NOTE ON A CRITERION OF SCHEERER

BY

PETER HILTON, GUIDO MISLIN, AND JOSEPH ROITBERG

1. Introduction

In [4] Scheerer considered principal G-bundles over S^n

$$G \rightarrow E_\alpha \xrightarrow{f_\alpha} S^n, \tag{1.1}$$

classified by $\alpha \in \pi_{n-1}(G)$, and proved the following theorem.

THEOREM 1.1. *Suppose the diagram*

$$\begin{array}{ccc} S^{n-1} \times G & \xrightarrow{\mu(\alpha \times 1)} & G \\ \Big\downarrow {\scriptstyle 1 \times k} & & \Big\downarrow {\scriptstyle k} \\ S^{n-1} \times G & \xrightarrow{\mu(k\alpha \times 1)} & G \end{array} \tag{1.2}$$

is (homotopy) commutative, where $k : G \rightarrow G$ is the k^{th} power map and $\mu : G \times G \rightarrow G$ is the multiplication. Then

$$k\alpha_0 \circ f_\alpha = 0 \tag{1.3}$$

where $\alpha_0 : S^n \rightarrow B_G$ is adjoint to α.

Now consider the pull-back diagram

$$\begin{array}{ccc} \bar{E} & \longrightarrow & E_{k\alpha} \\ \Big\downarrow & & \Big\downarrow {\scriptstyle f_{k\alpha}} \\ E_\alpha & \xrightarrow{f_\alpha} & S_n \end{array} \tag{1.4}$$

Then, of course, (1.3) guarantees that

$$\bar{E} = E_\alpha \times G, \tag{1.5}$$

so that Theorem 1.1 is highly relevant to the study of non-cancellation phenomena[1] in [1], [2], [3], [4]. Indeed in [2] it is shown that the hypothesis of Theorem 1.1 above is equivalent, in the case $G = S^3$, to the key condition

$$\tfrac{1}{2} k(k-1)\omega \circ \Sigma^3\alpha = 0 \in \pi_{n+2}(S^3) \tag{1.6}$$

Received February 3, 1972.

[1] A different, but related, approach to non-cancellation phenomena is due to A. Sieradski.

appearing in Theorem 2.4 of [3], where $\omega \in \pi_6(S^3)$ is the Blakers-Massey element generating $\pi_6(S^3)$.

In this note we give a different proof of Theorem 1.1, which leads to a new insight into the nature of the relation (1.3); we are also able to generalize the theorem. Roughly speaking, we show that, in the presence of the homotopy commutative square (1.2), we may replace a k-fold 'rolling' of the base S^n by a k-fold 'rolling' of the fibre in the opposite direction. More precisely, instead of the diagram

$$(1.7) \qquad \begin{array}{ccc} G & = & G \\ \downarrow & & \downarrow \\ E_{k\alpha} & \longrightarrow & E_\alpha \\ {\scriptstyle f_{k\alpha}}\downarrow & & \downarrow{\scriptstyle f_\alpha} \\ S^n & \xrightarrow{k} & S^n \end{array}$$

we may construct a diagram

$$(1.8) \qquad \begin{array}{ccc} G & \xrightarrow{k} & G \\ \downarrow & & \downarrow \\ E_\alpha & \longrightarrow & E_{k\alpha} \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f_{k\alpha}} \\ S^n & = & S^n \end{array}$$

It is, of course, trivial to deduce (1.3) from the diagram (1.8). Thus our aim in this note is to show that the Scheerer condition (1.2) guarantees the existence of a diagram (1.8). Our argument rests on a technical lemma whose statement and proof occupy Section 2. In Section 3 we state our main theorem, which is a corollary of the lemma, and offer some commentary.

The authors are indebted to Paul Olum for very helpful discussions in connection with the technical lemma.

2. A technical lemma

We work in the category T_0 of based spaces and based maps; homotopies will also be supposed to be based. We have cofibrations in this category and we represent a cofibration by

$$a \rightarrowtail c.$$

We also have push-outs in this category and the following proposition is well known.

