COBORDISM OF LINE BUNDLES WITH A RELATION

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In this paper a cobordism theory for line bundles over oriented manifolds, with $w_2(\text{Base}) = w_1^2(\text{bundle})$, is studied. The cobordism groups Λ_n are computed. A homomorphism

$$\Lambda_n \to \Omega_n^{Spin}(Z_2)$$

is given, and it is shown that this is a monomorphism mod torsion.

1. The classifying space

We reserve the term manifold for oriented, compact C^∞ manifolds, without boundary unless otherwise specified. Let BSO be the classifying space for stable oriented vector bundles, and BO_1 the classifying space for line bundles. Let

$$f : BSO \times BO_1 \to K(Z_2, 2)$$

be the map give by $f^*(\iota) = w_2 \otimes 1 + 1 \otimes t^2$, where $\iota \in H^2(K(Z_2, 2), Z_2)$ is the fundamental class, $t \in H^1(BO_1, Z_2)$ the generator, and $w_i \in H^i(BSO, Z_2)$ the ith universal Stiefel-Whitney class. Then f induces a fibration over $BSO \times BO_1$ from the path space over $M(Z_2, 2)$

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & PK(Z_2, 2) \arrow[d] \\ BSO \times BO_1 \arrow[r, "f"] & K(Z_2, 2). \end{tikzcd}$$

Given an oriented manifold M and a line bundle η over M, the classifying map ν of the stable normal bundle of M, and the classifying map η of η induce a map

$$\nu \times \eta : M \to BSO \times BO_1 .$$

Now $w_2(M) + (w_1(\eta))^2 = (\nu \times \eta)^*(w_2 \otimes 1 + 1 \otimes t^2) = (\nu \times \eta)^* f^*(\iota)$. So $\nu \times \eta$ lifts to a map $c : M \to E$ iff $w_2(M) + (w_1(\eta))^2 = 0$. Thus we have the following definition.

Define an equivalence relation on the set of triples (M^n, η, c), where M^n is an n-dimensional manifold, η a line bundle over M, and c a lifting of $\nu \times \eta$ to E as follows: (M_1^n, η_1, c_1) is equivalent to (M_2^n, η_2, c_2) if there is a triple (W, η, c) where W is an $(n + 1)$-dimensional manifold with boundary, η a line bundle over W, and c a lifting of $\nu_W \times \eta$ to E, such that

(1) $\partial W = M_1 + (-M_2)$
(2) $c \,|\, M_i = c_i \quad i = 1, 2.$

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Note that c determines η and so $\eta \mid M_i = \eta_i$ automatically. Let Λ_n denote the set of equivalence classes.

We can also do this entire procedure over $BSO_k \times BO_1$ and obtain E_k. But the map $BSO_k \to BSO_{k+1}$ lifts to a map $E_k \to E_{k+1}$. Hence, this is a special case of $'B, f'$-cobordism according to Lashof [3], where $B = E$ and f is the composition $E \to BSO \times BO_1 \to BSO$. So we have the following [3].

THEOREM 1. *Λ_n is a group with operation induced by disjoint union. Furthermore, setting $\Lambda_* = \oplus_n \Lambda_n$, we have $\Lambda_* \approx \pi_*(M(\xi))$, the stable homotopy of the Thom space of the bundle ξ over E induced by the composition $E \to BSO \times BO_1 \to BSO$ from the universal bundle over BSO.*

Also there is a product in Λ_*, given by cartesian product of manifolds, and tensor product of line bundles. Lifting the composition

$$E \times E \to BSO \times BO_1 \times BSO \times BO_1 \to BSO \times BSO \times BO_1 \xrightarrow{\oplus,\, \otimes} BSO \times BO_1$$

gives a map $\mu : E \times E \to E$ such that the diagram

$$\begin{array}{ccc} E \times E & \xrightarrow{\mu} & E \\ \downarrow{\scriptstyle \pi_1 \times \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ BSO \times BSO & \longrightarrow & BSO \end{array}$$

commutes. So μ induces $\mu : M(\xi) \wedge M(\xi) \to M(\xi)$, and Λ_* is a graded ring.

