INTERSECTION PROPERTIES OF CURVES OF CONSTANT WIDTH

BY

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Abstract. Two curves (boundaries of 2-dimensional disks) of constant width in the plane, whose interiors intersect, must meet in an even or infinite number of components. Some new constant width curves are constructed. Corollaries and partial converses suggest several open questions.

1. Introduction

For the purposes of this paper, a convex curve is the boundary of a compact convex disk with interior in the plane. Curves will be denoted by C_1, C_2 etc. and the disks they bound by D_1, D_2 etc. respectively. If S is any set, conv (S), the convex hull of S, is the minimal convex set containing S. A line m supports the convex set D (or the curve C) if it has a point in common with D and D is contained in one of the two closed halfplanes determined by m. Any convex set possesses two distinct support lines perpendicular to each direction; the distance between these support lines is the *width* of the set in the perpendicular direction. If its width is the same in every direction, a curve C (or the disk D) is said to have *constant width.* Two convex sets *share* a support line m if m supports both sets at a common point and both sets lie in the same halfplane determined by m. Points p and p' *correspond* on a convex curve C if they are on parallel support lines. The *diameter* of a convex curve C is the maximum distance between pairs of points on C. We also speak of a chord of maximal length of C as a *diameter.* Two convex sets *intersect properly* if their interiors intersect and neither is contained in the other.

A curve of constant width (we will take them all to have width w) is *complete* in the sense that the addition of any point increases its diameter. Hence, if C_1 and C_2 are curves of constant width w, and $D_1 \subset D_2$, it follows that $D_1 = D_2$. For, if $p \in D_2 - D_1$, we have

$$\text{diameter } [\text{conv}(D_1 \cup \{p\}] > w,$$

but, any subset of D_2 can have diameter at most w. Each support line to a curve of constant width meets it in a unique point and the chord joining corresponding points is always perpendicular to the support lines in question. The distance between corresponding points is always w and the chord joining them is always a diameter. Moreover, any two points at distance w are the ends of a diameter and must correspond. Two diameters of a constant width curve intersect inside or on the curve, and the latter occurs only at a corner (a point at which there is more than one support line). All the points corresponding to a corner of a curve of constant width w lie on a circle of radius w

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centered at the corner, and, if C contains an arc of a circle of radius w, all points of that arc correspond to the same corner of C. A set D of constant width w is the intersection of all circular disks of radius w centered in D. From this it follows that any arc of a curve of constant width which contains a pair of corresponding points determines the set completely.

Our main purpose is to establish the following curious property of curves of constant width.

THEOREM 1. *If C_1 and C_2 are curves of constant width w and D_1 and D_2 intersect properly, then the number of components of $C_1 \cap C_2$ is even or infinite.*

2. Proof of the theorem

If C_1 and C_2 are curves of constant width, a component I of $C_1 \cap C_2$ (necessarily a single point or an interval) is a *crossing component* if for every $\varepsilon > 0$ the neighborhood $N(I, \varepsilon)$ contains points of both $D_2 - D_1$ and $D_1 - D_2$. Otherwise I is a *non-crossing component.* If C_1 and C_2 intersect properly, we will denote the number of components of $C_1 \cap C_2$ by $\alpha(C_1, C_2)$. To show that, if finite, $\alpha(C_1, C_2)$ is even, we begin with a simple lemma.

LEMMA. *If $p \in I$, a non-crossing component of $C_1 \cap C_2$, C_1 and C_2 share a support line at p.*

Proof. Pick $\varepsilon > 0$, so that the neighborhood $N(I, \varepsilon)$ misses $D_1 - D_2$, and let $x \in \text{int}(D_1) \cap \text{int}(D_2)$. The segment px contains a point

$$y \in N(p, \varepsilon) \cap \text{int}(D_1) \cap \text{int}(D_2).$$

Clearly $N(p, \varepsilon) \cap D_1 \subset D_2$. If m is a support line to D_2 at p which fails to support D_1, there must be a point $q \in D_1$ on the opposite side of m from x. But then the segment yq meets $N(p, \epsilon)$ in a point of $D_1 - D_2$.

The converse of this lemma is not true, for two Reuleaux triangles may cross at a pair of corners and share support lines at each corner.

