m=1}^{2n+1} \xi_m(F(L_{m-1})A^*F(\tau_m) + F(L_m)A^*F(\tau_m))y\rangle.$$

We require two more formulae for η. To obtain the first of these, observe that by (3) and (5) we have $F(L_0) = F(L_{2n+1}) = 0$. By (2) and the lemma,

$$\langle x, F(L_m)A^*F(L_m)y\rangle = \langle x, A^*F(L_m)y\rangle = 0,$$
$$F(L_m)A^*F(\mathbf{C}\backslash(\tau_m \cup \tau_{m+1} \cup L_m)) = 0.$$

It follows from the last two equations and (2) that

$$\langle x, F(L_m)A^*F(\tau_m)y\rangle + \langle x, F(L_m)A^*F(\tau_{m+1})y\rangle = \langle x, F(L_m)A^*y\rangle = 0.$$

From these facts, we may rewrite equation (15) as follows.

$$(16)\qquad \eta = \langle x, \textstyle\sum_{m=1}^{2n} (\xi_m - \xi_{m+1})F(L_m)A^*F(\tau_m)y\rangle.$$

Again by the lemma, $F(L_m)A^*F(\tau_m \backslash \delta_m) = 0$. Therefore, (16) may be rewritten

(17) $$\eta = \langle x, \sum_{m=1}^{2n} (\xi_m - \xi_{m+1})F(L_m)A^*F(\delta_m)y\rangle.$$

Now, if $m \neq r$, $\bar{\delta}_m \cap L_r = \emptyset$, and so by the lemma we have $F(L_m)A^*F(\delta_r) = 0$. Also, if $m \neq r$, $\delta_m \cap \delta_r = \emptyset$ and $L_m \cap L_r = \emptyset$. Hence

$$-\eta = \eta_1 + \eta_2$$

where

$$\begin{aligned}\eta_1 &= \langle x, (2K/(2n+1)) \sum_{m=1}^{2n} (F(L_m)A^*F(\delta_m))y\rangle \\ &= \langle x, (2K/(2n+1))(F(\textstyle\bigcup_{m=1}^{2n} L_m)A^*F(\bigcup_{m=1}^{2n} \delta_m))y\rangle \\ \eta_2 &= \langle x, \textstyle\sum_{m=1}^{2n} (\xi_{m+1} - \xi_m - 2K/(2n+1))F(L_m)A^*F(\delta_m)y\rangle.\end{aligned}$$

Now let $M = \sup\{\|F(\tau)\| : \tau \in \Sigma\}$. Then $M < \infty$, and

$$|\eta_1| \leq (2K/(2n+1))\|A\| M^2 \|x\| \|y\|$$

$$|\eta_2| \leq (4nK/(2n+1)^2)\|A\| M^2 \|x\| \|y\|$$

using (6). Hence

(18) $$\begin{aligned}|\eta| &\leq (4K/(2n+1))M^2 \|A\| \|x\| \|y\| \\ &\leq (2KM^2/n)\|A\| \|x\| \|y\|.\end{aligned}$$

From (7) we obtain

(19) $$\sup_{z\in\Omega} |\operatorname{Re} z - \sum_{m=0}^{n} \xi_{2m+1}\chi(\tau_{2m+1}; z) - \sum_{m=1}^{n} \xi_{2m}\chi(\bar{\tau}_{2m}; z)| \leq K/n.$$

Now, if f is any bounded Borel measurable function on $\sigma(T)$, $x_0 \in X$ and $y_0 \in X^*$, then we have

(20) $$\left|\left\langle x_0, \int_{\sigma(T)} f(\lambda)F(d\lambda)y_0\right\rangle\right| \leq 4M\|x_0\| \|y_0\| \sup_{\lambda\in\sigma(T)} |f(\lambda)|.$$

Take $x_0 = Ax$, $y_0 = y$ and

$$f(z) = \operatorname{Re} z - \sum_{m=0}^{n} \xi_{2m+1}\chi(\tau_{2m+1}, z) - \sum_{m=1}^{n} \xi_{2m}\chi(\bar{\tau}_{2m}, z) \quad (z \in \sigma(T)).$$