Since $\pi_1 \, p_n : E_n \to BSO(n) \times BZ_2 \to BSO(n)$ is a mod p homotopy equivalence for odd primes p, it follows that $\pi_*(M(\xi))$ has no odd torsion.

2. $H^*(M(\xi); Z_2)$

For the rest of this paper, all homology and cohomology will be with coefficient group Z_2, $\mathcal{A}$ will denote the mod 2 Steenrod algebra, and $w_i \in H^i(BSO)$ the i^{th} Stiefel Whitney class.

THEOREM 2. *As a graded $\mathcal{A}$ algebra, $H^*(E)$ is isomorphic to the polynomial ring*

$$Z_2[(\pi_1 \, p)^*(w_i),\ i \neq 2^j + 1] \otimes Z_2[(\pi_2 \, p)^*(t)],$$

t being the generator in $H^1(BZ_2)$, with the extension given by $(\pi_2 \, p)^*(t^2) = (\pi_1 \, p)^*(w_2)$.

Proof. In the fibration $K(Z_2, 1) \to E \to BSO \times BZ_2$ the fundamental group of the base acts trivially on the cohomology of the fibre. The fundamental class

$$\iota_1 \in H^1(K(Z_2, 1))$$

transgresses to $p^*(w_2 \otimes 1 + 1 \otimes t^2)$. Hence ι_1^2 transgresses to

$$Sq^1 p^*(w_2 \otimes 1 + 1 \otimes t^2) = p^*(w_3 \otimes 1)$$

and $\iota_1^{2^i}$ transgresses to $p^*((w_{2^i+1} + \text{decomposables}) \otimes 1)$. Thus, by Borel's theorem, $H^*(E)$ is the required quotient of $H^*(BSO \times BZ_2)$.

Hereafter we drop the p^* from $p^*(w_1)$ and $p^*(t)$.

COROLLARY 1. *The bundle ξ over E has a $Spin^c$ structure, and its classifying map*

$$\hat{\xi} : E \to BSpin^c$$

induces a monomorphism on cohomology.

COROLLARY 2. *Let $U \in H^0(M(\xi))$ be the Thom class. Then the homomorphism*

$$\mathcal{A} \to H^*(M(\xi))$$

give by $a \to aU$ has kernel $\mathcal{A}/\mathcal{A}(Sq^1, Sq^3)$.

This follows from the corresponding fact for $H^*(MSpin^c)$ [7].

In order to compute $H^*(M(\xi))$ as an $\mathcal{A}$ module, we will need the following building blocks.

DEFINITION. Let M be the $\mathcal{A}$ module obtained from the direct sum

$$\mathcal{A}/\mathcal{A}Sq^1 \oplus \oplus_{i=1}^{\infty} \mathcal{A}$$

by the relations $Sq^2 x_0 = Sq^1 x_1$, $Sq^2 Sq^3 x_i = Sq^1 x_{i+1}$, where x_0 denotes the generator of the summand $\mathcal{A}/\mathcal{A}Sq^1$, and x_i the generator of the ith summand. Note deg $(x_0) = 0$, deg $(x_i) = 4i - 3$.

THEOREM 3. *Let $\mathcal{J}$ be the set of all non-decreasing sequences of integers $(j_1, \cdots, j_s)$ of finite length such that $j_r > 1$ for all r. Let Y be the graded Z_2 vector space with one generator Y_J for each $J \in \mathcal{J}$, with* deg $y_J = 4n(J) = 4\sum j_i$. *Then $H^*(M(\xi))$ is isomorphic as an $\mathcal{A}$ module to $M \otimes Y \oplus F$, where F is a free $\mathcal{A}$ module.*

The proof will occupy the remainder of this section. The homology of $M \otimes Y$ and $H^*(M(\xi))$ with respect to the differentials Q_0, and Q_1, induced by operation of Sq^1 and $Sq^3 + Sq^2Sq^1$, respectively will be computed. Then

$$f_* : H(M \otimes Y, Q_i) \to H(H^*(M(\xi)), Q_i)$$

will be shown to be an isomorphism and Theorem 5.1 of [6] will be applied. Note that the product in Λ_* gives $H^*(M(\xi))$ the structure of a coalgebra over $\mathcal{A}$.