Proof of the theorem. That the number of crossing components is even, if it is finite, is obvious even for arbitrary convex curves. We need only show that the number of non-crossing components is also even. This we will do by showing that to each non-crossing component I, there corresponds a unique different non-crossing component.

If $p \in I = [x, y]$, where the closed interval I may degenerate to a single point, we let m be a shared support line at p and p' the corresponding point on C_1. Since the distance $d(p, p') = w$, p' is also a corresponding point on C_2. Hence p' is on a component I' of $C_1 \cap C_2$, and $I' = [x', y']$, where x' and y' correspond to x and y respectively. Since all diameters intersect in or on the curve and some neighborhood $N(x, \varepsilon)$ contains points of $D_1 - D_2$ but not of $D_2 - D_1$, it follows that there is a neighborhood $N(x', \varepsilon')$ containing points of $D_2 - D_1$ but no points of $D_1 - D_2$. (If C_1 is "outside" C_2 at x, C_2 is "outside" C_1 at x'.) Moreover, if $I = I'$, it is an arc of both C_1 and C_2 containing a pair of corresponding points. But then I determines each curve completely, and $C_1 = C_2$ in contradiction to our proper intersection. Hence the component I' is crossing if and only if I is crossing, and $I \neq I'$. The non-crossing components can thus be counted in pairs and the number of all components is even.

It is worth noting that, although the result requires it, the pairing of non-crossing components is not dependent upon finiteness of α.

3. Corollaries and converses

COROLLARY 1. *If C is a curve of constant width and C' a congruent copy of C, then $\alpha(C, C')$ is even or infinite.*

COROLLARY 2. *If C is a curve of constant width w and S a circle of diameter w, $\alpha(C, S)$ is even or infinite.*

Each of these obvious statements offers a way of generalizing various properties of circles which appear to be shared by few other curves. No curves, other than those of constant width, are known to enjoy either of these properties.

The direction of generalization to higher dimension is most unclear both because the intersection of round spheres is connected in every dimension but 2 and because the necessary properties of diameters do not generalize in any obviously useful way.

We do know, however, that every even number $2k$ can occur as $\alpha(C, C')$ and $\alpha(C, S)$ for a proper choice of C. In the case where k is odd, C can be taken as a Reuleaux k-gon (we will consider the circle a Reuleaux 1-gon). The position of C' is determined by a 180° rotation about a point on an axis of symmetry and close to but different from the center of the set. The same point may be taken as the center of S. The situations for $k = 3$ are exhibited in Figure 1. For k even, C may be taken as a Reuleaux $(k + 1)$-gon, C' a

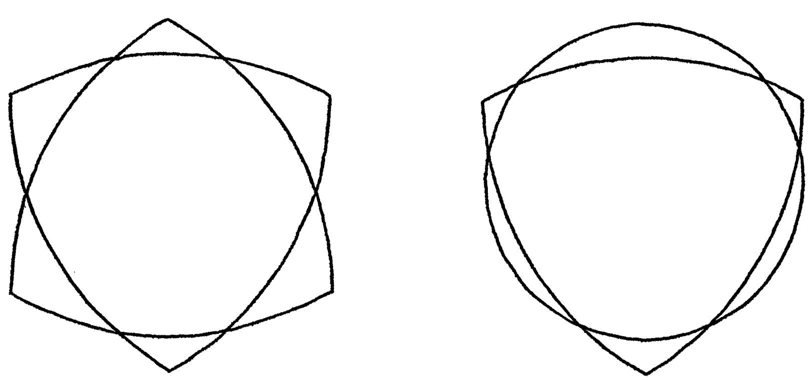

FIGURE 1.

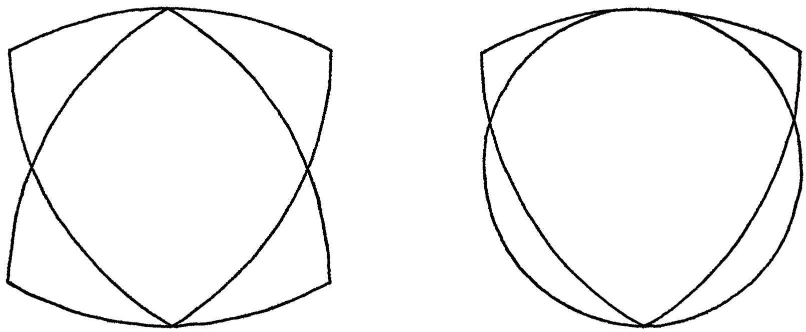

FIGURE 2.