PROPOSITION 2.1. *If*

$$\begin{array}{ccc} a & \xrightarrow{\phi} & b \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ c & \xrightarrow{\psi} & d \end{array}$$

is a push-out in T_0 and if μ is a cofibration, so is ν.

Moreover, every morphism $\phi : a \to b$ factorizes canonically, *via* the (based) mapping cylinder, as

$$a \xrightarrow{\bar{\phi}} \bar{b} \xrightarrow{\rho} b,$$

where $\bar{\phi}$ is a cofibration and ρ is a homotopy equivalence. Now suppose given a push-out square

$$\begin{array}{ccc} a & \xrightarrow{\phi} & b \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ c & \xrightarrow{\psi} & d \end{array} \quad . \tag{2.1}$$

We may then construct the diagram

$$\begin{array}{ccccc} a & \xrightarrow{\bar{\phi}} & \bar{b} & \xrightarrow{\rho} & b \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\bar{\nu}} & & \downarrow{\scriptstyle\nu} \\ c & \xrightarrow{\bar{\psi}} & \bar{d} & \xrightarrow{\sigma} & d \end{array} \quad , \tag{2.2}$$

factorizing (2.1), where the top row is the canonical factorization of ϕ and the left-hand square is again a push-out, so that σ is determined by the push-out property. Again we quote a well-known proposition.

PROPOSITION 2.2. *In the diagram* (2.2), *σ is a homotopy equivalence.*

We note that it is very easy to see that σ is a homology equivalence. However, with the help of the standard homotopy inverse, $\tilde{\rho}$, to ρ, one may construct an explicit homotopy inverse, $\tilde{\sigma}$, to σ.

With this preparation we announce the technical lemma. We consider the diagram (2.1) augmented by a map $\theta : d \to e$ such that $\theta\nu = 0$,

$$\begin{array}{ccc} a & \xrightarrow{\phi} & b \\ \mu \downarrow & & \downarrow \nu \\ c & \xrightarrow{\psi} & d \\ & & \searrow \theta \\ & & e \end{array} \qquad \theta\nu = 0. \tag{2.3}$$

Further, we consider a commutative diagram

$$\begin{array}{ccc} a' & \xrightarrow{\phi'} & b' \\ \mu' \downarrow & & \downarrow \nu' \\ c' & \xrightarrow{\psi'} & d' \\ & & \searrow \theta' \\ & & e' \end{array} \qquad \theta'\nu' = 0. \tag{2.4}$$

We suppose given maps

$$\alpha : a \to a', \quad \beta : b \to b', \quad \gamma : c \to c', \quad \varepsilon : e \to e',$$

such that

$$\beta\phi \simeq \phi'\alpha, \quad \gamma\mu = \mu'\alpha, \quad \varepsilon\theta\psi = \theta'\psi'\gamma'. \tag{2.5}$$

LEMMA 2.3. *Under these circumstances, there exists $\delta : d \to d'$ such that*

$$\delta\nu \simeq \nu'\beta, \quad \varepsilon\theta \simeq \theta'\delta.$$

Proof. Using (2.2), replace (2.3) by

$$\begin{array}{ccc} a & \xrightarrow{\bar\phi} & \bar b \\ \mu \downarrow & & \downarrow \bar\nu \\ c & \xrightarrow{\psi} & \bar d \\ & & \searrow \bar\theta \\ & & e \end{array} \tag{2.6}$$

where $\bar\theta = \theta\sigma$, and set $\bar\beta = \beta\rho : \bar b \to b'$. Then $\bar\theta\bar\nu = 0$ and

(2.7) $$\bar\beta\bar\phi = \beta\phi \simeq \phi'\alpha, \quad \gamma\mu = \mu'\alpha, \quad \varepsilon\bar\theta\bar\psi = \varepsilon\theta\psi = \theta'\psi'\gamma.$$

