DERIVATIONS ON $\mathcal{B}(\mathcal{H})$: THE RANGE

BY

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A derivation on a Banach algebra $\mathfrak{A}$ is a linear map $\Delta : \mathfrak{A} \to \mathfrak{A}$ which satisfies $\Delta(ab) = a\Delta(b) + \Delta(a)b$ for all $a, b \in \mathfrak{A}$. Let $\mathcal{B}(\mathcal{H})$ denote the bounded linear operators on a Hilbert space $\mathcal{H}$. It is known that every derivation Δ on $\mathcal{B}(\mathcal{H})$ is inner; that is, $\Delta = \Delta_A$ for some $A \in \mathcal{B}(\mathcal{H})$ where

$$\Delta_A : B \to AB - BA$$

for all $B \in \mathcal{B}(\mathcal{H})$. (In fact, every derivation on a von Neumann algebra is inner; see Kadison [8], Kaplansky [10], and Sakai [15].) Lumer and Rosenblum [12] have determined the spectrum of an inner derivation. They showed that

$$\sigma(\Delta_A) = \{\lambda_1 - \lambda_2 : \lambda_1, \lambda_2 \in \sigma(A)\}$$

It is known [17] that

$$\| \Delta_A \| = 2 \min \{ \| A - \lambda I \| : \lambda \text{ complex}\}.$$

(For the norm of a derivation in a von Neumann algebra see [5] and [9].)

We now turn our attention to the range of the derivation Δ_A. Specific questions about the size of $\Delta_A(\mathcal{B}(\mathcal{H}))$, raised in [2], [18] and [21], will be answered. (For a not unrelated question from the algebraist's point of view see [11], question 12.)

The basic tool in the main theorems is the following simple lemma. The essential spectrum of A, denoted by $\sigma_{\text{ess}}(A)$ is the spectrum of A in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}$ where $\mathcal{K}$ is the two sided ideal of compact operators.

LEMMA 1. *Let $A \in \mathcal{B}(\mathcal{H})$. Let $\lambda_0 \in \partial\sigma_{\text{ess}}(A)$. Then there exist mutually orthogonal sequences of unit vectors $\{f_n\}$, $\{g_n\}$ such that*

$$\| (A - \lambda_0)f_n \| \to 0 \quad \textit{and} \quad \| (A - \lambda_0)^* g_n \| \to 0.$$

Proof. If $\lambda_0 \in \partial\sigma_{\text{ess}}(A)$, then λ_0 is in the left essential spectrum of A and hence by [4] there exists an orthonormal sequence $\{f_n\}$ such that $\| (A - \lambda_0)f_n \| \to 0$. By the same reasoning there exists an orthonormal sequence $\{g_n\}$ such that $\| (A - \lambda_0)^* g_n \| \to 0$. By replacing $\{f_n\}$ and $\{g_n\}$ by appropriate linear combinations we easily achieve the desired result.

THEOREM 1. *Let $A \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{R}(\Delta_A)$, the range of Δ_A, is never norm dense in $\mathcal{B}(\mathcal{H})$.*

Proof. Choose λ_0, $\{f_n\}$, $\{g_n\}$ as in Lemma 1.

Received October 26, 1971.

[1] The author gratefully acknowledges the support of the National Science Foundation.

Define V as follows

$$V : f_n \to g_n$$
$$V : \text{clm}\,\{f_n\}^{\perp} \to \text{clm}\,\{g_n\}^{\perp} \text{ arbitrary but bounded.}$$

Then for any $T \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned} \| V - (AT - TA) \| &\geq | ([V - (AT - TA)]f_n, g_n) | \\ &= 1 + ((A - \lambda_0)f_n, T^*g_n) - (Tf_n, (A - \lambda_0)^*g_n) \\ &\to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

The proof is complete.

Remark. This answers a question raised in [18]. It is easy to modify the definition of V to make it unitary, self-adjoint, nilpotent or almost what you will.

By $\mathcal{R}(\Delta_A)^-$ we mean the norm closure of $\mathcal{R}(\Delta_A)$.