We get from (7) and (20)

$$\begin{aligned}\langle x, (A^*R^* - \textstyle\sum_{m=0}^{n} \xi_{2m+1}A^*F(\tau_{2m+1}) - \sum_{m=1}^{n} \xi_{2m}A^*F(\bar{\tau}_{2m}))y\rangle& \\ \leq (4MK/n)\|A\| \|x\| \|y\|&.\end{aligned}$$

Next, in (20) take $x_0 = x$ and $y_0 = A^*y$. Then, we obtain

$$\begin{aligned}\langle x, (R^*A^* - \textstyle\sum_{m=0}^{n} \xi_{2m+1}F(\tau_{2m+1})A^* - \sum_{m=1}^{n} \xi_{2m}F(\bar{\tau}_{2m})A^*)y\rangle& \\ \leq (4MK/n)\|A\| \|x\| \|y\|&.\end{aligned}$$

From the last two inequalities and (11) we obtain

(21) $$|\langle x, (A^*R^* - R^*A^*)y\rangle - \eta| \leq (8MK/n)\|A\| \|x\| \|y\|.$$

From (18) and (21) we get

$$|\langle x, (A^*R^* - R^*A^*)y\rangle| \leq (2MK \| A \| \| x \| \| y \|/n)(M + 4.)$$

Now n, x and y are arbitrary. Hence $A^*R^* = R^*A^*$ and so $AR = RA$. This completes the proof of the theorem.

THEOREM 2. *Let T be a prespectral operator on X, with a resolution of the identity $E(\cdot)$ of class Γ. Let A, in $L(X)$, satisfy $AT = TA$. Then*

(i) $$A \int_{\sigma(T)} f(\lambda)E(d\lambda) = \int_{\sigma(T)} f(\lambda)E(d\lambda)A \quad (f \in C(\sigma(T))).$$

(ii) *If $F(\cdot)$ is any resolution of the identity of T*

$$\int_{\sigma(T)} f(\lambda)E(d\lambda) = \int_{\sigma(T)} f(\lambda)F(d\lambda) \quad (f \in C(\sigma(T))).$$

(iii) *T has a unique resolution of the identity of class Γ.*

(iv) *T has a unique Jordan decomposition for resolutions of the identity of all classes.*

Proof. Define

$$R = \int_{\sigma(T)} \mathrm{Re}\lambda\, E(d\lambda), \qquad J = \int_{\sigma(T)} \mathrm{Im}\lambda\, E(d\lambda).$$

By Theorem 1, $AR = RA$. Similarly $AJ = JA$. Hence

$$A \int_{\sigma(T)} p(\lambda, \bar{\lambda})E(d\lambda) = \int_{\sigma(T)} p(\lambda, \bar{\lambda})E(d\lambda)A$$

for any polynomial p in λ and $\bar{\lambda}$. Therefore by the Stone-Weierstrass theorem

$$A \int_{\sigma(T)} f(\lambda)E(d\lambda) = \int_{\sigma(T)} f(\lambda)E(d\lambda)A \quad (f \in C(\sigma(T))),$$

and this proves (i). Next, define

$$R_0 = \int_{\sigma(T)} \mathrm{Re}\lambda\, F(d\lambda), \qquad J_0 = \int_{\sigma(T)} \mathrm{Im}\lambda\, F(d\lambda).$$

Then by (i), $RR_0 = R_0 R$, $RJ_0 = J_0 R$, $JR_0 = R_0 J$ and $JJ_0 = J_0 J$, since R_0 and J_0 commute with T. Since each of R, R_0, J, J_0 can be made hermitian by equivalent renorming of X [1; Theorem 2.5], and since these operators commute, it follows from Corollary 7 of [5; p. 78] that after some appropriate equivalent renorming of X they are simultaneously hermitian. We assume that this renorming has been carried out. Let $S + N$ and $S_0 + N_0$ be respectively the Jordan decompositions of T with respect to $E(\cdot)$ and $F(\cdot)$. Then

$$T = S + N = S_0 + N_0 \quad \text{and} \quad SS_0 = S_0 S.$$

Hence $NN_0 = N_0 N$. Consider the equations

$$N_0 - N = (R - R_0) + i(J - J_0), \tag{22}$$

$$i(N_0 - N) = (J_0 - J) + i(R - R_0). \tag{23}$$

The difference of two hermitian operators is hermitian. Also $N - N_0$, being the sum of two commuting quasinilpotents, is also quasinilpotent. By applying Lemma 15 of [5; p. 82] to (22) and (23) we obtain