Since $\bar\phi$ is a cofibration, $\bar\beta \simeq \bar{\bar\beta}$ with $\bar{\bar\beta}\bar\phi = \phi'\alpha$. Since the square in (2.6) is a push-out, there exists $\bar\delta : \bar d \to d'$, characterized by

(2.8) $$\bar\delta\bar\nu = \nu'\bar{\bar\beta}, \quad \bar\delta\bar\psi = \psi'\gamma.$$

We claim that

(2.9) $$\theta'\bar\delta = \varepsilon\bar\theta.$$

To prove this, it is sufficient to show that

$$\theta'\nu'\bar{\bar\beta} = \varepsilon\bar\theta\bar\nu, \quad \theta'\psi'\gamma = \varepsilon\bar\theta\bar\psi,$$

(see (2.8)). But the first relation in (2.10) follows since each side is zero, and the second is part of (2.7). Thus (2.9) holds. Now set $\delta = \bar\delta\tilde\sigma$, where $\tilde\sigma$ is a homotopy inverse to σ. Then, with $\tilde\rho$ homotopy inverse to ρ,

$$\delta\nu = \bar\delta\tilde\sigma\nu \simeq \bar\delta\bar\nu\tilde\rho = \nu'\bar{\bar\beta}\tilde\rho \simeq \nu'\bar\beta\tilde\rho \simeq \nu'\beta, \quad \theta'\delta = \theta'\bar\delta\tilde\sigma = \varepsilon\bar\theta\tilde\sigma \simeq \varepsilon\theta.$$

This completes the proof of the lemma.

Remark. The lemma clearly admits a dual formulation, in the sense of Eckmann-Hilton.

3. The main theorem

Let

$$F \xrightarrow{i} E \xrightarrow{f} \Sigma A$$

be a G-bundle classified by $u : A \to G$ and let

$$\tau : G \times F \to F$$

be the action of G on F. Similarly let

$$F' \xrightarrow{i'} E' \xrightarrow{f'} \Sigma A'$$

be a G'-bundle classified by $u' : A' \to G'$ with action

$$\tau' : G' \times F' \to F'.$$

Let $\kappa : F \to F'$, $\lambda : A \to A'$ be maps.

THEOREM 3.1. *If $\kappa\tau(u \times 1) \simeq \tau'(u'\lambda \times \kappa) : A \times F \to F'$, there is a map $\delta : E \to E'$ such that the diagram*

$$\begin{array}{ccc} F & \xrightarrow{\kappa} & F' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ E & \xrightarrow{\delta} & E' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \Sigma A & \xrightarrow{\Sigma\lambda} & \Sigma A' \end{array}$$

homotopy commutes.

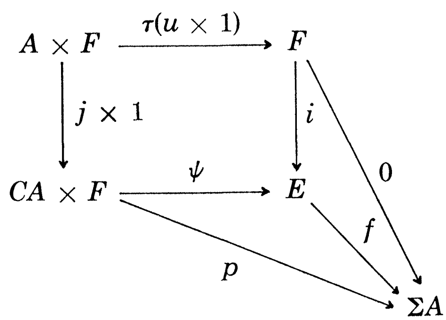
Proof. Consider the maps

$$\tau(u \times 1) : A \times F \to F, \quad j \times 1 : A \times F \to CA \times F.$$

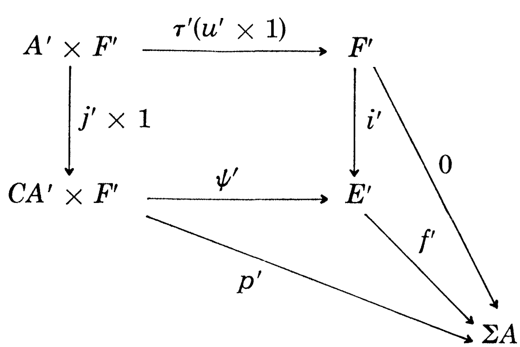
Then E is the push-out of these two maps, thus

$$\begin{array}{ccc} A \times F & \xrightarrow{\tau(u \times 1)} & F \\ \Big\downarrow{\scriptstyle j \times 1} & & \Big\downarrow{\scriptstyle i} \\ CA \times F & \xrightarrow{\psi} & E \end{array} .$$