The first step is to compute $H(H^*(M(\xi)), Q_0)$ and $H(H^*(M(\xi)), Q_1)$. Since $Q_0 U = 0$ and $Q_1 U = 0$ in $H^*(M(\xi))$, the Thom isomorphism $H^*(E) \to H^*(M(\xi))$ induces an isomorphism on Q_0 and Q_1 homologies. So $H(H^*(E), Q_i)$ will be computed.

LEMMA 1. *There are classes $u_{2^i} \in H^{2^i}(E)$ and $\tilde{u}_{2^i-2} \in H^{2^i-2}(E)$ such that*

$$(1) \qquad H(H^*(E), Q_0) \approx Z_2[w_{2i}^2, i \neq 2^j] \otimes Z_2[u_{2^j}, j > 1]$$

(2) $H(H^*(E), Q_1) \approx Z_2[w_{2i}^2, i \neq 2^j - 1] \otimes Z_2[u_{2^j-2}, j > 2] \otimes \Lambda(w_2)$.

Proof. (1) $Q_0(w_{2i}) = w_{2i+1}$, $Q_0 w_{2i+1} = 0$, $Q_0 t = t^2 = w_2$. There are classes

$$u'_{2^j} \in H^{2^j}(BSpin^c)$$

such that $u'_{2^j} = w'_{2^j} +$ decomposables, and $Q_0 u'_{2^j} = 0$ [7, pg. 316]. Hence we can write $H^*(E)$ as

$$Z_2[w_{2i}, Q_0 w_{2i}, i \neq 2^j] \otimes Z_2[u_{2^j}, j > 1] \otimes Z_2[t],$$

where u_{2^j} is the image in $H^*(E)$ of u'_{2^j}.

(2) $Q_1 w_{2i} = w_{2i+3}$, $Q_1 w_{2i+1} = 0$, $Q_1 t = t^4 = w_2^2$. Choose $\tilde{u}_{2^j-2}$ as in proof of (1). Then we get

$$H^*(E) \approx Z_2[w_{2i}, Q_1 w_{2i}, i \neq 2^j - 1] \otimes Z_2[\tilde{u}_{2^j-2}, j > 2] \otimes Z_2[t].$$

COROLLARY 3. *$H(H^*(E))$ has Q_0 homology only in dimensions congruent to 0 mod 4.*

LEMMA 2.

$$H^*(MSpin^c) \approx (\oplus_{J'} (\mathfrak{A}/\mathfrak{A}(Sq^1, Sq^3))x'_J) \oplus F$$

where F is a free module, $\deg x'_J = 4n(J)$, *and $\mathcal{J}'$ is the set of all finite non-decreasing sequences J of positive integers.*

Proof. [7, pg. 319].

We use this to construct the map $f : M \otimes Y \to H^*(M(\xi))$.

Let $z'_J U \in H^{4n(J)}(MSpin^c)$ be the generator of the $(\mathfrak{A}/\mathfrak{A}(Sq^1, Sq^3))x'_J$. Let

$$z_J U \in H^*(M(\xi))$$

be its image. Recall M is a quotient of the direct sum

$$\mathfrak{A}/\mathfrak{A}Sq^1 \oplus \mathfrak{A} \oplus \mathfrak{A} \oplus \mathfrak{A} \oplus \cdots$$

with generators $x_0, x_1, x_2, \cdots$. Let $f(x_0 \otimes y_J) = z_J U$. Then $Sq^1(z_J U) = 0$. Now

