$\mathfrak{F}$-SPEED, $\mathfrak{F}$-ABNORMAL DEPTH AND $(R, \mathfrak{F})$-CHAINS IN CERTAIN LOCALLY FINITE GROUPS

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1. Introduction

This paper is essentially a continuation of [5] to which we refer for most of our notation and terminology. In [5] we extended the theory of $\mathfrak{F}$-reducers to any QS-closed subclass $\mathfrak{R}$ of the class $\mathfrak{U}$ introduced in [2] and studied further in [6] and [7]. In this paper we consider certain invariants of $\mathfrak{R}$-groups and their subgroups which arise naturally from the study of $\mathfrak{F}$-reducers.

As in [5], $\mathfrak{R}$ will denote an arbitrary QS-closed subclass of $\mathfrak{U}$ and $\mathfrak{F} = \mathfrak{F}(\mathfrak{f})$ the saturated $\mathfrak{R}$-formation defined by the $\mathfrak{R}$-preformation function $\mathfrak{f}$ on the set of primes π. Furthermore, $\mathfrak{f}$ is assumed to be R_0-closed, i.e.,

$$\mathfrak{R} \cap R_0 \mathfrak{f}(p) = \mathfrak{f}(p) \quad \text{for all } p \in \pi. \tag{1.1}$$

It will be convenient to make two further assumptions which were not made in [5], namely

$$\pi \text{ is the set of all primes,} \tag{1.2}$$

and

$$\mathfrak{f}(p) = S\mathfrak{f}(p) \quad \text{for all primes} \quad p. \tag{1.3}$$

The majority of our results seem to require the presence of both these conditions though a few do hold without either and some with the presence of only one.

We shall also assume that all groups belong to the class $\mathfrak{U}$, unless the contrary is explicitly stated.

In section two we define two "convergence processes", similar to those given in Sections 3, 4 of [3], from the second of which we obtain the first of our invariants—the $\mathfrak{F}$-speed of a $\mathfrak{R}$-group. The convergence processes give ways of constructing an $\mathfrak{F}$-projector of an arbitrary $\mathfrak{R}$-group G as the limiting term of a "converging" series of subgroups of G. The first method is somewhat unsatisfactory in that at each stage one has to "construct" an $\mathfrak{F}$-projector of some subgroup of G. The second approach, which consists of successively taking $\mathfrak{F}$-normalizers and $\mathfrak{F}$-reducers, overcomes the previous objection but is, more often than not, too cumbersome for the actual computation of $\mathfrak{F}$-projectors. The processes we shall describe generalize not only

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those in [3] but also similar constructions for Carter subgroups of finite soluble groups due to Carter [1], Fischer (unpublished), Mann [10] and Rose [12]. The $\mathfrak{F}$-speed of a $\mathfrak{K}$-group G is, roughly speaking, the number of steps that have to be taken before the second convergence process becomes stationary, and is denoted by $i_{\mathfrak{F}}(G)$. The $\mathfrak{F}$-speed of G is easily seen to be an invariant of G, and is always finite even though the groups we consider may well be infinite. For each non-negative integer r the class $r(\mathfrak{K}, \mathfrak{F})$, of $\mathfrak{K}$-groups with $\mathfrak{F}$-speed at most r, is a $\mathfrak{K}$-formation (Theorem 2.9) containing the class $\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2r}\mathfrak{F}$. We have been unable to decide whether $r(\mathfrak{K}, \mathfrak{F})$ is a saturated $\mathfrak{K}$-formation (except in trivial cases) though we do have an example (2.13) to show that it need not be subgroup-closed.

Suppose $H \leq G \in \mathfrak{K}$. We say that a chain $(\Lambda_\sigma, V_\sigma ; \sigma \in \Omega)$ from H to G (cf. [§4, 5]) is *$\mathfrak{F}$-balanced* if V_σ is either $\mathfrak{F}$-abnormal or $\mathfrak{F}$-serial in Λ_σ, for each $\sigma \in \Omega$. Now a maximal subgroup of a $\mathfrak{K}$-group is either $\mathfrak{F}$-normal (and hence $\mathfrak{F}$-serial) or $\mathfrak{F}$-abnormal, so if $(\Lambda_\sigma, V_\sigma ; \sigma \in \Omega)$ is a maximal chain from H to G then it is $\mathfrak{F}$-balanced. Thus every subgroup can be joined to G by an $\mathfrak{F}$-balanced chain. We shall be primarily interested in subgroups H of G which can be joined to G by a finite $\mathfrak{F}$-balanced chain, i.e., a chain

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

such that H_i is either $\mathfrak{F}$-abnormal or $\mathfrak{F}$-serial in H_{i+1} $(0 \leq i < n)$. When such a chain exists we denote by $a^{\mathfrak{F}}(G:H)$ the minimal number of $\mathfrak{F}$-abnormal links in a finite $\mathfrak{F}$-balanced chain from H to G. $a^{\mathfrak{F}}(G:H)$ is called the *$\mathfrak{F}$-abnormal depth of H in G*. It seems unlikely that every subgroup of G should be joined to G by a finite $\mathfrak{F}$-balanced chain though we have no example to the contrary. However if $G \in \mathfrak{K} \cap \mathfrak{A}\mathfrak{F}$ then every subgroup of G has $\mathfrak{F}$-abnormal depth at most one in G (Theorem 3.2). More generally, every $\mathfrak{F}$-subgroup of $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^t\mathfrak{F}$ $(t \geq 0)$ has $\mathfrak{F}$-abnormal depth at most t in G; moreover if H is an $\mathfrak{F}$-ascendabnormal $\mathfrak{F}$-subgroup of G (in the sense of [7]) then $a^{\mathfrak{F}}(G:H) \leq t - 1$ (provided $t \geq 2$) (Theorems 3.1 and 3.3). The concepts $\mathfrak{F}$-balanced chain and $\mathfrak{F}$-abnormal depth generalize similar concepts considered by Rose [13] for finite soluble groups.

In Section 4 we consider $(R, \mathfrak{F})$-chains which may be thought of as canonical $\mathfrak{F}$-balanced chains. They generalize the Q-chains introduced by Mann in [11] and lead to our third and final invariant which we denote by $b^{\mathfrak{F}}(G:H)$ (when defined). It turns out that $a^{\mathfrak{F}}(G:H) \leq b^{\mathfrak{F}}(G:H)$ when the latter is defined and that the same bounds apply to $b^{\mathfrak{F}}(G:H)$ as apply to $a^{\mathfrak{F}}(G:H)$ in the cases mentioned earlier. Finally we generalize another of Rose's concepts and consider $\mathfrak{F}$-contranormal subgroups; a subgroup being *$\mathfrak{F}$-contranormal* in a $\mathfrak{K}$-group G if it is a subgroup of no proper $\mathfrak{F}$-serial subgroup of G. Using some rather elementary results on these subgroups we sharpen Theorems 3.8 and 4.8 to obtain the fact that if H is an $\mathfrak{F}$-subgroup of $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^t\mathfrak{A}\mathfrak{F}$ then $b^{\mathfrak{F}}(G:H)$ is at most t. Here, as usual, $\mathfrak{A}$ denotes the class of abelian groups.

2. The convergence processes and $\mathfrak{F}$-speed

The first convergence process.

THEOREM 2.1. *Let D be an arbitrary $\mathfrak{F}$-subgroup of the $\mathfrak{K}$-group G and define subgroups R_i, D_i of G inductively as follows:*

$$R_0 = G, \qquad D_0 = D,$$

and for $i \geq 0$,

$$R_{i+1} = R_G(D_i ; \mathfrak{F}),$$

and D_{i+1} any $\mathfrak{F}$-projector of R_{i+1}.

Then this process yields an $\mathfrak{F}$-projector of G; more precisely, if

$$G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^t\mathfrak{F} \quad (t \geq 0)$$

then D_{t+1} is an $\mathfrak{F}$-projector of G.

Proof. Since every $\mathfrak{K}$-group has finite $\mathrm{L}\mathfrak{N}$-length it is clearly sufficient to prove the final statement which we do by induction on t.

If $t = 0$ then $G \in \mathfrak{F}$ so by [5, 3.12] $G = R_G(D; \mathfrak{F})$, i.e., $G = R_1$. Thus D_1 is an $\mathfrak{F}$-projector of G by construction, so the induction begins.

If $t \geq 1$ let $R = \rho(G)$, the Hirsch-Plotkin radical of G. It is immediate, from [5, 3.4] and the "homomorphism invariance" of $\mathfrak{F}$-projectors, that the subgroups $D_i(G/R) = D_iR/R$ and $R_i(G/R) = R_i R/R$ are the ith terms in a convergence process for G/R, the first term in this series being the $\mathfrak{F}$-subgroup DR/R of G/R. Now $G/R \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{t-1}\mathfrak{F}$ so by induction $D_t R/R$ is an $\mathfrak{F}$-projector of G/R. Since $D_t \in \mathfrak{F}$ by construction, there is an $\mathfrak{F}$-projector E of $D_t R$ containing D_t by [2, 5.12]. Moreover E is an $\mathfrak{F}$-projector of G by Gaschütz Lemma [2, 5.3]. By 1.3 and [5, 3.11 (ii)], $E \leq R_G(D_t ; \mathfrak{F}) = R_{t+1}$. Thus E is an $\mathfrak{F}$-projector of R_{t+1} and hence is conjugate in R_{t+1} to D_{t+1}. In particular therefore D_{t+1} is an $\mathfrak{F}$-projector of G, as claimed. Notice that $D_{t+1} = D_i = R_j$ for each $i \geq t + 1$ and $j \geq t + 2$ by [5, 3.18 (i)] and 1.2.

The second convergence process.