COROLLARY. *Let $A \in \mathcal{B}(\mathcal{H})$. Then $\mathcal{B}(\mathcal{H})/\mathcal{R}(\Delta_A)^-$ is not separable.*

Proof. As in the previous proof choose λ_0, $\{f_n\}$ and $\{g_n\}$. Assume without loss of generality that $\lambda_0 = 0$ and that $\text{clm}\,\{f_n\}^{\perp} = F$ and $\text{clm}\,\{g_n\}^{\perp} = G$ are infinite dimensional. Let α be a subset of $\mathbf{Z}^+$ and define

$$U_\alpha f_n = \begin{cases} +g_n & \text{for } n \in \alpha, \\ -g_n & \text{for } n \notin \alpha. \end{cases}$$

Extend U_α to a map of F onto G so that U_α is unitary. Clearly $\| U_\alpha - U_\beta \| = 2$ for $\alpha \neq \beta$. Define an equivalence relation on the set $\{U_\alpha\}$ as follows:

$U_\alpha \sim U_\beta$ if they differ at only a finite number of the f_n's. Clearly there are an uncountable number of distinct classes. Moreover

$$\| (AT - TA) + (U_\alpha - U_\beta) \| \geq 2$$

for U_α, U_β in distinct equivalence classes by the argument in the previous theorem.

Since $\inf \{ \| L + (U_\alpha - U_\beta) \| : L \in \mathcal{R}(\Delta_A)^- \} = 2$ for U_α, U_β in distinct equivalence classes, it follows that $\mathcal{B}(\mathcal{H})/\mathcal{R}(\Delta_A)^-$ can not be separable.

Remark. Note that $\mathcal{R}(\Delta_A)$ can itself be non-separable. For example if A is the operator valued matrix

$$\begin{vmatrix} I & 0 \\ 0 & 2I \end{vmatrix},$$

on $\mathcal{H} \oplus \mathcal{H}$ then $\mathcal{R}(\Delta_A)$ is already norm closed and consists of all operators of the form

$$\begin{vmatrix} 0 & S \\ R & 0 \end{vmatrix}$$

where R, S are arbitrary operators in $\mathcal{B}(\mathcal{H})$. On the other hand for A compact, $\mathcal{R}(\Delta_A)^-$ is always separable.

In problem 49, page 479 of [21], J. Daleckii asks whether

$$\mathcal{R}(\Delta_A)^{=} + \{A\}' = \mathcal{B}(\mathcal{H})$$

for all self-adjoint $A \in \mathcal{B}(\mathcal{H})$. (Here $\{A\}'$ denotes the commutant of A.) If we set $A\varphi_n = (1/n)\varphi_n$ where φ_n is an orthonormal basis for H, it is not hard to see we do not obtain equality. In fact $\mathcal{B}(\mathcal{H})/[\mathcal{R}(\Delta_A)^{=} + \{A\}']$ is not even separable in this case.

If A is not self adjoint then even more striking behavior can occur. Let A be the Donoghue shift: $A\varphi_n = 2^{-n}\varphi_{n+1}$ where $\{\varphi_n\}_1^\infty$ is an orthonormal basis for $\mathcal{H}$. Then by a result of Nordgren [22], $\{A\}'$ consists of compact operators (in fact, any operator in $\{A\}'$ is the norm limit of polynomials in A). Thus $\mathcal{R}(\Delta_A)^{=} + \{A\}'$ is a subset of the compact operators and hence is separable.

THEOREM 2. *Let A, $G \in \mathcal{B}(\mathcal{H})$ be fixed where $G \neq 0$. Then there exists a unitary operator U such that $U^*GU \notin \mathcal{R}(\Delta_A)$; that is $\mathcal{R}(\Delta_A)$ contains no unitarily invariant subset of operators.*

Proof. If $G = \lambda I$ then $G \notin \mathcal{R}(\Delta_A)$ since I is not a commutator by a well known result of Wintner [20]. If $G \neq \lambda I$ then there exists a basis $\{\varphi_n\}$ for H such that

$$(G\varphi_n, \varphi_m) \neq 0 \quad \text{for } n, m = 1, 2, \cdots \qquad \text{(see [13])}.$$