Moreover the fibre projection $f : E \to \Sigma A$ is characterized by the conditions $fi = 0, f\psi = p$, where $p : CA \times F \to \Sigma A$ first projects onto CA and then passes to the quotient $CA/A = \Sigma A$. We also remark that $j \times 1$ is a cofibration. Thus we have



and similarly,



We are thus in a position to apply the technical lemma of Section 2, with $\alpha = \lambda \times \kappa$, $\beta = \kappa$, $\delta = C\lambda \times \kappa$, $\varepsilon = \Sigma\lambda$. The first condition in (2.5) is then precisely the hypothesis of the theorem, and the other two conditions are automatically satisfied. Thus we find $\delta : E \to E'$ with $\delta i \simeq i'\kappa$, $f'\delta \simeq (\Sigma\lambda)f$.

Remark. With the fibration $F \to E \to \Sigma A$ is associated a map $v : A \to F$ in the usual way; in fact, $v = \tau \iota_1 u$, where ι_1 embeds G in $G \times F$. Similarly we associate $v' : A' \to F'$ with $F' \to E' \to \Sigma A'$. Then we note that the hypothesis of Theorem 3.1 implies the homotopy commutativity of the square

$$(3.1)\qquad \begin{array}{ccc} A & \xrightarrow{\lambda} & A' \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v'} \\ F & \xrightarrow{\kappa} & F' \end{array} .$$

Let us consider, in particular, principal bundles over ΣA, with $\lambda = 1$. Thus $F = G$, $F' = G'$, $\tau = \mu$, the multiplication on G, $\tau' = \mu'$, the multiplication on G'. Then we have the following corollary.

COROLLARY 3.2. *Suppose given principal bundles*

$$G \xrightarrow{i} E \xrightarrow{f} \Sigma A, \quad G' \xrightarrow{i'} E' \xrightarrow{f'} \Sigma A,$$

classified by $u : A \to G$, $u' : A \to G'$ *respectively, and a map* $\kappa : G \to G'$. *If*

$$\kappa\mu(u \times 1) \simeq \mu'(u' \times \kappa),$$

there is a map $\delta : E \to E'$ *such that the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\kappa} & G' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i'} \\ E & \xrightarrow{\delta} & E' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \Sigma A & = & \Sigma A \end{array}$$

homotopy commutes. Consequently $u_0' f \simeq 0$, *where* $u_0' : \Sigma A \to B_G$ *is adjoint to* u'.

PROPOSITION 3.3. *If the hypothesis of Corollary 3.2 holds, then* $\kappa u \simeq u'$. *Conversely, if* $\kappa u \simeq u'$ *and* $\kappa : G \to G'$ *is an* H*-map, then the hypothesis of Corollary 3.2 holds.*

Proof. The first statement is a special case of the Remark following the proof of Theorem 3.1 (see (3.1)). Conversely, consider the diagram

$$\begin{array}{ccc} A \times G & \xrightarrow{1 \times \kappa} & A \times G' \\ \Big\downarrow u \times 1 & & \Big\downarrow u' \times 1 \\ G \times G & \xrightarrow{\kappa \times \kappa} & G' \times G' \\ \Big\downarrow \mu & & \Big\downarrow \mu' \\ G & \xrightarrow{\kappa} & G' \end{array}$$

The homotopy-commutativity of the top square is the condition $\kappa u \simeq u'$, and that of the bottom square is the condition that κ be an H-map. The homotopy commutativity of the composite square is the hypothesis of Corollary 3.2.