$$Sq^1(Sq^2 z_J \cdot U) = Sq^3 z_J \cdot U = 0.$$

So by Corollary 3, there is a $z'_J \in H^{4n(J)+1}(E)$ with $Sq^1 z'_J = Sq^2 z_J$. Let

$$f(x_i \otimes y_J) = z_J w_2^{2i-2} t U + z'_J w_2^{2i-2} U.$$

Now

$$Sq^1 f(x_i \otimes y_J) = z_J w_2 U + Sq^1 z'_J U = z_J w_2 U + Sq^2 z_J U = Sq^2 f(x_0 \otimes y_J)$$

and

$$Sq^2 Sq^3 f(x_i \otimes y_J) = w_2^{2i+1} z_J U + (Sq^2 z_J) w_2^{2i} U$$
$$= Sq^1(w_2^{2i} t z_J U + w_2^{2i} z'_J U) = Sq^1 f(x_{i+1} \otimes y_J)$$

and hence $f : M \otimes Y \to H^*(M(\xi))$ is defined.

Let $\mathfrak{a}_1$ be the sub Hopf-algebra of $\mathfrak{a}$ generated by $Sq^0 = 1$, Sq^1 and Sq^2. Define $\hat{M}$ as the quotient of the direct sum $\mathfrak{a}_1/\mathfrak{a}_1 Sq^1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \cdots$ by the relations

$$Sq^2 x_0 = Sq^1 x_1, \quad Sq^2 Sq^3 x_i = Sq^1 x_{i+1}.$$

Then $M = \mathfrak{a} \oplus_{\mathfrak{a}_1} \hat{M}$. Let $\hat{M}^i$ be the elements in $\hat{M}$ of degree i, and $\hat{M}^{(i)}$ the sub $\mathfrak{a}_1$ module of $\hat{M}$ generated by $\hat{M}^j$, $j \leq 1$. Then $\hat{M}^{(0)} \subset \hat{M}^{(1)} \subset \cdots$ defines an increasing filtration on $\hat{M}$, and $M^{(i)} = \mathfrak{a} \otimes_{\mathfrak{a}_1} \hat{M}^{(i)}$ gives an increasing filtration on M.

LEMMA 3. *The inclusion $M^{(0)} \to M$ induces an epimorphism*

$$H(M^{(0)}, Q_i) \to H(M, Q_1).$$

Proof. It is enough to show this for $\hat{M}$. Now

$$\hat{M}^{(0)} \approx \mathfrak{a}_1/\mathfrak{a}_1(Sq^1, Sq^3) \quad \text{and} \quad \hat{M}^{(4i+1)}/\hat{M}^{(4i-3)} \approx \mathfrak{a}_1/\mathfrak{a}_1 Sq^1.$$

In the spectral sequence for $H(\hat{M}, Q_i)$, E^1 is isomorphic to

$$H(\hat{M}^{(0)}, Q_i) \oplus \oplus_{j=1}^{\infty} H(\mathfrak{a}_1/\mathfrak{a}_1 Sq^1, Q_i).$$

In the case $i > 0$, $H(\mathfrak{a}_1/\mathfrak{a}_1 Sq^1, Q_0) \approx Z_2 \oplus Z_2$, given by the classes of the generator x_i and $Sq^2 Sq^3 x_i$. Now $d_1 x_i = Sq^2 Sq^3 x_{i-1}$ if $i > 1$, and $d_1 x_1 = Sq^2 x_0 = Sq^2 x_0 \neq 0$. So only the $H(\hat{M}^{(0)}, Q_0)$ term survives. Since $H(\mathfrak{a}_1/\mathfrak{a}_1 Sq^1, Q_1) = 0$, the result follows.

To conveniently express $H(M, Q_i)$, note that dualization following an application of the cannonical antiautomorphism X of the Steenrod algebra induces an isomorphism of Q_i homology. Let $\xi_i \in \mathfrak{a}^*$ be the usual generator of degrees $2^i - 1$.