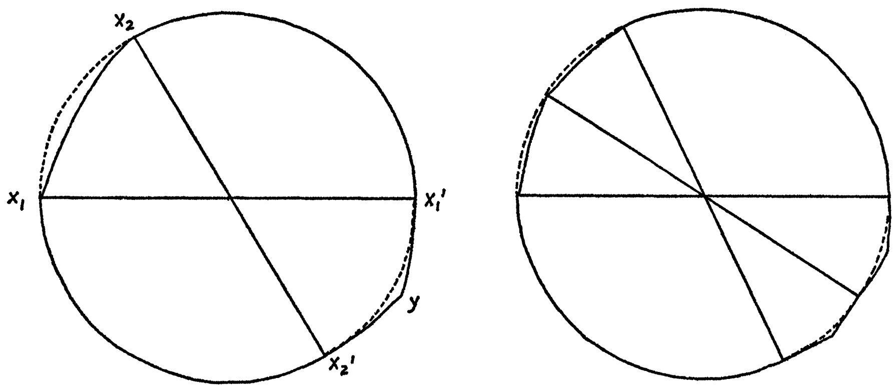

FIGURE 3.

rotation through 180° about the center, and S concentric with C. The situations for $k = 2$ are exhibited in Figure 2.

It is worth noting here that a curve of constant width may have any number of corners, and those for which the number of corners is even provide another class of examples to achieve the results above. To construct these curves, it is necessary only to repeat judiciously the following construction of Sallee [6].

Pick two points x_1 and x_2 on S, a circle centered at 0 of diameter w (The circle is not crucial here. Any constant width curve would work, but we need here only the simplest situation.) and such that the angle $x_1 0 x_2 \leq \pi/3$. Construct a circular arc of radius w centered at x_1, passing through the corresponding point x_1' and lying on the opposite side of the line $x_1 x_1'$ from x_2. Perform the corresponding construction with x_2 and let the arcs determined this way intersect at y. With y as center, construct a circular arc of radius w joining x_1 and x_2 and lying inside the circle S. The new curve S_1 is formed by replacing the smaller arc $x_1 x_2$ on S by this new arc and the smaller arc $x_1' x_2'$ on S by the small arcs $x_1' y$ and $y x_2'$ (see Figure 3.) It is easy to see that S_1 has constant width and corners at each of the x_i's.

Choosing small intervals $x_1 x_2$ and $x_2 x_3$, the construction of S_1 will not disturb $x_2 x_3$, so that we may, using it again, construct a new curve S_2 with precisely 5 corners (see Figure 3). Clearly any odd number is possible. Now if $2k \geq 6$, we may take small disjoint intervals on S and reiterate the construction so that one interval contributes 3 corners and the other $2k - 3$ to the final curve.

To construct curves with 1 or 2 corners is somewhat more difficult, but can be accomplished by the addition of curves with more corners, since the sum curve has a corner on a support line in the direction ϑ if and only if each of the summands has a corner on the support line in the corresponding direction and on the corresponding side. Clearly a Reuleaux triangle and an "isosceles triangle" with a very small base (the curve S_1 is one) can be arranged (let the axes of symmetry coincide) so that precisely two corners correspond. The sum curve will have one and only one corner. The same trick can be used to construct a curve with precisely two corners. A much simpler construction for curves with precisely four corners will follow immediately from Theorem 2.

Clearly, the fundamental construction could be applied on a sequence of disjoint intervals converging to a point x_0 on S to produce a curve of constant width S_∞ with infinitely many corners and such that each of $\alpha(S_\infty, S'_\infty)$ and $\alpha(S_\infty, S)$ can be made infinite. See Figure 4.