Now the hypothesis of Theorem 1.1 is just the hypothesis of Corollary 3.2 in the case $A = S^{n-1}$, $G = G'$, $\kappa = k^{\text{th}}$ power map, $u = \alpha$, $u' = ku = k\alpha$. (Note that the hypothesis of Corollary 3.2 implies, in this case, $u' = ku$.) Thus the conclusion we wish to draw—the existence of (1.8) and hence the truth of (1.3)—is a special case of Corollary 3.2. Proposition 3.3 is relevant in drawing explicit attention to the fact that (1.3) holds if $k : G \to G$ is an H-map (as observed by Scheerer).

Remark 1. Since, if $G = S^3$, the Scheerer condition is equivalent to (1.6), we have, in that case, a complete grasp of the values of k for which the Scheerer condition holds, modulo knowledge of homotopy groups of spheres. Thus, in particular, if we take $n = 7$, $\alpha = m\omega$, $0 \leq m \leq 6$, we find that the appropriate values of k are the following

$$\text{(3.3)} \qquad \begin{array}{ll} k \text{ arbitrary} & \text{if } m \equiv 0 \bmod 3 \\ k \equiv 0 \text{ or } 1 \bmod 3 & \text{if } m \not\equiv 0 \bmod 3. \end{array}$$

Thus, for the values of k listed in (3.3), we have maps

$$\text{(3.4)} \qquad \begin{array}{ccc} S^3 & \xrightarrow{k} & S^3 \\ \Big\downarrow & & \Big\downarrow \\ E_{m\omega} & \longrightarrow & E_{km\omega} \\ \Big\downarrow & & \Big\downarrow \\ S^7 & = & S^7 \end{array}$$

in addition to the maps

$$\text{(3.5)}\qquad \begin{array}{ccc} S^3 & = & S^3 \\ \downarrow & & \downarrow \\ E_{km\omega} & \longrightarrow & E_{m\omega} \\ \downarrow & & \downarrow \\ S^7 & \xrightarrow{k} & S^7 \end{array}$$

which we have for all values of m, k. It is possible that the maps (3.4), perhaps in conjunction with the maps (3.5), could yield further information, or a new insight into existing information, on the manifolds $E_{m\omega}$. For example, one gets immediate information on the attaching map for the top cell. A similar program is in principle possible for the total spaces of arbitrary sphere bundles over spheres.

Remark 2. Corollary 3.2 shows that, under the given hypothesis, the principal bundles $G \to E \to \Sigma A$, $G' \to E' \to \Sigma A$ are p-fibre-homotopy-equivalent for any prime p for which κ_p is a homotopy equivalence (assuming p-fibre homotopy-equivalence makes sense for these bundles). Thus if we suppose A connected and assume $u : A \to G$ to be of finite order n; and if we further suppose that $G = G'$ is 1-connected and $\kappa_p : G_p \to G_p$ belongs to $H(G_p)$ if $p \mid n$, then the hypothesis of Corollary 3.2 implies that the bundles belong to the same genus in the sense of [2] and hence, by [1, Theorem 3.4], $E \times G \cong E' \times G$. The same result would follow directly from arguments in [2], but here we have the additional information that there is a map $\delta : E \to E'$ yielding a localized equivalence δ_p for $p \mid n$.

BIBLIOGRAPHY

1. PETER HILTON, GUIDO MISLIN AND JOSEPH ROITBERG, *Sphere bundles over spheres and non-cancellation phenomena*, J. Lond. Math. Soc., vol. 6 (1972), pp. 15–23; see also Lecture notes in mathematics, vol. 249, Springer (1971), pp. 34–46.
2. ———, *H-spaces of rank two and non-cancellation phenomena*, Inventiones Math., vol. 16 (1972),
3. PETER HILTON AND JOSEPH ROITBERG, *On principal S^3-bundles over spheres*, Ann. of Math., vol. 90 (1969), pp. 91–107.
4. H. SCHEERER, *On principal bundles over spheres*, Indag. Math., vol. 32 (1970), pp. 353–355.