LEMMA 2.2. *Let D be the $\mathfrak{F}$-normalizer associated with the Sylow basis $\mathbf{S}$ of of the $\mathfrak{K}$-group of G. Then D is contained in the $\mathfrak{F}$-normalizer of $R_G(D; \mathfrak{F})$ associated with the Sylow basis $\mathbf{S} \cap R_G(D; \mathfrak{F})$.*

Proof. By [2, 2.13 (ii)], $\mathbf{S}$ reduces into D, so by [5, 3.3], $\mathbf{S}$ reduces into every subgroup of G containing $R_G(D)$. In particular, by [5, 3.1], $\mathbf{S}$ reduces into $R_G(D; \mathfrak{F})$. The result is now immediate from [2, 4.10].

The second convergence is defined in the following way. Let $\mathbf{S}$ be a Sylow basis of the $\mathfrak{K}$-group G and let D be the $\mathfrak{F}$-normalizer of G associated with $\mathbf{S}$. Put $D_0 = D_1 = D$ and $R_0 = G$. Let D_2 be the $\mathfrak{F}$-normalizer of $R_1 = R_G(D; \mathfrak{F})$ associated with the Sylow basis $\mathbf{S} \cap R_1$. Then $D_0 = D_1 \leq D_2$ by 2.2. The same argument shows that $D_2 \leq D_3$, the $\mathfrak{F}$-normalizer of $R_2 = R_{R_1}(D_2 ; \mathfrak{F})$

associated with the Sylow basis $(\mathbf{S} \cap R_1) \cap R_2 = \mathbf{S} \cap R_2$. Continuing in this way we obtain two sequences of subgroups of G,

$$(1) \qquad D = D_0 = D_1 \leq D_2 \leq D_3 \leq \cdots$$

$$(2) \qquad G = R_0 \geq R_1 \geq R_2 \geq R_3 \geq \cdots$$

where for each $i \geq 1$, $R_i = R_{R_{i-1}}(D_i ; \mathfrak{F})$ and D_i is the $\mathfrak{F}$-normailzer of R_{i-1} associated with the Sylow basis $\mathbf{S} \cap R_{i-1}$, i.e.

$$D_i = \bigcap_p N_{R_{i-1}}(S_{p'} \cap C_p(R_{i-1})).$$

LEMMA 2.3 *For each* $i \geq 0$, $D_i \leq D_{i+1} \leq R_{i+1} \leq R_i$.

Proof. We certainly have $D_i \leq D_{i+1} \leq D_{i+2}$ and $R_{i+1} \leq R_i$ for each $i \geq 0$. But by construction D_{i+2} is an $\mathfrak{F}$-normalizer of R_{i+1}, so in particular $D_{i+2} \leq R_{i+1}$. The result is now clear.

We therefore have

$$(3) \quad D = D_0 = D_1 \leq D_2 \leq D_3 \leq \cdots \leq R_3 \leq R_2 \leq R_1 \leq R_0 = G$$

and, as in the proof of 2.2, $\mathbf{S}$ reduces into each D_i and R_i. Thus by [7, 5.8] (cf. also [5, 2.9])

(4) $\mathbf{S}^{\mathfrak{F}}$ strongly $\mathfrak{F}$-reduces into D_i, R_i for each $i \geq 0$.

Remark. Where we want to specify the group G in the above process we shall write $D_i = D_i(G)$ and $R_i = R_i(G)$. It is clear from the conjugacy of Sylow bases that the series obtained above, in some sense, is an invariant of G; for the corresponding series for the Sylow basis $\mathbf{S}^x$, of G is just the conjugate, by x, of the series (3).

As an immediate consequence of [5, 3.4] and the "homomorphism invariance" of $\mathfrak{F}$-normalizers we have

LEMMA 2.4. *If* $N \lhd G \in \mathfrak{K}$ *then*

$$D_i(G/N) = D_i(G)N/N \text{ and } R_i(G/N) = R_i(G)N/N \quad \text{for each } i \geq 0.$$

LEMMA 2.5. *The sequences* (3) *converges. More precisely:*

(1) *if* $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2t}\mathfrak{F}$ $(t \geq 0)$ *then* $D_i = D_t = R_t = R_i$ *for all* $i \geq t$,

(2) *if* $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2t+1}\mathfrak{F}$ $(t \geq 0)$ *then* $D_i = D_{t+1} = R_{t+1} = R_i$ *for all* $i \geq t + 1$.

Proof. Since every $\mathfrak{K}$-group has finite $\mathrm{L}\mathfrak{N}$-length it suffices to prove (1) and (2) which we do by a simultaneous induction on t.

Case (a). $t = 0$. (1) In this case $G \in \mathfrak{F}$ so that $D = G$ and $R_1 = G$. Hence $D_i = D_0 = G = R_0 = R_i$ for $i \geq 0$, as required.

(2) Here $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}$ so that D is an $\mathfrak{F}$-projector of G by [2, 5.1]. Therefore, by 1.2 and [5, 3.18(i)], $R_1 = D$ and hence $D_i = D_1 = R_1 = R_i$ for $i \geq 1$.

Case (b). $t > 0$. (1) By [5, 5.6],

$$R_1 = R_G(D; \mathfrak{F}) \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2(t-1)}\mathfrak{F}.$$

Thus by induction, $D_j(R_1) = D_{t-1}(R_1) = R_{t-1}(R_1) = R_j(R_1)$ for $j \geq t - 1$. Now it is clear that, if we begin the construction for R_1 with the Sylow basis $\mathsf{S} \cap R_1$ of R_1 then $D_j(R_1) = D_{j+1}(G)$ and $R_j(R_1) = R_{j+1}(G)$ for each $j \geq 0$. Therefore $D_i = D_t = R_t = R_i$ for each $i \geq t$, as required.

(2) In this case $R_1 = R_G(D; \mathfrak{F}) \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2(t-1)+1}\mathfrak{F}$ and a similar argument to that in case (1) gives $D_i = D_{t+1} = R_{t+1} = R_i$ for each $i \geq t + 1$, which completes the induction argument.

Remark. We shall show later that the results in 2.5 are best possible in the sense that there exists a QS-closed subclass $\mathfrak{K}$ of $\mathfrak{U}$, a saturated $\mathfrak{K}$-formation $\mathfrak{F}$ satisfying (1.1), (1.2), and (1.3), and groups G_{2t}, G_{2t+1}, in $\mathfrak{K}$ such that

$$G_{2t} \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2t}\mathfrak{F} \quad \text{but} \quad D_{t-1} \neq R_{t-1},$$

$$G_{2t+1} \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2t+1}\mathfrak{F} \quad \text{but} \quad D_t \neq R_t,$$

for each $t \geq 1$.

LEMMA 2.6. *For each integer* $i \geq 1$, $R_i = R_G(D_i; \mathfrak{F})$.

Proof. We again argue by induction on i. If $i = 1$ then $R_1 = R_G(D_1; \mathfrak{F})$ by definition so we may assume that $i > 1$ and that, by induction, $R_{i-1} = R_G(D_{i-1}; \mathfrak{F})$. Now $D_{i-1} \leq D_i \in \mathfrak{F}$, so by 1.3 and [5, 3.11(ii)], $R_G(D_i; \mathfrak{F}) \leq R_{i-1}$. Now S reduces into R_{i-1} and D_i, so by [5, 2.6],

$$R_G(D_i; \mathfrak{F}) = \langle x \in G; \mathsf{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} D_i \rangle$$

$$R_i = R_{R_{i-1}}(D_i, \mathfrak{F}) = \langle y \in R_{i-1}; (\mathsf{S} \cap R_{i-1})^{y\mathfrak{F}} \searrow_{\mathfrak{F}} D_i \rangle.$$

If $x \in G$ and $\mathsf{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} D_i$ then, from above, $x \in R_{i-1}$. Since $\mathsf{S}^{x\mathfrak{F}}$ clearly $\mathfrak{F}$-reduces into R_{i-1} to $(\mathsf{S} \cap R_{i-1})^{x\mathfrak{F}}$ it follows, from [5, 2.16], that $(\mathsf{S} \cap R_{i-1})^{x\mathfrak{F}}$ $\mathfrak{F}$-reduces into D_i. Thus $x \in R_i$ and hence $R_G(D_i; \mathfrak{F}) \leq R_i$. From [5, 3.10] we now obtain the result.

If $G \in \mathfrak{K}$ we let $E(\mathsf{S})$ be the limit of the sequence (3), i.e. if $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2t}\mathfrak{F}$ then $E(\mathsf{S}) = R_t = D_t$ and if G is a $\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2t+1}\mathfrak{F}$-group then $E(\mathsf{S}) = R_{t+1} = D_{t+1}$. Since each subgroup D_i belongs to $\mathfrak{F}$ we have, from 2.6,

$$E(\mathsf{S}) = R_G(E(\mathsf{S}); \mathfrak{F}) \in \mathfrak{F}. \tag{5}$$

Thus, by [5, 3.18(ii)], we have

THEOREM 2.7. $E(\mathsf{S})$ *is a* $\mathfrak{F}$*-projector of* G.

COROLLARY 2.8. $E(\mathsf{S})$ *is the unique* $\mathfrak{F}$*-projector of* G *into which* S *reduces and into which* $\mathsf{S}^{\mathfrak{F}}$ $\mathfrak{F}$*-reduces. Moreover* $\mathsf{S}^{\mathfrak{F}}$ *strongly* $\mathfrak{F}$*-reduces into* $E(\mathsf{S})$.

Proof. By construction S reduces and $\mathsf{S}^{\mathfrak{F}}$ strongly $\mathfrak{F}$-reduces into each D_i and R_i. In particular therefore S reduces and $\mathsf{S}^{\mathfrak{F}}$ strongly $\mathfrak{F}$-reduces into

$E(\mathbf{S})$. If $x \in G$ and $\mathbf{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into $E(\mathbf{S})^x$ then $x \in R_G(E(\mathbf{S}); \mathfrak{F}) = E(\mathbf{S})$, so that $E(\mathbf{S})^x = E(\mathbf{S})$. Thus $E(\mathbf{S})$ is the unique $\mathfrak{F}$-projector of G into which $\mathbf{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces. Finally, by [5, 2.6(iii)], $E(\mathbf{S})$ is the unique $\mathfrak{F}$-projector of G into which $\mathbf{S}$ reduces.