Let $(G\varphi_{3n}, \varphi_{3n+1}) = z_n$. Choose λ_0, $\{f_n\}$, $\{g_n\}$ as in Theorem 1. We assume $\lambda_0 = 0$. By passing to a subsequence we can guarantee that

$$\| Af_n \| \leq n^{-1}z_n \quad \text{and} \quad \| A^*g_n \| \leq n^{-1}z_n.$$

We define U as follows

$$U : f_n \rightarrow \varphi_{3n}$$
$$U : g_n \rightarrow \varphi_{3n+1}$$
$$U : \operatorname{clm}\{f_n, g_n\}^{\perp} \rightarrow \operatorname{clm}\{\varphi_{3n+2}\} \quad \text{1-1, onto, and isometric.}$$

Clearly U is unitary. Assume $AT - TA = U^*GU$ for some $T \in \mathcal{B}(\mathcal{H})$. then

$$| z_n | = | (U^*GUf_n, g_n) | = | (AT - TA)f_n, g_n) | \leq 2 \| T \| \, | z_n |/n.$$

Hence $\| T \| \geq n/2$ for all n which is absurd.

COROLLARY. *For $A \in \mathcal{B}(\mathcal{H})$, $\mathcal{R}(\Delta_A)$ does not contain all operators of rank one and hence does not contain any ideal in $\mathcal{B}(\mathcal{H})$.*

This corollary answers a question raised in [2].

COROLLARY. *Let A, $G_k \in \mathcal{B}(\mathcal{H})$ for $k = 1, 2, \cdots$. Then there exists a unitary operator U such that $U^*G_k U \notin \mathcal{R}(\Delta_A)$ for $k = 1, 2, \cdots$.*

Proof. Assume without loss of generality that each $G_k \neq \lambda I$. An easy modification of the argument in [13] enables us to choose an orthonormal basis $\{\varphi_n\}_1^\infty$ such that $(G_k\varphi_n, \varphi_m) \neq 0$ for $k, n, m = 1, 2, \cdots$. Set $\mathcal{H} = \Sigma \oplus \mathcal{H}_n$

where each $\mathcal{H}_n$ is infinite dimensional and has a subset of the φ_n's as an orthonormal basis. For each G_k it is possible to repeat the argument in the theorem (on $\mathcal{H}_k$) to attain the desired consequence.

Let K be compact and let $\lambda_1 \geq \lambda_2 \geq \cdots$ be the eigenvalues of $(K^*K)^{1/2}$. Then $K \in \mathcal{C}_p$ (Schatten p-class) if $\Sigma \mid \lambda_n \mid^p < \infty$.

LEMMA 2. *There exists a compact operator K which does not commute with any operator of Schatten p-class.*

Proof. Let $\{\varphi_n\}_1^\infty$ be an orthonormal basis for $\mathcal{H}$. Define $K\varphi_n = a_n \varphi_{n+1}$ where $a_n = 1/\log n$ for $n > 2$ and $a_1 = a_2 = 1$. Assume B commutes with K. Let $B\varphi_j = \Sigma_1^\infty b_{k,j}\varphi_k$. If $B \neq 0$ then $b_{k,1} \neq 0$ for some k since φ_1 is a cyclic vector for K. Let m be the smallest k for which $b_{k,1}$ does not vanish. Assume $b_{m,1} = 1$. A routine calculation shows that

$$b_{m+j,j+1} = \frac{a_m \cdots a_{m+j-1}}{a_1 \cdots a_j} \quad \text{for } j = 1, 2, \cdots.$$

Hence $\mid b_{m+j,j+1} \mid \geq \mid a_{m+j-1} \mid^{m-1}$ for $j \geq m$.

Thus for any $p \geq 1$

$$\sum_{j=1}^\infty \| B\varphi_j \|^p \geq \sum_{j=m}^\infty \mid a_{m+j-1} \mid^{p(m-1)} = \sum_{j=2m-1}^\infty (\log j)^{-p(m-1)} = \infty.$$

Hence B can not be of Schatten p-class since if $p \geq 2$ then $\| B \|_p^p \geq \Sigma \| B\psi_j \|^p$ for any orthonormal basis $\{\psi_j\}$ (see Gohberg and Krein [6, page 95]).