LEMMA 4.

$$H(M, Q_0) \approx Z_2[\xi_1^{4k}], \qquad H(M, Q_1) \approx \Lambda[\xi_i^2, i \geq 1].$$

Proof.

$$H(\mathfrak{a}/\mathfrak{a}(Sq^1, Sq^3), Q_0) \approx Z_2[\xi_1^{2k}], \quad H(\mathfrak{a}/\mathfrak{a}(Sq^1, Sq^3), Q_1) \approx \Lambda[\xi_1^2, i \geq 1]$$

by [7]. In the E^1 term of the spectral sequence for $H(M, Q_0)$, the term corresponding to ξ_1^{4j+2} is a boundary, by proof of Lemma 3. For the same reason, the spectral sequence for $H(M, Q_1)$ collapses.

LEMMA 5. *f induces an isomorphism $f_* : H(N, Q_i) \to H(H^*(M(\xi)), Q_i)$ for $i = 1, 2$.*

Proof. This is analogous to the corresponding state for $MSpin^c$ [7, Lemma 1, p. 320]. We can consider $H(H^*(E), Q_0)$ as the free $Z_2[w_{2i}^2, i > 1]$ module on generators

$$u_{2j(1)} \cdots u_{2j(s)}, \quad 1 < j(1) < j(2) < \cdots < j(s).$$

Write $w^2_{2i(1)} \cdots w^2_{2i(k)}$, $u_{2j(1)} \cdots u_{2j(s)}$ as $w^2_I u_S$, where

$$I = (i(1), \cdots, i(k)), \quad S = (j(1), \cdots, j(s)).$$

Partially order the monomials $w^2_I u_S$ by $w^2_I u_S < w^2_{I'} u_{S'}$ if dimension $w^2_I <$ dimension $w^2_{I'}$. Then

$$H(M \otimes Y, Q_0) = H(M) \otimes Y.$$

Let $\alpha_{4k} \in H(M, Q_0)$ correspond to ξ_1^{4k}. Then, exactly as in [7], $f(\xi_1^{4k} \otimes y_J) = w_J u_S \cdot U + \sum_{J'} w_{J'} u_{S'} U$ where $w_{J'} > w_J$ and $S = (j(1), \cdots, j(s)$ is the dyadic expansion of $4k$. Thus f induces an isomorphism on Q_0 homology.

To show f induces an isomorphism on Q_1 homology, write $H(H^*(E), Q_1)$ as the free $Z_2[w^2_{2j}, j > 1]$ module on generators

$$\tilde{u}_{2j(1)-2}\, \tilde{u}_{2j(2)-2} \cdots \tilde{u}_{2j(s)-2}$$

where $1 < j(1) < j(2) < \cdots j(s)$, setting $\tilde{u}_2 = w_2$. If $K = (k_1, k_2, \cdots, k_r)$ is a finite sequence of 0's and 1's, let $\alpha_K \in H(M, Q_1)$ be the homology class corresponding to $\xi_1^{2k_1}\xi_2^{2k_2} \cdots \xi_r^{2k_r}$. Then [7]

$$f_*(\alpha_K \otimes y_J) = w_J u_S U + \textstyle\sum_{J'} w_{J'} u_{S'}$$

where

$$w_{J'} > w_J \quad \text{and} \quad u_S = (\tilde{u}_{2^2-2})^{k_1} \cdot (u_{2^3-2})^{k_2} \cdots (u_{2^{s+1}-2})^{k_s}.$$

Proof of Theorem 3. By a theorem of Peterson [6, Theorem 5.1], since f induces an isomorphism $f_* : H(N, Q_i) \to H(H^*(M(\xi)), Q_i)$ for $i = 0, 1$, Theorem 3 will follow if we verify the following:

Let $x \in N$, degree $x = n$, such that x is not in the submodule of N generated over $\mathfrak{A}$ by terms of degree less than n. Then there is an element $b \in \mathfrak{A}_1$, $b \neq 0$, such that $bx = 0$. But this is trivial, since x must be $\sum \alpha_i x_i \otimes y_{J_i}$ where $\alpha_i \in Z_2$. But $Sq^3 Sq^1$ does the job.