If G is a $\mathfrak{K}$-group we define the $\mathfrak{F}$-*speed* of G to be the least integer $i = i_{\mathfrak{F}}(G)$ such that $R_i(G) = E(\mathbf{S})$. It is immediate from the conjugacy of Sylow bases, that $i_{\mathfrak{F}}(G)$ is an invariant of G.

If r is a non-negative integer we define $r(\mathfrak{K}, \mathfrak{F})$ to be the class of all $\mathfrak{K}$-groups with $\mathfrak{F}$-speed at most r, i.e.,

$$r(\mathfrak{K}, \mathfrak{F}) = \{G \in \mathfrak{K}; i_{\mathfrak{F}}(G) \leq r\}.$$

THEOREM 2.9. *$r(\mathfrak{K}, \mathfrak{F})$ is a $\mathfrak{K}$-formation for each non-negative integer r.*

Proof. Suppose $\mathbf{S}$ is a Sylow basis and N a normal subgroup of a $\mathfrak{K}$-group G. Then, by 2.4,

$$E(\mathbf{S}N/N) = E(\mathbf{S})N/N. \tag{6}$$

Since the sequence (3) becomes stationary when it reaches $E(\mathbf{S})$, it is clear that the $\mathfrak{K}$-group G belongs to $r(\mathfrak{K}, \mathfrak{F})$ if and only if $R_r(G) = E(\mathbf{S})$.

We show first that $r(\mathfrak{K}, \mathfrak{F})$ is Q-closed. Indeed let N be a normal subgroup of the $r(\mathfrak{K}, \mathfrak{F})$-group G. Then $R_r(G) = E(\mathbf{S})$ so, by 2.4 and (6), $R_r(G/N) = E(\mathbf{S}N/N)$. Thus by our previous remarks, $G/N \in r(K, \mathfrak{F})$. This shows that $r(\mathfrak{K}, \mathfrak{F})$ is Q-closed.

If $G \in \mathfrak{K} \cap R(r(\mathfrak{K}, \mathfrak{F}))$ then there exist normal subgroups N_λ of $G (\lambda \in \Lambda)$ such that $G/N_\lambda \in r(\mathfrak{K}, \mathfrak{F})$ for each $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda} N_\lambda = 1$. Thus, by 2.4 and (6) we have $R_r(G)N_\lambda = E(\mathbf{S})N_\lambda$ for each $\lambda \in \Lambda$. Hence, by [2, 3.6(i)], $R_r(G) \leq \bigcap_{\lambda \in \Lambda}(E(\mathbf{S})N_\lambda) = E(\mathbf{S})$. But $E(\mathbf{S}) \leq R_r(G)$ by construction, so we must have $R_r(G) = E(\mathbf{S})$. Thus $G \in r(\mathfrak{K}, \mathfrak{F})$ and hence $\mathfrak{K} \cap R(r(\mathfrak{K}, \mathfrak{F})) = r(\mathfrak{K}, \mathfrak{F})$. This shows that $r(\mathfrak{K}, \mathfrak{F})$ is a $\mathfrak{K}$-formation, as claimed.

Thus for every saturated $\mathfrak{K}$-formation $\mathfrak{F}$ satisfying (1.1), (1.2) and (1.3) we obtain a series of $\mathfrak{K}$-formations

$$0(\mathfrak{K}, \mathfrak{F}) \leq 1(\mathfrak{K}, \mathfrak{F}) \leq 2(\mathfrak{K}, \mathfrak{F}) \leq \cdots \tag{7}$$

and it is immediate, from Lemma 2.5, that for each $t \geq 0$,

$$\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2t}\mathfrak{F} \leq t(\mathfrak{K}, \mathfrak{F}), \qquad \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{2t+1}\mathfrak{F} \leq t + 1(\mathfrak{K}, \mathfrak{F}). \tag{8}$$

Since every $\mathfrak{K}$-group has finite $\mathrm{L}\mathfrak{N}$-length we also have

$$\mathfrak{K} = \bigcup_{n=0}^{\infty} n(\mathfrak{K}, \mathfrak{F}) \tag{9}$$

LEMMA 2.10. (1) $0(\mathfrak{K}, \mathfrak{F}) = \mathfrak{F}$

(2) *$1(\mathfrak{K}, \mathfrak{F})$ contains the class of $\mathfrak{K}$-groups in which the $\mathfrak{F}$-normalizer and $\mathfrak{F}$-projectors coincide.*

(3) *$\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^2 t(\mathfrak{K}, F) \leq t + 1(\mathfrak{K}, \mathfrak{F})$ for each $t \geq 0$.*

Proof. (1) $G \in 0(\mathfrak{K}, \mathfrak{F}) \Leftrightarrow G = R_0 = E(\mathbf{S})$. Therefore $0(\mathfrak{K}, \mathfrak{F}) = \mathfrak{F}$.

(2) If the $\mathfrak{F}$-normalizers and $\mathfrak{F}$-projectors of the $\mathfrak{K}$-group G coincide,

then with the usual notation, $D = E(\mathbf{S})$. Thus $R_1(G) = R_G(D; \mathfrak{F}) = D = E(\mathbf{S})$ by [5, 3.18 (i)]. Hence $G \in 1(\mathfrak{K}, \mathfrak{F})$.

(3) Suppose $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^2 t(\mathfrak{K}, \mathfrak{F})$ and let F denote the $(\mathrm{L}\mathfrak{N})^2$-radical of G, i.e., $F/\rho(G) = \rho(G/\rho(G))$. Then $G/F \in t(\mathfrak{K}, \mathfrak{F})$ since this class is Q-closed by 2.9. Thus, by 2.4 and (6), $R_t(G)F = E(\mathbf{S})F$. Hence

$$R_t(G) \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^2\mathfrak{F}.$$

Now $D_{t+1}(G)$ is an $\mathfrak{F}$-normalizer of $R_t(G)$ so, by [5, Theorem 5.2],

$$R_{t+1}(G) = R_{R_t(G)}(D_{t+1}(G); \mathfrak{F})$$

is an $\mathfrak{F}$-projector of $R_t(G)$. Since $E(\mathbf{S})$ is an $\mathfrak{F}$-projector of $R_t(G)$ contained in $R_{t+1}(G)$ we must have $R_{t+1}(G) = E(\mathbf{S})$. Thus $G \in t + 1(\mathfrak{K}, \mathfrak{F})$ as required.

COROLLARY 2.11. *If* $t \geq 0$ *then* $t(\mathfrak{K}, \mathfrak{F}) < t + 1(\mathfrak{K}, \mathfrak{F})$ *if and only if* $t(\mathfrak{K}, \mathfrak{F}) < \mathfrak{K}$.

Proof. It is clear that we need only show that $t(\mathfrak{K}, \mathfrak{F})$ is a proper subclass of $t + 1(\mathfrak{K}, \mathfrak{F})$ if it is a proper subclass of $\mathfrak{K}$. Suppose that this is not the case. Then, by 2.10 (3), we have

$$\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^2 t(\mathfrak{K}, \mathfrak{F}) \leq t + 1(\mathfrak{K}, \mathfrak{F}) = t(\mathfrak{K}, \mathfrak{F})$$

whence

$$\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^2 t(\mathfrak{K}, \mathfrak{F}) = t(\mathfrak{K}, \mathfrak{F}).$$

An easy induction argument now shows that for each $n \geq 0$,

$$\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^n t(\mathfrak{K}, \mathfrak{F}) = t(\mathfrak{K}, \mathfrak{F}).$$

Since every $\mathfrak{K}$-group has finite $\mathrm{L}\mathfrak{N}$-length it follows that $\mathfrak{K} \leq t(\mathfrak{K}, \mathfrak{F})$ contradicting the hypothesis that $t(\mathfrak{K}, \mathfrak{F})$ is a proper subclass of $\mathfrak{K}$. This contradiction completes the proof.

From 2.10 and 2.11 we see that the ascending sequence (7) commences at $\mathfrak{F}$ and becomes stationary only when it reaches $\mathfrak{K}$.

COROLLARY 2.12. $G \in 1(\mathfrak{K}, \mathfrak{F})$ *if and only if* $E(\mathbf{S})$ *is the strong* $\mathfrak{F}$*-serializer of* D *in* G.

Proof. Suppose firstly that $G \in 1(\mathfrak{K}, \mathfrak{F})$. Then $R_G(D; \mathfrak{F}) = R_1 = E(\mathbf{S})$. Now D $\mathfrak{F}$-ser $E(\mathbf{S})$ by 1.2 and [5, 5.9(i)], so in this case D is $\mathfrak{F}$-serial in $R_G(D; \mathfrak{F})$. Thus, by [5,41.7], $E(\mathbf{S})$ is the strong $\mathfrak{F}$-serializer of D in G.

If on the other hand $E(\mathbf{S})$ is the strong $\mathfrak{F}$-serializer of D in G then $E(\mathbf{S}) = R_G(D; \mathfrak{F})$ by definition (cf. [5, §4]). Thus $\mathrm{R}_1 = E(\mathbf{S})$ and $G \in 1(\mathfrak{K}, \mathfrak{F})$ as required.

From 1.2 and 2.10 we see that $0(\mathfrak{K}, \mathfrak{F})$ is both subgroup-closed and saturated. We have been unable to decide whether $t(\mathfrak{K}, \mathfrak{F})$ is saturated for $t \geq 1$. However, we now give an example to show that the classes $t(\mathfrak{K}, \mathfrak{F})$

are in general not subgroup-closed. This example also shows that the sequence (7) can be strictly ascending and that (1), (2) in Lemma 2.5 are best possible.

Example 2.13. We take $\mathfrak{K} = \mathfrak{S}^*$ the class of finite soluble groups and $\mathfrak{F} = \mathfrak{N}^*$ the class of finite nilpotent groups. Certainly these satisfy (1.1), (1.2) and (1.3) and in this case the $\mathfrak{N}^*$-reducers are exactly the reducers.