THEOREM 3. *There exists a (compact) operator K such that $\mathcal{R}(\Delta_K)^= = \mathcal{K}$ the ideal of compact operators.*

Proof. We choose K to be the operator constructed in the previous lemma. Since K is compact $KT - TK \in \mathcal{K}$ for all $T \in \mathcal{B}(\mathcal{H})$ and hence $\mathcal{R}(\Delta_K) \subset \mathcal{K}$. On the other hand by Theorem 3 of [19], if A does not commute with an operator of trace class then $\mathcal{R}(\Delta_A)^= \supset \mathcal{K}$. Hence $\mathcal{R}(\Delta_K)^= = \mathcal{K}$.

Remark. If $A \neq \lambda I +$ compact, then $\mathcal{R}(\Delta_A)$ contains a non-compact operator. This result, which admits a variety of proofs, can be found in [3].

We now turn our attention to one of the major unsolved problems on the range of a derivation: Is $I \in \mathcal{R}(\Delta_A)^=$ for any $A \in \mathcal{B}(\mathcal{H})$? The following statements are easily seen to be equivalent:

(i) $I \in \mathcal{R}(\Delta_A)^=$
(ii) there exists an invertible operator B in $\{A\}'$ such that $B \in \mathcal{R}(\Delta_A)^=$
(iii) $\mathcal{R}(\Delta_A)^=$ contains all the invertible operators in $\{A\}'$.

Our partial answer to the question indicates that it is no mean feat for $\mathcal{R}(\Delta_A)^=$ to contain the identity. We begin with the following:

LEMMA 3. *If $\| A \| \leq 1$ and $\| (AT - TA) - I \| < \varepsilon$ then*

$$\| (A^{n+1}T - TA^{n+1}) - (n + 1)A^n \| < 3^n\varepsilon.$$

Proof. We proceed by induction. Assume $A^nT - TA^n = nA^{n-1} + 3^{n-1}\delta_n$ where $\| \delta_n \| < \varepsilon$. Multiplying fore, then aft by A and adding we obtain

$$(A^{n+1}T - TA^{n+1}) + (A^nTA - ATA^n) = 2nA^n + 2\cdot 3^{n-1}\delta_{n+1}$$

where $\| \delta_{n+1} \| < \varepsilon$. But

$$(A^nTA - ATA^n) = A(A^{n-1}T - TA^{n-1})A = (n-1)A^n + 3^{n-2}\delta_{n-1}$$

where $\| \delta_{n-1} \| < \varepsilon$. Thus $A^{n+1}T - TA^{n+1} = (n+1)A^n + 2\cdot 3^{n-1}\delta_{n+1} + 3^{n-2}\delta_{n-1}$ which completes the proof.

THEOREM 4. *If $A^k = 0$, then $I \notin \mathcal{R}(\Delta_A)^-$.*

Proof. We may and do assume $\| A \| \leq 1$. Choose $T \in \mathcal{B}(\mathcal{H})$ such that $\| (AT - TA) - I \| < \varepsilon$. Then by the lemma

$$\| (A^kT - TA^k) - kA^{k-1} \| < 3^{k-1}\varepsilon.$$

Hence $\| A^{k-1} \| < k^{-1}3^{k-1}\varepsilon$ and since ε was arbitrary it follows that $A^{k-1} = 0$. By repeating the argument we are led, inexorably, to the conclusion that $A = 0$, which is absurd.

COROLLARY. *If A^k is compact, (that is, A is nilpotent in the Calkin algebra) then $I \notin \mathcal{R}(\Delta_A)^-$.*

Proof. The argument given above is valid in a $\mathcal{C}^*$ algebra.

Because we will have occasion to appeal to the next lemma several times, we state it here explicitly. The proof is left to the reader.

LEMMA 4. *Let $A \in \mathcal{B}(\mathcal{H})$ be similar to an operator of the form*

$$\begin{vmatrix} S & 0 \\ 0 & T \end{vmatrix}$$

on $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$. If $I_{\mathcal{H}_1} \notin \mathcal{R}(\Delta_S)^-$ then $I \notin \mathcal{R}(\Delta_A)^-$.