3. $\pi_*(M(\xi))$

We now obtain information on $\pi_*(M(\xi))$ via the Adams Spectral sequence. Since

$$H^*(M(\xi)) \approx M \otimes Y \oplus F,$$

to compute the E_2 term it is sufficient to compute $\mathrm{Ext}_{\mathfrak{A}}(M; Z_2)$. Since $M = \mathfrak{A} \otimes_{\mathfrak{A}_1} \hat{M}$ as an $\mathfrak{A}$ module, $\mathrm{Ext}_{\mathfrak{A}}(M, Z_2) = \mathrm{Ext}_{\mathfrak{A}_1}(\hat{M}, Z_2)$ [4]. Note $\mathrm{Ext}_{\mathfrak{A}_2}(\hat{M}, Z_2)$ is an $\mathrm{Ext}_{\mathfrak{A}_2}(Z_2, Z_2)$ module. Let

$$h_0 \in \mathrm{Ext}^{1,1}_{\mathfrak{A}_1}(Z_2, Z_2) \quad \text{and} \quad h_1 \in \mathrm{Ext}^{1,2}_{\mathfrak{A}_1}(Z_2, Z_2)$$

be the elements coming from the relation $Sq^1 1 = 0$ and $Sq^2 1 = 0$.

THEOREM 4. *For each interger $i > 0$, there are elements*

$$x_i \in \mathrm{Ext}^{0,4i-3}(\hat{M}, Z_2) \quad \text{and} \quad \gamma_i \in \mathrm{Ext}^{2i,4i}(\hat{M}, Z_2)$$

such that the only nonzero elements are h_0^j, $h_0^{k_i} x_i$ and $h_0^j \gamma_i$ where $j \geq$ and $0 \leq$

$k_i < 2i+1$. *There is a relation given by* $h_1\gamma_i = h_0^{2i+1}x_{i+1}$ *and since* $h_0 h_1 = 0$, $h_0^{2i+2}x_{i+1} = 0$.

Proof. Construct a resolution. This is a simple computation.

COROLLARY 4. $\mathrm{Ext}_{\mathfrak{a}}^{s,t}(M; Z_2) = 0$ *unless* $t - s \equiv 0, 1 \bmod 4$.

THEOREM 5. $E_2 = E_\infty$.

Proof. Since in $\mathrm{Ext}_{\mathfrak{a}}^{**}(M \otimes Y; Z_2)$ there are only entries for $t - s \equiv 0$, 1 mod 4, and the elements with $t - s \equiv 0$ mod 4 build infinite towers, the only non-zero differentials can come from the summand $\mathrm{Ext}_{\mathfrak{a}}^{**}(F; Z_2) = \mathrm{Ext}_{\mathfrak{a}}^{0,*}(F, Z_2)$. We compute F to see that this cannot occur.

Let $k : BSpin \to E$ be a lifting of the composition

$$BSpin \to BSO \to BSO \times pt \to BSO \times BZ_2 .$$

Since $k^!(\xi)$ is the universal bundle over $BSpin$, k induces a map

$$k : MSpin \to M(\xi)$$

with $k^*(w_i U) = w_i U$ $(i \neq 2^j + 1)$, and so k^* induces an isomorphism

$$\hat{k} : H^*(M(\xi))/(t) \to H^*(MSpin),$$

where $H^*(M(\xi))/(t)$ is quotient by the submodule generated by (powers of $t) \cdot U$.