If G is an A-group (that is a finite soluble group with abelian Sylow p-subgroups for each prime p) then the basis (i.e. system) normalizers of G are pronormal in G [12, 2.4]. Thus if D is a basis normalizer of G, then by [5, 3.22], $R_G(D) = N_G(D)$. Inspection now shows, using [1, Theorem 6] that for A-groups our second convergence process reduces to that defined by Carter [1, §4].

In [1], Carter constructs an A-group G_j for each $j \geq 1$, in the following way. Let $p_1, p_2, p_3, \cdots$ be a sequence of distinct primes and inductively define

$$G_1 = C_{p_1}, G_k = C_{p_k} \wr G_{k-1} \quad (k > 1)$$

where C_{p_i} denotes a cyclic group of order p_i. Carter shows [1, Theorem 12], that, for each $n \geq 1$, G_{2n} is an A-group of nilpotent length $2n$ in which $F_{n-1} \neq E$ (i.e. in our notation $R_{n-1} \neq E(S)$) and G_{2n+1} is an A-group of nilpotent length $2n + 1$ in which $D_n \neq E$ (i.e. in our notation $D_n \neq E(S)$). In the latter case $R_{n-1} \neq E(S)$ since D_n is a basis normalizer of R_{n-1}. Thus, by (8), we have

$$(10) \qquad G_{2n+1} \in (\mathfrak{N}^*)^{2n+1} \cap (n(\mathfrak{S}^*, \mathfrak{N}^*) - n - 1(\mathfrak{S}^*, \mathfrak{N}^*)),$$

$$G_{2n+2} \in (\mathfrak{N}^*)^{2n+2} \cap (n + 1(\mathfrak{S}^*, \mathfrak{N}^*) - n(\mathfrak{S}^*, \mathfrak{N}^*)).$$

Hence equations (1), (2), in 2.5 are best possible. Also the sequence (7) is strictly ascending in this case, i.e.,

$$\mathfrak{N}^* = 0(\mathfrak{S}^*, \mathfrak{N}^*) < 1(\mathfrak{S}^*, \mathfrak{N}^*) < 2(\mathfrak{S}^*, \mathfrak{N}^*) < \cdots.$$

For each $t \geq 1$, the $\mathfrak{S}^*$-formation $t(\mathfrak{S}^*, \mathfrak{N}^*)$ is not subgroup closed. For, by 2.10(2), $1(\mathfrak{S}^*, \mathfrak{N}^*)$ contains all SC-groups, i.e., finite soluble groups in which the basis normalizers and Carter subgroups coincide. Now the Alperin-Thompson Theorem [9, page 747] states that every finite soluble group can be embedded in an SC-group. Thus if $t(\mathfrak{S}^*, \mathfrak{N}^*)$ were subgroup-closed for some $t \geq 1$ then we would have to have $t(\mathfrak{S}^*, \mathfrak{N}^*) = \mathfrak{S}^*$, contradicting (10) above. Thus $t(\mathfrak{S}^*, \mathfrak{N}^*)$ is not subgroup-closed for each $t \geq 1$.

Suppose f_i $(i = 1, 2)$ is a $\mathfrak{K}$-preformation function on the set of all primes satisfying (1.1) and (1.3), and $\mathfrak{F}_i$ is the saturated $\mathfrak{K}$-formation defined by $\mathfrak{f}_i$. We close this section with examples that show that

(a) $\mathfrak{F}_1 \leq \mathfrak{F}_2$ does not imply $t(\mathfrak{K}, \mathfrak{F}_1) \leq t(\mathfrak{K}, \mathfrak{F}_2)$ $(t > 0)$,
(b) $\mathfrak{F}_1 \leq \mathfrak{F}_2$ does not imply $t(\mathfrak{K}, \mathfrak{F}_2) \leq t(\mathfrak{K}, \mathfrak{F}_1)$ $(t > 0)$.

In fact (b) follows easily from our previous example. For if we take $\mathfrak{K} = \mathfrak{F}_2 = \mathfrak{S}^*$ and $\mathfrak{F}_1 = \mathfrak{N}^*$ then, by (10), $t(\mathfrak{S}^*, \mathfrak{N}^*) < t(\mathfrak{S}^*, \mathfrak{S}^*) = \mathfrak{S}^*$ for all $t \geq 0$.

Our example for (a) is somewhat more complex.

Example 2.14. We in fact consider the group G which Hawkes [8] constructed as follows:

Let $Q = \langle a, b;\, a^4 = 1, a^2 = b^2 = [a, b]\rangle$ be a quaternion group of order 8 and S a subgroup of the automorphism group of Q isomorphic to the symmetric group of degree 3. S is chosen so as to contain an involution x whose action on Q is defined by $a^x = b$, $b^x = a$. Let $R = QS$ be the semidirect product of Q by S. We write $z = [a, b]$ and $Z = \langle z\rangle$; Z is the centre of both Q and R. We let T denote the normal subgroup of index 2 in S.

Now set $K = \langle (12)(35), (12345)\rangle$, a dihedral group of order 10 considered as a subgroup of the alternating group of degree 5, and let $H = \langle (12345)\rangle$ be the normal subgroup of index 2 in K. Let $G = R \wr K$, the wreath product of R by K according to this permutation representation. Let $\sigma_i : R \rightarrow R_i$ denote an isomorphism $(i = 1, -, 5)$ and let $D = R_1 \times \_ \times R_5$ be the base group of G. Using the suffix i to denote images under σ_i Hawkes sets

$$\bar{x} = x_1x_2x_3x_4x_5, \quad \bar{z} = z_1z_2z_3z_4z_5 \quad \text{and} \quad k = (12)(35).$$

He also defines the following subgroups of G:

$$\bar{A} = \langle \bar{z}\rangle; \qquad A = Z_1 \times \cdots \times Z_5;$$

$$B = Q_1 \times \cdots \times Q_S; \qquad C = B(T_1 \times \cdots \times T_5);$$

$$\bar{D} = C\langle \bar{x}\rangle; \qquad \bar{S} = S_1 \times \cdots \times S_5;$$

$$E_1 = \langle \bar{x}\rangle \times \langle \bar{z}\rangle \times \langle \bar{k}\rangle; \qquad F = (A \times \bar{S})\langle k\rangle.$$

Hawkes also considers the saturated $\mathfrak{S}^*$-formation $\mathfrak{F}$ defined by the $\mathfrak{S}^*$-formation function

$$\mathfrak{f}(p) = \{1\} \quad \text{for} \quad p \neq 3$$

$$\mathfrak{f}(3) = \mathfrak{S}_2^*, \quad \text{the class of finite 2-groups.}$$

It is easy to see that the upper nilpotent series of G is $1 < B < C < D < DH < G$ so that G has nilpotent length 5 and belongs to $(\mathfrak{N}^*)^4\mathfrak{F}$ but not to $(\mathfrak{N}^*)^3\mathfrak{F}$.

Hawkes showed in his paper that E_1 is both an $\mathfrak{F}$-normalizer and basis normalizer of G and that F is an $\mathfrak{F}$-projector of G.

Let $S_2 = B(\langle x_1\rangle \times \cdots \times \langle x_5\rangle)\langle k\rangle$, $S_3 = T_1 \times \cdots \times T_5$, $S_5 = H$. Then $\mathbf{S} = \{S_2, S_3, S_5\}$ is a Sylow basis of G which reduces into both E_1 and F. Thus in our usual terminology $F = E(\mathbf{S})$. The p-complement system of G

associated with $\mathbf{S}$ is $\{S_{2'}, S_{3'}, S_{5'}\}$ where

$$S_{2'} = (T_1 \times \_ \times T_5)H,$$
$$S_{3'} = B(\langle x_1 \rangle \times \cdots \times \langle x_5 \rangle)K,$$
$$S_{5'} = D\langle k \rangle.$$

We calculate the $\mathfrak{F}$-reducer of E_1 in G. Since $\mathbf{S}$ reduces into E_1 we have $R_G(E_1\,;\mathfrak{F}) = \langle y \in G;\ \mathbf{S}^{y\mathfrak{F}} \searrow_{\mathfrak{F}} E_1 \rangle$ by [5, 2.6]. Now the $\mathfrak{f}(p)$-residual $E_1^{\mathfrak{p}}$ of E_1 is E_1 for $p \neq 3$ and 1 if $p = 3$. Also $C_p(G) = G$ for $p \neq 3$. Thus

$$\begin{aligned}
\mathbf{S}^{y\mathfrak{F}} \searrow_{\mathfrak{F}} E_1 &\Leftrightarrow S_{p'}^{y} \cap E_1 \in Syl_{p'}(E_1) \quad \text{for each } p \neq 3\\
&\Leftrightarrow S_{5'}^{y} \text{ reduces into } E_1 \quad (\text{since } E_1 \text{ is a 2-group})\\
&\Leftrightarrow E_1 \leq (D\langle k \rangle)^{y}\\
&\Leftrightarrow D\langle k \rangle = (D\langle k \rangle)^{y} \quad (\text{since } D \lhd G)\\
&\Leftrightarrow y \in N_G(D\langle k \rangle) = D\langle k \rangle.
\end{aligned}$$

Thus $R_G(E_1\,;\mathfrak{F}) = D\langle k \rangle$. Since $D\langle k \rangle$ is not an $\mathfrak{F}$-projector of G we therefore have $G \notin 1(\mathfrak{S}^*, \mathfrak{F})$.

We now calculate the reducer of E_1 in G. Since

$$R_G(E_1) \leq R_G(E_1\,;\mathfrak{F}) = D\langle k \rangle$$

by [5, 3.1], we have $R_G(E_1) = R_{D\langle k \rangle}(E_1)$. Now $\mathbf{S}$ reduces into $D\langle k \rangle$ so that $R_{D\langle k \rangle}(E_1)$ is generated by those elements $y \in D\langle k \rangle$ such that $(\mathbf{S} \cap D\langle k \rangle)^{y}$ reduces into E_1. But E_1 is a 2-group so it follows that

$$R_G(E_1) = R_{D\langle k \rangle}(E_1) = \langle y \in D\langle k \rangle;\ S_2^{y} \quad \text{reduces into} \quad E_1 \rangle.$$

Hence $S_2 \leq R_{D\langle k \rangle}(E_1)$.