THEOREM 5. *Let $A \in \mathcal{B}(\mathcal{H})$. Let $f(A) = N$ where N is normal and f is analytic on an open set containing $\sigma(T)$. Then $I \notin \mathcal{R}(\Delta_A)^-$.*

Proof. We must consider two cases. The first when $\sigma(A)$ has infinite cardinality; the second when it has not. Let $\sigma(A)$ be infinite and let $z_1, \cdots, z_n$ be the zeros of f'. Let $W = f^{-1}[f(\bigcup_1^n z_i)]$. Choose a closed disc γ such that $\gamma \cap W = \emptyset$ and $\gamma \cap \sigma(A) \neq \emptyset$. Thus f' never vanishes on γ. Let $N = \int \lambda \, dE(\lambda)$. Since A commutes with N and hence with $E(\cdot)$ we may write

$$A = \begin{vmatrix} A_1 & 0 \\ 0 & A_2 \end{vmatrix} \quad \text{on } E(f(\gamma))\mathcal{H} \oplus E(f(\gamma)')\mathcal{H}$$

and

$$f(A) = \begin{vmatrix} f(A_1) & 0 \\ 0 & f(A_2) \end{vmatrix} = \begin{vmatrix} N_1 & 0 \\ 0 & N_2 \end{vmatrix}$$

(here $'$ denotes set complementation). Since $f(A_1)$ is normal and f' never vanishes on $\sigma(A_1) \subset f^{-1}[f(\delta)] \subset W'$, it follows from [1], that A_1 is scalar on $E(f(\gamma))\mathcal{H}$; that is, similar to a normal operator. Thus $\mathcal{R}(\Delta_{A_1})^-$ does not contain the identity, and hence by the previous lemma, neither does $\mathcal{R}(\Delta_A)^-$. (It is easy to see that $I \notin \mathcal{R}(\Delta_B)^-$ for B normal [16]. This point will be discussed again shortly.)

Now let $\sigma(A)$ be finite. Choose $z_0 \in \sigma(A)$ and let $f(z_0) = \zeta_0$. Then A is similar to an operator of the form

$$\begin{vmatrix} A_1 & 0 \\ 0 & A_2 \end{vmatrix} \quad \text{on } \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$$

where $\sigma(A_1) = \{z_0\}$ and $\sigma(A_2) = \sigma(A)\backslash\{z_0\}$ (see [14] Chapter XI). By the normality of N, the spectral mapping theorem, and any one of several arguments (one of which was used in the first case) $f(A_1) = \zeta_0 I$ where I here is the identity on $\mathcal{H}_1$. Hence $g(A_1) = f(A_1) - \zeta_0 I_{\mathcal{H}_1} = 0$. After factoring f as $(z - z_0)^n h(z)$ we conclude that $(A_1 - z_0)^n = 0$. Thus $I_{\mathcal{H}_1} \notin \mathcal{R}(\Delta_{A_1})^-$ by Theorem 4 and hence $I \notin \mathcal{R}(\Delta_A)^-$ by the previous lemma.

Corollary. *If A is of the following form (or similar to an operator of the following form) then $I \notin \mathcal{R}(\Delta_A)^-$:*

(1) $f(A) =$ *normal*
(2) $A =$ *hyponormal* $+$ *compact*
(3) $A =$ *Toeplitz* $+$ *compact*
(4) $\| (A - \lambda) \| =$ *spectral radius of* $(A - \lambda)$ *for some* λ.

Proof. Actually operators of the form (2), (3) or (4) are all in $\bar{\mathcal{R}}_1$, that is, they all possess an approximate reducing eigenvector. More precisely, given $\varepsilon > 0$, there exists a λ_0 and a unit vector f such that

$$\| (A - \lambda_0)f \| < \varepsilon \quad \text{and} \quad \| (A - \lambda_0)^* f \| < \varepsilon$$

(see [16]). It is easy to see that the conclusion follows from this condition.