$$R = R_0, \quad J = J_0 \quad \text{and} \quad N = N_0.$$

The last equation suffices to prove (iv). Now, by the standard properties of the integral with respect to a spectral measure

$$\int_{\sigma(T)} p(\lambda, \bar{\lambda}) E(d\lambda) = \int_{\sigma(T)} p(\lambda, \bar{\lambda}) F(d\lambda)$$

for any polynomial p in λ and $\bar{\lambda}$. Therefore by the Stone-Weierstrass theorem

$$\int_{\sigma(T)} f(\lambda) E(d\lambda) = \int_{\sigma(T)} f(\lambda) F(d\lambda) \quad (f \in C(\sigma(T))).$$

This proves (ii). Finally, if $E(\cdot)$ and $F(\cdot)$ are both of class Γ, then the conclusion $E(\cdot) = F(\cdot)$ follows at once from (ii) and Lemma 3.2 of [2; p. 295]. This completes the proof of the theorem.

We observe that it was shown in §6.3 of [2; p. 309] that the sum of a scalar-type prespectral operator and a commuting quasinilpotent need not be prespectral of any class. However, we do have the following three results pertaining to such operators. In the statement of the first theorem, the operators S, N, S_0 and N_0 act on X.

THEOREM 3. *Let S be a scalar-type prespectral operator and N a quasinilpotent operator with $SN = NS$. Suppose that A, in $L(X)$, commutes with $S + N$. Then A commutes with each of S and N. Moreover, if $S + N = S_0 + N_0$, where S_0 is a scalar-type prespectral operator, N_0 is a quasinilpotent operator and $S_0 N_0 = N_0 S_0$, then $S = S_0$ and $N = N_0$.*

Proof. Let $E(\cdot)$ be a resolution of the identity for S. Then, by Theorem 2 (i) and the hypothesis $NS = SN$ we obtain

$$N \int_{\sigma(S)} f(\lambda) E(d\lambda) = \int_{\sigma(S)} f(\lambda) E(d\lambda) N \quad (f \in C(\sigma(S))).$$

By Theorem 3.7 of [2; p. 297], $(S + N)^*$ is prespectral on X^* of class X, with a Jordan decomposition $S^* + N^*$. Similarly $S_0^* + N_0^*$ is a Jordan decomposition for $(S_0 + N_0)^* = (S + N)^*$, and so the second statement of the theorem follows from Theorem 2 (iv). Since A^* commutes with the prespectral operator $(S + N)^*$, the first statement of the theorem follows readily from Theorem 2 (i).

THEOREM 4. *Let S, in $L(X)$, be a scalar-type prespectral operator. Let N, in $L(X)$, be a quasinilpotent operator with $SN = NS$. Then if $T = S + N$ is prespectral, every resolution of the identity for T is also a resolution of the identity for S. Also, $T = S + N$ is the unique Jordan decomposition for T. Moreover, N commutes with every resolution of the identity for T.*

Proof. Let $S_0 + N_0$ be the Jordan decomposition for the prespectral operator T. Then from the definition of Jordan decomposition [2; p. 297], and Theorem 3 we obtain $S = S_0$, $N = N_0$. The other statements of the theorem now follow from Theorem 3.5 of [2; p. 296].

THEOREM 5. *Let S be a scalar-type prespectral operator with resolution of the identity $E(\cdot)$ of class Γ. Let N be a quasinilpotent operator with $SN = NS$. Then $S + N$ is prespectral of class Γ if and only if*

$$NE(\tau) = E(\tau)N \quad (\tau \in \Sigma).$$

Proof. The sufficiency of the condition follows from Theorem 3.5 of [2; p. 296]. Now let $S + N$ be prespectral with resolution of the identity $F(\cdot)$ of class Γ. By the previous theorem, $F(\cdot)$ is a resolution of the identity of class Γ for S, and

$$NF(\tau) = F(\tau)N \quad (\tau \in \Sigma).$$

By Theorem 2 (iii), S has a unique resolution of the identity $E(\cdot)$ of class Γ. Hence $F(\cdot) = E(\cdot)$ and

$$NE(\tau) = E(\tau)N \quad (\tau \in \Sigma).$$

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