Suppose $y \in D\langle k \rangle$ and S_2^{y} reduces into E_1. Now

$$D\langle k \rangle = S_2(T_1 \times\!-\!\times T_5)$$

so that $y = uv$ where $u \in S_2$ and $v \in T_1 \times \cdots \times T_5$. Thus $S_2^{v} = S_2^{y}$ reduces into E_1. Therefore $E_1 \leq S_2^{v}$ and, since $\bar{x}$ normalizes $T_1 \times \cdots \times T_5$,

$$[\bar{x}, v] \in S_2^{v} \cap (T_1 \times \cdots \times T_5) = 1.$$

Now the centralizer of x_i in T_i is the identity, so it follows that the centralizer of $\bar{x}$ in $T_1 \times \cdots \times T_5$ is also the identity. Thus $v = 1$ and $y = u \in S_2$. Hence $R_{D\langle k \rangle}(E_1) \leq S_2$ and from our previous inequality we have $R_G(E_1) = S_2$.

It follows, for example from [5, 3.9 and 3.19], that S_2 is a Carter subgroup of G. Therefore the $\mathfrak{N}^*$-convergence process for G takes one step, i.e., $G \in 1(\mathfrak{S}^*, \mathfrak{N}^*)$.

Now it is clear that $\mathfrak{N}^* \leq \mathfrak{F}$ so we have an example to show (a) since $G \in 1(\mathfrak{S}^*, \mathfrak{N}^*) - 1(\mathfrak{S}^*, \mathfrak{F})$.

3. $\mathfrak{F}$-abnormal depth

We recall that, when defined, the *$\mathfrak{F}$-abnormal depth*, $a^{\mathfrak{F}}(G{:}H)$, of a subgroup H in a $\mathfrak{K}$-group G is the minimal number of $\mathfrak{F}$-abnormal links in a finite *$\mathfrak{F}$-balanced chain* from H to G, i.e., a chain

$$H = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_n = G$$

in which H_i is either $\mathfrak{F}$-abnormal or $\mathfrak{F}$-serial in H_{i+1} $(0 \leq i < n)$.

If we take $\mathfrak{K} = \mathfrak{S}^*$ and $\mathfrak{F} = \mathfrak{N}^*$ then the concepts "$\mathfrak{F}$-balanced chain", "$\mathfrak{F}$-abnormal depth" reduce to the concepts "balanced chain", "abnormal depth" defined by Rose [13]. If H is a subgroup of a finite soluble group G then the abnormal depth of H in G is, as in [13], denoted $a(G{:}H)$. The first three theorems in this section generalize similar results of Rose [13].

THEOREM 3.1. *Suppose H is an $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^t\mathfrak{F}$-group G $(t \geq 0)$. Then $a^{\mathfrak{F}}(G{:}H) \leq t$.*

Proof. Since $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^t\mathfrak{F}$ there is a series

$$1 = U_0 \leq U_1 \leq U_2 \leq \cdots \leq U_t \leq U_{t+1} = G$$

of normal subgroups U_i of G such that $U_i/U_{i-1} \in \mathrm{L}\mathfrak{N}$ for $1 \leq i \leq t$ and $G/U_t \in \mathfrak{F}$. Set $H_i = HU_i$. Then

$$H = H_0 \leq H_1 \leq \cdots \leq H_t \leq H_{t+1} = G.$$

Let $i \in \{0, \cdots, t-1\}$. Then $H_{i+1}/U_i = U_{i+1}/U_i$. H_i/U_i and $H_i/U_i \in \mathfrak{F}$, so, by [2, 5.12], there is an $\mathfrak{F}$-projector F_i/U_i of H_{i+1}/U_i containing H_i/U_i. Now F_i/U_i is $\mathfrak{F}$-abnormal in H_{i+1}/U_i by [7, 3.5], and H_i/U_i is $\mathfrak{F}$-serial in F_i/U_i by 1.3 and [5, 5.9(i)]. Thus

$$H_i \ \mathfrak{F}\text{-}ser\ F_i \rtimes_{\mathfrak{F}} H_{i+1},$$

and we have

$$(1) \qquad H = H_0 \ \mathfrak{F}\text{-}ser\ F_0 \rtimes_{\mathfrak{F}} H_1 \ \mathfrak{F}\text{-}ser\ F_1 \rtimes_{\mathfrak{F}} H_2 \cdots \rtimes_{\mathfrak{F}} H_t \leq G$$

Now $G/U_t \in \mathfrak{F}$, so, by [5, 5.9], H_t/U_t $\mathfrak{F}$-*ser* G/U_t. Thus H_t $\mathfrak{F}$-*ser* G and the chain (1) is $\mathfrak{F}$-balanced. Since there are (at most) t $\mathfrak{F}$-abnormal links in this chain we have $a^{\mathfrak{F}}(G{:}H) \leq t$, as required.

THEOREM 3.2. *If $G \in \mathfrak{K} \cap \mathfrak{A}\mathfrak{F}$ and H is any subgroup of G then $a^{\mathfrak{F}}(G{:}H) \leq 1$.*

Proof. Let $A = G^{\mathfrak{F}}$, the $\mathfrak{F}$-residual of G. Then A is abelian by hypothesis and it is clear that

$$(2) \qquad a^{\mathfrak{F}}(G{:}H) \leq a^{\mathfrak{F}}(G{:}AH) + a^{\mathfrak{F}}(AH{:}H).$$

Now $G/A \in \mathfrak{F}$ so, by [5, 5.9 (i)], AH $\mathfrak{F}$-*ser* G. Thus $a^{\mathfrak{F}}(G{:}AH) = 0$. Since A is an abelian normal subgroup of G, $A \cap H$ is a normal subgroup of AH. It follows that

$$a^{\mathfrak{F}}(AH{:}H) = a^{\mathfrak{F}}(AH/A \cap H{:}AH/A \cap H).$$

Since $\mathfrak{F}$ is a subgroup-closed and $H/A \cap H \cong AH/A$, we have $H/A \cap H \in \mathfrak{F}$. Thus by Theorem 3.1, $a^{\mathfrak{F}}(AH/A \cap H:H/A \cap H) \leq 1$. From (2) and our previous remarks we now deduce $a^{\mathfrak{F}}(G:H) \leq 1$, as required.

Remark. In his paper, [13], Rose shows that for each integer $n \geq 1$ there is a finite supersoluble group G with a subgroup H such that $a(G:H) = n$. Such a group G is necessarily metanilpotent so Theorem 3.2 cannot be extended to the case where $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}$.

THEOREM 3.3. *Suppose H is an $\mathfrak{F}$-ascendabnormal $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^t\, \mathfrak{F}$-group G $(t \geq 2)$. Then $a^{\mathfrak{F}}(G:H) \leq t - 1$.*

Proof. As in the proof of 3.1 we have a series

$$1 = U_0 \leq U_1 \leq U_2 \leq \cdots \leq U_t \leq U_{t+1} = G$$

of normal subgroups of G such that $U_i/U_{i-1} \in \mathrm{L}\mathfrak{N}$ for $1 \leq i \leq t$ and $G/U_t \in \mathfrak{F}$. Now $HU_{t-2} \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^{t-2}\mathfrak{F}$ so, by Theorem 3.1, we have $a^{\mathfrak{F}}(HU_{t-2} : H) \leq t - 2$. Since

$$a^{\mathfrak{F}}(G:H) \leq a^{\mathfrak{F}}(G:HU_{t-2}) + a^{\mathfrak{F}}(HU_{t-2} :H)$$

it suffices to prove that $a^{\mathfrak{F}}(G:HU_{t-2}) \leq 1$.

By [7, 4.1], H contains an $\mathfrak{F}$-normalizer of G, so HU_{t-2}/U_{t-2} contains some $\mathfrak{F}$-normalizer D/U_{t-2} of G/U_{t-2}. Since $H \in \mathfrak{F}$,

$$HU_{t-2}/U_{t-2} \leq R_{G/U_{t-2}}(D/U_{t-2}\,;\mathfrak{F}) = X/U_{t-2}$$

by [5, 3.11]. Now $G/U_{t-2} \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^2\mathfrak{F}$ so, by [5, 5.2], X/U_{t-2} is an $\mathfrak{F}$-projector of G/U_{t-2}. In particular therefore

$$X/U_{t-2} \rtimes_{\mathfrak{F}} G/U_{t-2}\,.$$

Also HU_{t-2}/U_{t-2} $\mathfrak{F}$-*ser* X/U_{t-2} by [5, 5.9(i)]. Thus

$$HU_{t-2}\ \mathfrak{F}\text{-}ser\ X \rtimes_{\mathfrak{F}} G$$

and we have $a^{\mathfrak{F}}(G:HU_{t-2}) \leq 1$, as required.

Remarks 1. Theorem 3.3 does not hold if $t = 1$. For if H is an $\mathfrak{F}$-ascendabnormal $\mathfrak{F}$-subgroup of $G \in K \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}$ then, by [7, 4.1], H contains an $\mathfrak{F}$-normalizer D of G which, by [2, 5.1], is also an $\mathfrak{F}$-projector of G. Since $H \in \mathfrak{F}$ we must then have $H = D$, an $\mathfrak{F}$-abnormal subgroup of G by [7, 3.5]. Thus $a^{\mathfrak{F}}(G:H) = 1$ and 3.3 does not hold.

2. If H is a subgroup of an $\mathfrak{F}$-projector E of the $\mathfrak{K}$-group G then, by [5, 5.9(i)] and [7, 3.5], H $\mathfrak{F}$-*ser* $E \rtimes_{\mathfrak{F}} G$ whence $a^{\mathfrak{F}}(G:H) \leq 1$. In particular if D is an $\mathfrak{F}$-normalizer of G then $a^{\mathfrak{F}}(G:D) \leq 1$.