Remark. If $\mathfrak{A}$ is a von Neumann algebra and $A \in \mathfrak{A}$ then $\Delta_A : \mathfrak{A} \to \mathfrak{A}$. We mention that Theorem 1 is valid in this context, that is, $\Delta_A(\mathfrak{A})$ is never norm dense in $\mathfrak{A}$. Since any von Neumann algebra can be written as the direct sum of algebras of the various types it suffices to consider the case when $\mathfrak{A}$ itself is of fixed type. The algebras of type I_n or II, are easily handled by a trace argument. Using powerful results from his work on von Neumann algebras, Herbert Halpern has taken care of the remaining algebras (the properly infinite ones) thus completing the proof. (See [7] for details.)

Added in proof. Joel H. Anderson has recently shown that there exists a strange and wondrous operator A for which $I \in R(\Delta_A)^-$. His paper, "The identity and the range of a derivation", will appear in the Bulletin of the American Mathematical Society.

References

1. C. Apostol, *On the roots of spectral operator valued analytic functions*, Rev. Math. Pures. Appl., vol. 13 (1968), pp. 587–589.
2. A. Brown and C. Pearcy, *Compact restrictions of operators*, Acta Sci. Math., vol. 32 (1971), pp. 271–282.
3. Calkin, *Two sided ideals and congruences in the ring of bounded operators on Hilbert space*, Ann. of Math., vol. 42 (1941), pp. 839–873.
4. P. A. Fillmore, J. G. Stampfli and J. P. Williams, *On the essential spectrum, the essential numerical range and a problem of Halmos*, Acta Sci. Math., vol. 33 (1972), pp. 179–192.
5. P. Gajendragakar, *The norm of a derivation on a von Neumann algebra*, Trans. Amer. Math. Soc., vol. 170 (1972), pp. 165–170.
6. I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Translations Math. Monographs, vol. 18, Amer. Math. Soc., Providence, R. I., 1969.
7. H. Halpern, *Essential central spectrum and range for elements of a von Neumann algebra*, Pacific J. Math., vol. 43 (1972), pp. 349–380.
8. R. V. Kadison, *Derivations of operator algebras*, Ann. of Math., vol. 83 (1966), pp. 280–293.
9. R. V. Kadison, E. C. Lance and J. R. Ringrose, *Derivations and automorphisms of operator algebras II*, J. Functional Anal., vol. 1 (1967), pp. 204–221.
10. I. Kaplansky, *Modules over operator algebras*, Amer. J. Math., 75 (1953), pp. 839–859.
11. ———, *Problems in the theory of rings, revisited*, Amer. Math. Monthly, vol. 77 (1970), pp. 445–454.
12. G. Lumer and M. Rosenblum, *Linear operator equations*, Proc. Amer. Math. Soc., vol. 10 (1959), pp. 32–41.
13. H. Radjavi and P. Rosenthal, *Matrices for operators and generators of* $\mathcal{B}$ (H), J. London Math. Soc., vol. 2 (1970), pp. 557–560.
14. F. Riesz and B. Sz-Nagy, *Functional Analysis*, Ungar, New York, 1955.
15. S. Sakai, *Derivations of W^*-algebras*, Ann. of Math., vol. 83 (1966), pp. 273–279.
16. J. G. Stampfli, *On hyponormal and Toeplitz operators*, Math. Ann , vol 183 (1969), pp. 328–336.
17. ———, *The norm of a derivation*, Pacific J. Math., vol. 33 (1970), pp. 737–747.
18. J. P. Williams, *Finite operators*, Proc. Amer. Math. Soc., vol. 26 (1970), pp. 129–136.
19. ———, *On the range of a derivation*, Pacific J. Math., vol. 38 (1971), pp. 273–279.
20. A. Winter, *The unboundedness of quantum mechanical matrices*, Phys. Rev., vol. 71 (1947), pp. 738–739.
21. *Proceedings of the International Colloquium on Nuclear Spaces and Ideals in Operator Algebras*, Warsaw, 18–25 June 1969, Studia Math., vol. 38 (1970).
22. E A. Nordgren, *Closed operators commuting with a weighted shift*, Proc. Amer. Math. Soc., vol. 24 (1970), pp. 424–428.

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