Since the number of corners of any convex curve is always countable, and since each corner corresponds to an arc of a circle, the worst possible set of corners of a constant width curve C would be dense in an arc on C whose endpoints correspond. It has apparently not been noted that this situation can indeed occur. To construct the set we begin with a circle S centered at the

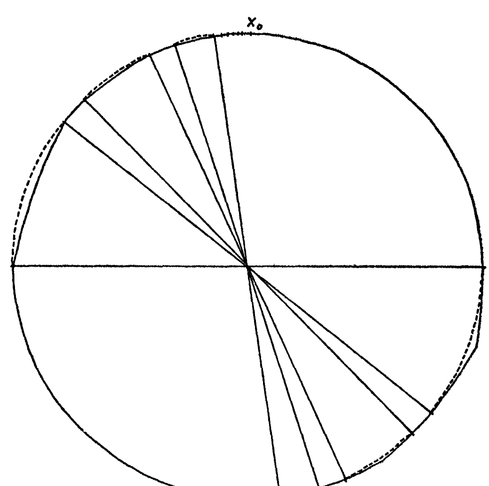

FIGURE 4.

origin and an interval $x_0 x_1$ on S so that x_0 is the point with polar coordinates $(w/2, 0)$ and the angle $x_0 0 x_1 = \varphi \leq \pi/3$. We will construct a curve C with a corner on each of the lines $\vartheta = \varphi m$, $0 \leq m \leq 1$, m a dyadic fraction. This curve C has corners at a dense subset of its arc $[x_0 x_1]$, and, if $\varphi = \pi/3$, x_0 and x_1 correspond.

In step 1, we construct a new curve S_1, replacing the arc $x_0 x_1$ on S by a new arc A_1 lying inside S, using of course the same fundamental construction. A third circular arc is constructed between A_1 and S and also lying on x_0 and x_1. Halfway between this arc and A_1 on the line $\vartheta = \frac{1}{2}\varphi$, we pick the point $x_{1/2}$. With $x_{1/2}$ as center we construct a circle of radius w which cuts the arcs $x_0' y$ and $y x_1'$, added in the construction of S_1, in points y_0 and y_1 respectively.

In step 2, we modify S_1 by the fundamental construction, adding the new arc $y_0 y_1$ and arcs of circles centered at y_0 (and y_1) and passing through x_0 and $x_{1/2}$ (and $x_{1/2}$ and x_1). Clearly the new curve S_2 has corners at x_0, $x_{1/2}$, and x_1. See Figure 5.

There is a circle on these three points. Between it and the arc $x_0 x_{1/2}$ we construct a circular arc on x_0 and $x_{1/2}$. Halfway between this arc and S_2 we pick the point $x_{1/4}$ on the line $\vartheta = \frac{1}{4}\varphi$. Similarly, we find the point $x_{3/4}$, and, applying the fundamental construction twice more, produce a new curve S_3 with corners at x_0, $x_{1/4}$, $x_{1/2}$, $x_{3/4}$, x_1.

The limit curve of the sequence $\{S_n\}$ has the desired property since there

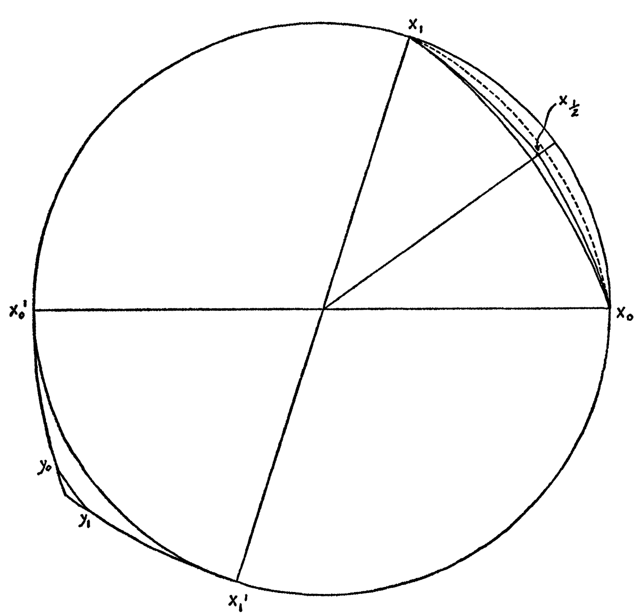

FIGURE 5.

are distinct support lines to each corner of each curve in the sequence which remain support lines to each succeeding curve.