LEMMA 3.4. *Suppose D is an $\mathfrak{F}$-normalizer of the $\mathfrak{K}$-group G. Then $a^{\mathfrak{F}}(G:H) = 0$ if and only if $G \in \mathfrak{F}$. Hence $a^{\mathfrak{F}}(G:D) = 1$ if and only if $G \notin \mathfrak{F}$.*

Proof. Suppose $a^{\mathfrak{F}}(G:D) = 0$. Then there is a series

$$D = D_0 \leq D_1 \leq \cdots \leq D_n = G$$

with D_i $\mathfrak{F}$-*ser* D_{i+1} for $0 \leq i \leq n - 1$. Clearly this implies that D $\mathfrak{F}$-*ser* G. It now follows, from [5, 4.10], that $G = R_G(D; \mathfrak{F})$. From [5, 5.6] we now deduce that G belongs either to $\mathfrak{F}$ or to $\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}$. If $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}$ then, by [2, 5.1] and [5, 3.18(i)], $G = D \in \mathfrak{F}$. Thus in either case we have $G \in \mathfrak{F}$, as required.

If conversely $G \in \mathfrak{F}$ then $G = D$ and clearly $a^{\mathfrak{F}}(G:D) = 0$. We have therefore shown that $a^{\mathfrak{F}}(G:D) = 0$ if and only if $G \in \mathfrak{F}$. The last part of the lemma now follows from the second remark prior to the statement of the result.

4. $(R, \mathfrak{F})$-chains

LEMMA 4.1. *Suppose H is a subgroup of the $\mathfrak{K}$-group G. Then there is a unique smallest $\mathfrak{F}$-serial subgroup of G containing H.*

Proof. Let **B** be the collection of all $\mathfrak{F}$-serial subgroups of G containing H; **B** is non-empty since $G \in$ **B**. From [5, 2.21 and 4.10] it follows that intersection B of all the members of **B** is also $\mathfrak{F}$-serial in G. Clearly B is the unique smallest $\mathfrak{F}$-serial subgroup of G containing H.

If H is a subgroup of a $\mathfrak{K}$-group G we denote by $S^{\mathfrak{F}}(G:H)$ the unique smallest $\mathfrak{F}$-serial subgroup of G containing H.

LEMMA 4.2. *Suppose $H \leq G \in \mathfrak{K}$ and $N \lhd G$. Then*

$$S^{\mathfrak{F}}(G/N:HN/N) = S^{\mathfrak{F}}(G:H)N/N.$$

Proof. Let $S^{\mathfrak{F}}(G/N:HN/N) = X/N$. Then $HN/N \leq X/N$ $\mathfrak{F}$-*ser* G/N so that $H \leq X$ $\mathfrak{F}$-*ser* G. Thus $S^{\mathfrak{F}}(G:H) \leq X$ by definition and hence

$$S^{\mathfrak{F}}(G:H)N/N \leq X/N.$$

But $S^{\mathfrak{F}}(G:H)N/N$ $\mathfrak{F}$-*ser* G/N by [5, 4.11], so we must have $S^{\mathfrak{F}}(G:H)N/N = X/N$, as claimed.

Suppose H is a subgroup of the $\mathfrak{K}$-group G. We define subgroups $S_i = S_i(G:H:\mathfrak{F})$ and $R_i = R_i(G:H:\mathfrak{F})$ of G containing H inductively as follows:

$$S_1 = S^{\mathfrak{F}}(G:H); \qquad R_1 = R_{S_1}(H; \mathfrak{F});$$

$$S_{i+1} = S^{\mathfrak{F}}(R_i:H); \qquad R_{i+1} = R_{S_{i+1}}(H; \mathfrak{F}) \qquad (i \geq 1).$$

In this way we obtain a chain

$$G \geq S_1 \geq R_1 \geq S_2 \geq R_2 \geq \cdots \tag{3}$$

of subgroups of G containing H. It seems possible that in the most general cases the series (3) may not reach H after a finite number of steps, though we have no example to verify this. However, by [5, 3.8 and 3.13(ii)] we do have

$$H \leq \cdots R_2 \rtimes_{\mathfrak{F}} S_2 \ \mathfrak{F}\text{-}ser\ R_1 \rtimes_{\mathfrak{F}} S_1 \ \mathfrak{F}\text{-}ser\ G \tag{4}$$

Thus when the chain (4) is finite and reaches H it is an $\mathfrak{F}$-balanced chain from H to G. We call (3) the *$(R, \mathfrak{F})$-chain of H in G*, and when it reaches H after a finite number of steps we denote by $b^{\mathfrak{F}}(G:H)$ the number of $\mathfrak{F}$-abnormal links in it. It is clear that $a^{\mathfrak{F}}(G:H) \leq b^{\mathfrak{F}}(G:H)$ when defined. It is immediate that if $t \geq 0$ then $b^{\mathfrak{F}}(G:H) = t$ if and only if $S_{t+1}(G:H:\mathfrak{F}) = H$.

Our $(R, \mathfrak{F})$-chains generalize Mann's Q-chains [11] and our first aim is to show that at least in finite $\mathfrak{R}$-groups they have some meaning, i.e. if H is a subgroup of a finite $\mathfrak{R}$-group G then the $(R, \mathfrak{F})$-chain of H in G reaches H (after a finite number of steps). To do this we require two lemmas.

LEMMA 4.3. *Suppose H is a subgroup of the finite $\mathfrak{R}$-group G and*

$$H < R_G(H; \mathfrak{F}) = G.$$

Then H lies in an $\mathfrak{F}$-normal maximal subgroup of G. Hence $S^{\mathfrak{F}}(G:H) < G$.

Proof. We argue by induction on the order of G. Since H is a proper subgroup of G, G is nontrivial. Let N be a minimal normal subgroup of G; then $R_{G/N}(HN/N; \mathfrak{F}) = G/N$ by [5, 3.4]. If HN/N is a proper subgroup of G/N then, by induction, HN/N lies in an $\mathfrak{F}$-normal maximal subgroup M/N of G/N. Thus H is contained in the $\mathfrak{F}$-normal maximal subgroup M of G. If $HN = G$ then H is a maximal subgroup of G and, by [5, 3.13(i)], must be $\mathfrak{F}$-normal in G. Thus in either case H lies in an $\mathfrak{F}$-normal maximal subgroup M of G. By definition, $S^{\mathfrak{F}}(G:H) \leq M$ so the final statement of the lemma is immediate.

LEMMA 4.4. *If $H \leq G \in \mathfrak{R}$ and $X = R_G(H; \mathfrak{F})$ then $R_X(H; \mathfrak{F}) = X$.*

Proof. Let $\mathbf{S}$ be a Sylow basis of G which reduces into both H and X. Then, by [5, 2.6(iii)],

$$X = R_G(H; \mathfrak{F}) = \langle x \in G;\ \mathbf{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H \rangle,$$

$$R_X(H; \mathfrak{F}) = \langle y \in X;\ (\mathbf{S} \cap X)^{y\mathfrak{F}} \searrow_{\mathfrak{F}} H \rangle.$$

Suppose $x \in G$ and $\mathbf{S}^{x\mathfrak{F}}$ $\mathfrak{F}$-reduces into H. Then $x \in X$ and, since $\mathbf{S}^{x\mathfrak{F}}$ clearly $\mathfrak{F}$-reduces into X to $(\mathbf{S} \cap X)^{x\mathfrak{F}}$, we have $(\mathbf{S} \cap X)^{x\mathfrak{F}}$ $\mathfrak{F}$-reduces into H, by [5, 2.16]. Thus $X \leq R_X(H:\mathfrak{F})$ and the result now follows.

Suppose now that H is a subgroup of the finite $\mathfrak{R}$-group G. Then, by 4.3 and 4.4, every containment in the chain (3) is strict (except possibly $G \geq S_1$) until H is reached. Thus the $(R, \mathfrak{F})$-chain of H in G reaches H and $b^{\mathfrak{F}}(G:H)$ is defined.

We have been unable to decide whether Lemma 4.3 holds in general.

Our aim now is to improve Theorems 3.1, 3.2 and 3.3 by showing that

$b^{\mathfrak{F}}(G:H)$ may replace $a^{\mathfrak{F}}(G:H)$ in each of the statements. Techniques similar to those employed by Mann [11] can be used to prove these extensions but we give here alternative proofs which use our work on $\mathfrak{F}$-reducers [5].

We shall require the following

LEMMA 4.5. *Suppose H is an $\mathfrak{F}$-ascendabnormal subgroup of the $\mathfrak{K}$-group G. Then $S^{\mathfrak{F}}(G:H) = G$.*

Proof. Let $S = S^{\mathfrak{F}}(G:H)$. Since H is $\mathfrak{F}$-ascendabnormal in G there is an ordinal σ and a chain $(H_\beta\,;\,\beta \leq \sigma)$ of subgroups of G such that $H = H_0$, $H_\beta \bowtie_{\mathfrak{F}} H_{\beta+1}$ for $\beta < \sigma$, $H_\lambda = \bigcup_{\beta<\lambda} H_\beta$ for limit ordinals $\lambda \leq \sigma$, and $H\sigma = G$. We prove by transfinite induction that $H_\beta \leq S$ for each $\beta \leq \sigma$. This will show that $G = H_\sigma \leq S$, proving the result.

If $\beta = 0$ then $H = H_0 \leq S$ by definition; therefore the induction begins.