From [1] we have

$$H^*(MSpin) = \mathfrak{a}/\mathfrak{a}(Sq^1, Sq^2) \otimes Y' \oplus \mathfrak{a}/\mathfrak{a}Sq^3 \otimes Y'' \oplus F'$$

where Y' and Y'' are the subspaces of Y generated by those $J \in \mathcal{J}$ with $n(J)$ even and $n(J)$ odd, respectively. For $n(J)$ even, we have

$$k^*(M \otimes y_J) = \mathfrak{a}/\mathfrak{a}(Sq^1, Sq^2) \otimes y_J' .$$

For $n(J)$ odd, $1 \in M$, $k^*(1 \otimes y_J) = Sq^2 \otimes y_J''$. (Recall for $n(J)$ odd, $\deg(y_J'') = n(J) - (2)$. Let

$$\hat{l} : (MSpin) \to H^*(M(\xi))$$

be defined by $\hat{l}(w_{i_1} \cdots w_{i_k} U) = w_{i_1} \cdots w_{i_k} U$. If $\alpha_J \in H^{4(n(J))-2}(M(\xi))$ is given by $\hat{l}(1 \otimes y_J'') = \alpha_J$, then α_J is the generator of a free $\mathfrak{a}$ module. So is $t\alpha_J$, and $t^2\alpha_J = w_2\alpha_J$ generates a copy of M, i.e. $w_2\alpha_J = 1 \otimes y_J$. Similarly, we identify all the $t^k\alpha_J$. These cannot support differentials, since they either come from infinite cycles in $E_2^{**}(MSpin)$, or are products of other zero-dimensional elements in E_2. An analogous argument gives the results for elements of the form $t^k\hat{l}(Z_i)$, where Z_i generates a free $\mathfrak{a}$-module in $MSpin$.

Thus one can read off Λ_n. A short table is as follows:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Λ_n	Z	Z_4	0	0	Z	Z_{16}	0	0	$Z \oplus Z$	$Z_{64} \oplus Z_4$	Z_2	Z_2	$Z \oplus Z \oplus Z$	$Z_{256} \oplus Z_{16} \oplus Z_4$

4. Relation to $\Omega^{Spin}(Z_2)$

If $(M, \eta, c) \in \Lambda_n$, then the sphere bundle $S(\eta)$ of η admits a natural free orientation preserving involution, and also a Spin structure, since

$$w_2(S(\eta)) = p^*(w_2(M)) = p^*(\eta(t))^2 = 0.$$

So there is a homomorphism $\Lambda_n \to \Omega_n^{Spin}(Z_2)$, the cobordism group of oriented manifolds, with free, orientation preserving Z_2 action. By forgetting the Z_2 action, and using the natural inclusions, we get a diagram

$$\begin{tikzcd} \Lambda_n \arrow[r, "s"] \arrow[d, "\alpha"'] & \Omega_n^{Spin^c} \\ \Omega_n^{Spin}(Z_2) \arrow[r, "\beta"'] & \Omega_n^{Spin} . \arrow[u, "\gamma"'] \end{tikzcd}$$

Then $2s(x) = \gamma\beta\alpha(x)$ for all a $\epsilon\ \Lambda_n$. Now s maps the integral summands of Λ_{4n} monomorphically, as a look at the map s^* in cohomology shows. Hence α is a monomorphism on Λ_n/torsion.

References

1. D. W. Anderson, E. H. Brown, Jr., and F. P. Peterson, *The structure of the Spin cobordism ring*, Ann. of Math., vol 86 (1967), pp. 271–298.
2. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Springer-Verlag, New York, 1964.
3. R. Lashof, *Some theorems of Browder and Novikov*, Mimeographed Notes, University of Chicago.
4. A. Liulevicius, *Notes on homotopy of Thom spectra*, Amer. J. Math., vol. 86 (1964), pp. 1–16.
5. J. W. Milnor, *Spin structures on manifolds*, Enseignment Math., vol. 9 (1963), pp. 198–203.
6. F. P. Peterson, *Cobordism theory*, Proc. of Symposia in Pure Math., Madison, Wisconsin, 1970.
7. R. E. Stong, *Notes on cobordism theory*, Princeton University Press, Princeton, New Jersey, 1968.

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