Given two intersecting curves of constant width, another construction leads to the following:

THEOREM 2. *If C_1 and C_2 are curves of constant width w and $\alpha(C_1, C_2) = n \geq 4$, there is a curve C_3 of constant width w such that*

$$D_1 \cap D_2 \subset D_3, \quad \alpha(C_1, C_3) = n - 2 \quad \text{and} \quad \alpha(C_2, C_3) = 2.$$

Proof. First fix an endpoint x_1 of a component I_1 of $C_1 \cap C_2$ and, in the obvious fashion, order the points of C_1 by arc length measured in the counterclockwise direction. We order the points of C_2 similarly; the orderings coincide on $C_1 \cap C_2$. We choose x_1 so that I_1 is a closed (possibly degenerate) interval $[x_1, y_1]$, $x_1 \leq y_1$. Proceeding around C_1 in the counterclockwise direction, we let $I_2 = [x_2, y_2]$ be the component adjacent to I_1, so that $x_1 \leq y_1 < x_2 \leq y_2$.

Each of the intervals I_1 and I_2 contains no pair of corresponding points, but, more importantly, we can choose them so that the entire interval $[x_1, y_2]$ contains no such pair from either curve. For, if I_1 and I_2 contain a pair of corresponding points, we take the next component I_3 and consider it with I_2. If these two contain a pair of corresponding points, there is a point of I_2 to which correspond points of both I_1 and I_3. This is impossible since these two components are disjoint. Hence, if I_1 and I_2 contain such a pair of points, I_2 and I_3 do not, and we can choose them to start our construction. We assume now that I_1 and I_2 do not contain a pair of corresponding points.

If $p \in C_1 \cap C_2$, we will denote a point corresponding to p on C_1 by p' and write $p \sim p'$. Similarly a point corresponding to p on C_2 is p'' and we write $p \sim p''$. We assume without loss of generality, that C_2 lies outside of C_1 between y_1 and x_2 (i.e. if $z \in C_1$, $y_1 < z < x_2$, there is a neighborhood of z contained in D_2.) Since there may be more than one point $y_1' \sim y_1$, we pick

$$y_1^{**} = \max\{y_1'' \mid y_1'' \sim y_1\}, \qquad y_1^{*} = \min\{y_1' \mid y_1' \sim y_1\},$$
$$x_2^{*} = \max\{x_2' \mid x_2' \sim x_2\}, \qquad x_2^{**} = \min\{x_2'' \mid x_2'' \sim x_2\}.$$

With y_1 as center we construct a circular arc A_1 on y_1^{**} and y_1^{*}. With x_2 as center we construct a circular arc A_2 on x_2^{*} and x_2^{**}. The curve C_3 consists of the arcs

$$y_1 x_2 \text{ on } C_1, \quad x_2 y_1^{**} \text{ on } C_2, \ A_1, \quad y_1^{*} x_2^{*} \text{ on } C_1, \ A_2, \quad x_2^{**} y_1 \text{ on } C_2.$$

(Note that if, say, $y_1^{**} \in C_1 \cap C_2$, we may have $y_1^{*} < y_1^{**}$ on C_1. To avoid any difficulty here we will simply adopt the convention that arcs swept out twice in opposite directions are disregarded. See Figure 6.) From this construction it is clear that C_3 is a curve of constant width w and that $D_1 \cap D_2 \subset D_3$.

Since I_1 and I_2 are adjacent and C_1 lies inside C_2 between them, and since

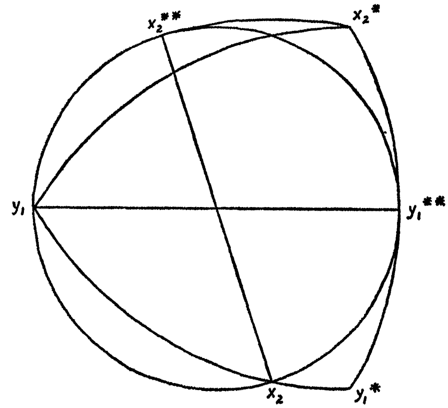

FIGURE 6.

C_3 and C_1 coincide between y_1 and x_2, C_3 lies inside C_2 on that interval. By construction C_3 lies outside C_2 between y_1^{**} and x_2^{**}. Since C_3 coincides with C_2 elsewhere, the only components of $C_2 \cap C_3$ are the intervals $[x_2, y_1^{**}]$ and $[x_2^{**}, y_1]$. Hence $\alpha(C_2, C_3) = 2$.

We have left only the computation of $\alpha(C_1, C_3)$. To do this note that C_3 differs from C_2 only on the intervals $[y_1, x_2]$ and $[y_1^{**}, x_2^{**}]$. Noting first that two components I_1 and I_2 have become one in $C_1 \cap C_3$ by the addition of $[y_1, x_2]$, we distinguish three cases.