Suppose $\beta = \alpha + 1$ for some $\alpha < \sigma$ and $H_\alpha \leq S$. Then $H_\alpha \leq S \cap H_{\alpha+1}$. Now $H_\alpha \bowtie_{\mathfrak{F}} H_{\alpha+1}$ so that $S \cap H_{\alpha+1} \bowtie_{\mathfrak{F}} H_{\alpha+1}$. Also

$$S \cap H_{\alpha+1}\ \mathfrak{F}\text{-}ser\ H_{\alpha+1} \quad \text{by [5, 4.3 (i)]}.$$

A proper subgroup of a $\mathfrak{K}$-group cannot be both $\mathfrak{F}$-abnormal and $\mathfrak{F}$-serial so we must have $S \cap H_{\alpha+1} = H_{\alpha+1}$. Thus $H_\beta = H_{\alpha+1} \leq S$ and the induction goes through in this case.

If $\lambda \leq \sigma$ is a limit ordinal and $H_\beta \leq S$ for each $\beta < \lambda$ then certainly $H_\lambda = \bigcup_{\beta<\lambda} H_\beta \leq S$. This completes the induction argument and the proof.

As an immediate consequence of 4.5 and [7, 4.1] we have

COROLLARY 4.6. *If D is an $\mathfrak{F}$-normalizer of a $\mathfrak{K}$-group G then $S^{\mathfrak{F}}(G:D) = G$.*

LEMMA 4.7. *Suppose H is an $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}$-group G. Then $b^{\mathfrak{F}}(G:H) \leq 1$.*

Proof. Let $R = \rho(G)$, the Hirsch-Plotkin radical of G. Then $G/R \in \mathfrak{F}$ so, by [5, 5.9 (i)], HR $\mathfrak{F}$-*ser* G. Therefore

$$S_1 = S_1(G:H:\mathfrak{F}) = S^{\mathfrak{F}}(G:H) \leq HR.$$

Since $H \leq S_1$ the modular law gives $S_1 = H(S_1 \cap R)$. Now $H \in \mathfrak{F}$ so, by [5, 4.21], H $\mathfrak{F}$-*ser* $R_{S_1}(H;\ \mathfrak{F}) = R_1(G:H:\mathfrak{F})$. Thus

$$H = S^{\mathfrak{F}}(R_1:H) = S_2(G:H:\mathfrak{F}).$$

Hence

$$H = S_2\ \mathfrak{F}\text{-}ser\ R_1 \bowtie_{\mathfrak{F}} S_1\ \mathfrak{F}\text{-}ser\ G$$

and $b^{\mathfrak{F}}(G:H) \leq 1$.

THEOREM 4.8. *Suppose H is an $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^t\mathfrak{F}$-group G $(t \geq 0)$. Then $b^{\mathfrak{F}}(G:H) \leq t$.*

Proof. We argue by induction on t. If $t = 0$ then $G \in \mathfrak{F}$ and, by [5, 5.9 (i)], H $\mathfrak{F}$-*ser* G. Thus $S_1(G:H:\mathfrak{F}) = H$ and $b^{\mathfrak{F}}(G:H) = 0$.

If $t > 0$ set $R = \rho(G)$. Then HR/R is an $\mathfrak{F}$-subgroup of G/R so by induction $b^{\mathfrak{F}}(G/R:HR/R) \leq t - 1$ and hence

$$S_t(G/R:HR/R:\mathfrak{F}) = HR/R.$$

Now it is clear, from [5, 3.4] and 4.2, that

$$S_i(G/R:HR/R:\mathfrak{F}) = S_i(G:H:\mathfrak{F})R/R,$$

$$R_i(G/R:HR/R:\mathfrak{F}) = R_i(G:H:\mathfrak{F})R/R$$

for each $i \geq 1$, i.e. that the $(R, \mathfrak{F})$-chain of HR/R in G/R is the image in G/R of the $(R, \mathfrak{F})$-chain of H in G. Thus

$$S_t = S_t(G:H:\mathfrak{F}) \leq HR$$

and in particular

$$S_t \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}.$$

Therefore, by 4.7, $b^{\mathfrak{F}}(S_t:H) \leq 1$. Now it is clear from the definitions that $S_1(S_t:H:\mathfrak{F}) = S_t$, $R_1(S_t:H:\mathfrak{F}) = R_t$, $S_2(S_t:H:\mathfrak{F}) = S_{t+1}$. Since $b^{\mathfrak{F}}(S_t:H) \leq 1$ we have $H = S_2(S_t:H:\mathfrak{F}) = S_{t+1}$ and hence $b^{\mathfrak{F}}(G:H) \leq t$, as claimed.

THEOREM 4.9. *If $G \in \mathfrak{K} \cap \mathfrak{A}\mathfrak{F}$ and $H \leq G$ then $b^{\mathfrak{F}}(G:H) \leq 1$.*

Proof. Let $A = G^{\mathfrak{F}}$, the $\mathfrak{F}$-residual of G; A is abelian by hypothesis. Therefore $H \cap A$ is a normal subgroup of AH and, as in the proof of 4.8,

$$b^{\mathfrak{F}}(AH/H \cap A:H/A \cap H) = b^{\mathfrak{F}}(AH:H).$$

Now $H/A \cap H$ is isomorphic to a subgroup of the $\mathfrak{F}$-group G/A so, by 1.3, $H/A \cap H \in \mathfrak{F}$. Thus, by 4.8, $b^{\mathfrak{F}}(AH/H \cap A:H/A \cap H) \leq 1$. Hence

$$b^{\mathfrak{F}}(AH:H) \leq 1.$$

Now $G/A \in \mathfrak{F}$ so, by [5, 5.9(i)], AH $\mathfrak{F}$-*ser* G. Therefore $S^{\mathfrak{F}}(G:H) \leq AH$ and it follows that $S^{\mathfrak{F}}(G:H) = S^{\mathfrak{F}}(AH:H)$. Thus the $(R, \mathfrak{F})$-chain of H in G coincides with the $(R, \mathfrak{F})$-chain of H in AH, so that

$$b^{\mathfrak{F}}(G:H) = b^{\mathfrak{F}}(AH:H) \leq 1,$$

as required.

Remark. Since $a^{\mathfrak{F}}(G:H) \leq b^{\mathfrak{F}}(G:H)$ when the latter is defined, it follows, from the remark after the proof of 3.2, that we cannot hope to extend 4.9 to the case where G is a $\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}$-group.

LEMMA 4.10. *Suppose H is an $\mathfrak{F}$-ascendabnormal $\mathfrak{F}$-subgroup of*

$$G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})^2\mathfrak{F}.$$

Then $b^{\mathfrak{F}}(G:H) \leq 1$.

Proof. By [7, 4.1], H contains an $\mathfrak{F}$-normalizer D of G, and since $H \in \mathfrak{F}$ we have $R_G(H; \mathfrak{F}) \leq R_G(D; \mathfrak{F})$ by [5, 3.11(ii)]. Now $\mathfrak{F}$ is subgroup-closed and $R_G(D; \mathfrak{F})$ is an $\mathfrak{F}$-projector of G by [5, 5.2]. Therefore $R_G(H; \mathfrak{F}) \in \mathfrak{F}$ and, by [5, 5.9(i)], H $\mathfrak{F}$-*ser* $R_G(H; \mathfrak{F})$. Now $S_1(G:H:\mathfrak{F}) = G$ since H is $\mathfrak{F}$-ascendabnormal in G by (4.5), so $R_1(G:H:\mathfrak{F}) = R_G(H; \mathfrak{F})$. Therefore

$$S_2(G:H:\mathfrak{F}) = S^{\mathfrak{F}}(R_G(H:\mathfrak{F}):H) = H.$$

Thus $b^{\mathfrak{F}}(G:H) \leq 1$, as required.

THEOREM 4.11. *Suppose H is an $\mathfrak{F}$-ascendabnormal $\mathfrak{F}$-subgroup of the*

$$\mathfrak{R} \cap (\mathrm{L}\mathfrak{N})^t\mathfrak{F}\text{-}group\ G \qquad (t \geq 2).$$

Then $b^{\mathfrak{F}}(G:H) \leq t - 1$.

Proof. We argue by induction on t, the case $t = 2$ being covered by 4.10. If $t > 2$ and $R = \rho(G)$ then HR/R is an $\mathfrak{F}$-ascendabnormal $\mathfrak{F}$-subgroup of G/R by [7, 4.5] so by induction $b^{\mathfrak{F}}(G/R:HR/R) \leq t - 2$. In particular therefore $S_{t-1}(G/R:HR/R:\mathfrak{F}) = HR/R$. As in the proof of 4.8 we now obtain $S_{t-1}(G:H:\mathfrak{F}) \leq HR$ and in particular

$$S_{t-1}(G:H:\mathfrak{F}) \in \mathfrak{R} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}.$$

The argument used to complete the proof of 4.8 now shows that $S_t(G:H:\mathfrak{F}) = H$. Thus $b^{\mathfrak{F}}(G:H) \leq t - 1$, as claimed.

To show the difference between the invariants $a^{\mathfrak{F}}(G:H)$ and $b^{\mathfrak{F}}(G:H)$ we have

THEOREM 4.12. *Suppose D is an $\mathfrak{F}$-normalizer of the $\mathfrak{R}$-group G. Then $a^{\mathfrak{F}}(G:D) = b^{\mathfrak{F}}(G:D)$ if and only if D has a strong $\mathfrak{F}$-serializer in G.*

Proof. If $G \in \mathfrak{F}$ then $D = G$ and $a^{\mathfrak{F}}(G:D) = b^{\mathfrak{F}}(G:D) = 0$, so there is nothing to prove. We may therefore suppose that $G \notin \mathfrak{F}$. Then $a^{\mathfrak{F}}(G:D) = 1$ by 3.4.

By 4.6, $S_1(G:D:\mathfrak{F}) = G$ so that

$$R_1(G:D:\mathfrak{F}) = R_G(D:\mathfrak{F}) \quad \text{and} \quad S_2(G:D:\mathfrak{F}) = S^{\mathfrak{F}}(R_G(D:\mathfrak{F}):D).$$

Thus

$$\begin{aligned} a^{\mathfrak{F}}(G:D) = b^{\mathfrak{F}}(G:D) &\Leftrightarrow b^{\mathfrak{F}}(G:D) = 1 \\ &\Leftrightarrow S_2(G:D:\mathfrak{F}) = D \\ &\Leftrightarrow D\ \mathfrak{F}\text{-}ser\ R_G(D; \mathfrak{F}) \\ &\Leftrightarrow D \text{ has a strong } \mathfrak{F}\text{-serializer in } G\ [5, 4.17] \end{aligned}$$

Suppose D is the $\mathfrak{F}$-normalizer of the $\mathfrak{R}$-group G associated with the Sylow basis $\mathbf{S}$ of G. The $(R, \mathfrak{F})$-chain of D in G is, in some respects, similar to the second convergence process of G for the Sylow basis $\mathbf{S}$. We consider briefly the question of whether there is any relation between $b^{\mathfrak{F}}(G:D)$ and $i_{\mathfrak{F}}(G)$.

Firstly $i_{\mathfrak{F}}(G)$ is not bounded in terms of $b^{\mathfrak{F}}(G:D)$. For take $\mathfrak{R} = \mathfrak{S}^*$

and $\mathfrak{F} = \mathfrak{N}^*$. If D is a basis normalizer of an A-group G then D is pronormal in G by [12, 2.4] so, by [5, 3.22], $R_G(D) = N_G(D)$. Thus, by 4.6,

$$S_1(G:D:\mathfrak{N}^*) = G, \qquad R_1(G:D:\mathfrak{N}^*) = R_G(D),$$

$$S_2(G:D:\mathfrak{N}^*) = S^{\mathfrak{N}^*}(R_G(D):D) = D$$

as $D \lhd R_G(D)$. Hence $b^{\mathfrak{N}^*}(G:D) \leq 1$. Now the groups G_i, in Example 2.13, are A-groups and for each $n \geq 0$, $i_{\mathfrak{N}^*}(G_{2n+1}) = n$. Thus if D_i is a basis normalizer of G_i then we have $b^{\mathfrak{N}^*}(G_i:D_i) \leq 1$ for all i but the $\mathfrak{N}^*$-speeds $i_{\mathfrak{N}^*}(G_i)$ are unbounded. Thus $i_{\mathfrak{F}}(G)$ is in general not bounded by some function of $b^{\mathfrak{F}}(G:D)$. We leave open the question of whether $b^{\mathfrak{F}}(G:D)$ is bounded in terms of $i_{\mathfrak{F}}(G)$.

We close this section by considering briefly a generalization of another of Rose's concepts [14].

If $H \leq G \in \mathfrak{K}$ we say H is *$\mathfrak{F}$-contranormal* in G if $S^{\mathfrak{F}}(G:H) = G$, i.e. if H is a subgroup of no proper $\mathfrak{F}$-serial subgroup of G.

By 4.5 every $\mathfrak{F}$-ascendabnormal subgroup is $\mathfrak{F}$-contranormal.

LEMMA 4.13. *If H is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}$-group G then H lies in some $\mathfrak{F}$-projector of G.*

Proof. Let $R = \rho(G)$. Then $G/R \in \mathfrak{F}$ so, by [5, 5.9(i)], HR $\mathfrak{F}$-*ser* G. Since H is $\mathfrak{F}$-contranormal in G we must have $HR = G$. The result is now immediate from [2, 5.10].

LEMMA 4.14. *If $G \in \mathfrak{K} \cap \mathfrak{A}\mathfrak{F}$ then the $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroups of G are precisely the $\mathfrak{F}$-projectors of G.*

Proof. The $\mathfrak{F}$-projectors of G are $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroups of G by [7, 3.5] and 4.5. On the other hand suppose H is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of G. Let $A = G^{\mathfrak{F}}$, the $\mathfrak{F}$-residual of G; by hypothesis A is abelian. Now $G/A \in \mathfrak{F}$ so, by [5, 5.9(i)], HA $\mathfrak{F}$-*ser* G. Since H is $\mathfrak{F}$-contranormal in G we must therefore have $G = HA$. Now H is contained in some $\mathfrak{F}$-projector E of G by 4.13, so by the modular law $E = H(E \cap A)$. But E complements A in G by [2, 4.12 and 5.1]. Therefore $E = H$, and the proof is complete.

Remark. If G is a finite soluble group then a subgroup H of G is $\mathfrak{N}^*$-contranormal in G if and only if H lies in no proper normal subgroup of G, i.e. if and only if the normal closure H^G of H in G is G. Thus for $\mathfrak{S}^*$-groups the concepts "$\mathfrak{N}^*$-contranormal" and "contranormal" (as defined in [14]) coincide.

In his paper, [14], Rose gives an example to show that $(\mathfrak{N}^*)^2$-groups may have nilpotent contranormal subgroups which are not Carter subgroups. We cannot therefore hope to improve 4.14 to the case where $G \in \mathfrak{K} \cap (\mathrm{L}\mathfrak{N})\mathfrak{F}$.

Using Lemma 4.14 we can sharpen 4.13 to give

LEMMA 4.15. *If H is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\text{L}\mathfrak{N})\mathfrak{A}\mathfrak{F}$-group G then H lies in some $\mathfrak{F}$-projector of G.*

Proof. Let $R = \rho(G)$. Then HR/R is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of G/R by 4.2. Thus, by 4.14, HR/R is an $\mathfrak{F}$-projector of G/R. By [2, 5.10], H lies in an $\mathfrak{F}$-projector E of HR, and since E is an $\mathfrak{F}$-projector of G by [2, 5.3] we have the desired result.

Our final result sharpens 3.1 and 4.8.

THEOREM 4.16. *If H is an $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\text{L}\mathfrak{N})^t\mathfrak{A}\mathfrak{F}$-group $G\,(t \geq 1)$ then $b^{\mathfrak{F}}(G:H) \leq t$.*

Proof. We argue by induction on t.

If $t = 1$, then H is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of the

$$\mathfrak{K} \cap (\text{L}\mathfrak{N})\mathfrak{A}\mathfrak{F}\text{-group } S_1 = S_1(G:H:\mathfrak{F}) = S^{\mathfrak{F}}(G:H).$$

Thus, by 4.15, H lies in an $\mathfrak{F}$-projector E of S_1. Let R be the Hirsch-Plotkin radical of S_1. Then, by 4.2, HR/R is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of S_1/R so, by 4.14, $HR = ER$. Hence $E = H(E \cap R)$. Now applying [5, 2.17 (iii)] we have $R_{S_1}(H; \mathfrak{F}) \leq R_{S_1}(E; \mathfrak{F})$. From [5, 3.18 (i)] we obtain $R_{S_1}(H; \mathfrak{F}) \leq E$. Now $E \in \mathfrak{F}$ so, by [5, 3.11 (i)], $E \leq R_{S_1}(H; \mathfrak{F})$. Thus

$$R_1 = R_1(G:H:\mathfrak{F}) = R_{S_1}(H; \mathfrak{F}) = E.$$

But H $\mathfrak{F}$-*ser* E by [5, 5.9 (i)], so we have $S_2(G:H:\mathfrak{F}) = H$, whence $b^{\mathfrak{F}}(G:H) \leq 1$ and the induction begins.

If $t > 1$ and $Y = \rho(G)$ then HY/Y is an $\mathfrak{F}$-subgroup of the

$$\mathfrak{K} \cap (\text{L}\mathfrak{N})^{t-1}\mathfrak{A}\mathfrak{F}\text{-group } G/Y,$$

so by induction $b^{\mathfrak{F}}(G/Y:HY/Y) \leq t - 1$. Thus

$$S_t(G/Y:HY/Y:\mathfrak{F}) = HY/Y.$$

The argument at the end of the proof of 4.8 now shows that $S_{t+1}(G:H:\mathfrak{F}) = H$. Thus $b^{\mathfrak{F}}(G:H) \leq t$, which completes the proof.

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REFERENCES

1. R. W. CARTER, *Nilpotent self-normalizing subgroups and system normalizers*, Proc. London Math. Soc. (3), vol. 12 (1962), pp. 535–563.
2. A. D. GARDINER, B. HARTLEY AND M. J. TOMKINSON, *Saturated formations and Sylow structure in locally finite groups*, J. Algebra, vol. 17 (1971), pp. 177–211.

3. C. J. GRADDON, *$\mathfrak{F}$-reducers in finite soluble groups*, J. Algebra, vol. 18 (1971), pp. 574–587.
4. ———, *The relation between $\mathfrak{F}$-reducers and $\mathfrak{F}$-subnormalizers in finite soluble groups*, J. London Math. Soc. (2), vol. 4 (1971), pp. 51–61.
5. ———, *$\mathfrak{F}$-reducers and $\mathfrak{F}$-serializers in certain locally finite groups*, to appear.
6. C. J. GRADDON AND B. HARTLEY, *Basis normalizers and Carter subgroups in a class of locally finite groups*, Proc. Cambridge Philos. Soc., vol. 71 (1972), pp. 189–198.
7. B. HARTLEY, *$\mathfrak{F}$-abnormal subgroups of certain locally finite groups*, Proc. London Math. Soc. (3), vol. 23 (1971), pp. 128–158.
8. T. O. HAWKES, *An example in the theory of soluble groups*, Proc. Cambridge Philos. Soc., vol. 67 (1970), pp. 13–16.
9. B. HUPPERT, *Endliche Gruppen I*, Springer-Verlag, Berlin-Heidelberg, 1967.
10. A. MANN, *System normalizers and subnormalizers*, Proc. London Math. Soc. (3), vol. 20 (1970), pp. 123–143.
11. A. MANN, *On subgroups of finite soluble groups II*, to appear.
12. J. S. ROSE, *Finite soluble groups with pronormal system normalizers*, Proc. London Math. Soc. (3), vol. 17 (1967), pp. 447–469.
13. ———, *Abnormal depth and hypereccentric length in finite soluble groups*, Math. Zeitschr., vol. 90 (1965), pp. 29–40.
14. ———, *Nilpotent subgroups of finite soluble groups*, Math. Zeitschr., vol. 106 (1968), pp. 97–112.

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