Case 1. $y_1^{**} \neq y_1^*$; $x_2^* \neq x_2^{**}$. In this case, the angles $y_1^{**}y_1y_1^*$ and $x_2^*x_2x_2^{**}$ each contain one component of $C_1 \cap C_2$. These are lost in the construction, but $[y_1^*, x_2^*]$ is added so that $C_1 \cap C_3$ has precisely two fewer components than $C_1 \cap C_2$.

Case 2. $y_1^{**} \neq y_1^*$; $x_2^* = x_2^{**}$ (or vice versa). Now only one component, that in the angle $y_1^{**}y_1y_1^*$, is lost, but, since $x_2^* \in C_1 \cap C_2$, the component $[y_1^*, x_2^*]$ is not new. Again the number of components is reduced by precisely two.

Case 3. $y_1^{**} = y_1^*$; $x_2^* = x_2^{**}$. In this instance, the addition of $[y_1^*, x_2^*]$ to construct C_3 connects two existing components of $C_1 \cap C_2$, reducing the number of components by one and making the total reduction precisely two.

Thus in any case $\alpha(C_1, C_3) = n - 2$. This completes the proof.

Figure 7 illustrates the construction in the case where C_1 is a circle, C_2 a Reuleaux triangle, and $n = 6$. In this case the intervals I_1 and I_2 are degenerate. It is interesting to note this method incidentally produces an easy ruler and compass (actually compass alone) construction of a curve of constant width with precisely four corners.

Finally, we wish to note that converses to the two corollaries are available for two special classes of curves.

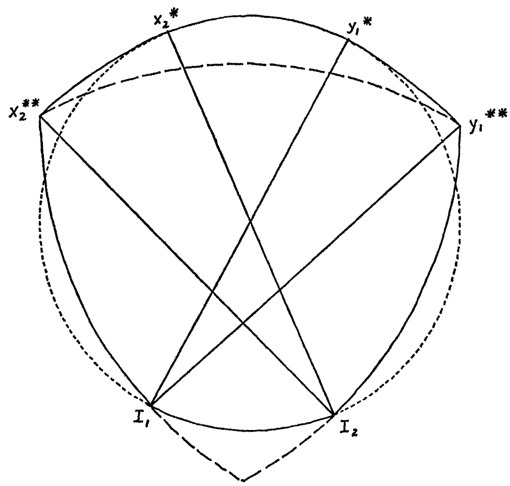

FIGURE 7.

THEOREM 3. *If C is a curve with two unequal chords of lateral symmetry (i.e. the reflection of C in the chord is C), then there is a congruent copy C' of C for which $\alpha(C, C')$ is odd.*

Proof. We need only place the smaller chord on the larger one so that a pair of end points coincide. Except for this point, the other components of $C \cap C'$ are paired by the double symmetry, so that the total number is odd.

THEOREM 4. *If P is a convex polygon there is a congruent copy P' of P such that $\alpha(P, P')$ is odd.*

Proof. Place a vertex v of Q, a congruent copy of P, at the midpoint of a side m of P so that the two sides of Q adjacent to v lie on the same side of m. The polygon Q may be rotated about v to a position Q' such that no side of Q' is parallel to a side of P and the sides of Q' adjacent to v are still on the same side of m. A slight translation of Q' parallel to m, will move Q' to a position P' such that no vertex of P', other than v, is on P. Hence the only components of $P \cap P'$ are crossing components, the number of which is necessarily even, and $\{v\}$. Note that if P is an infinite convex polygon the theorem still holds in the sense that, although possibly infinite, $P \cap P'$ consists of one vertex plus a set of points which can be counted in pairs. This completes the proof.

Whether the curves of constant width are the only curves with the properties of Corollaries 1 and 2 is an open question. An affirmative answer would generalize results of Fujiwara [1], Kojima [3], Kubota [4], and Hombu [2], to the effect that the circle is the only convex curve which coincides with itself as soon as three points coincide.

The referee has kindly pointed out that material related especially to the properties in Section 1 and to Theorem 1 appears in a paper of E. Meissner, Vierteljahrsschrift Naturforsch. Ges. Zurich 56, 1911.

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