The purpose of this paper is to relate several generalizations of the notion of the Heegaard genus of a closed 3-manifold to compact, orientable 3-manifolds with connected, nonempty boundary.

All spaces considered will be polyhedra and all maps will be piecewise linear. By a solid torus of genus $n$ we mean a space homeomorphic to a regular neighborhood in $\mathbb{R}^3$ of a compact, connected graph with Euler characteristic $1 - n$. The Euler characteristic of any space $X$ will be denoted $\chi(X)$. If $D$ is a 2-cell, then $N(D)$ will denote a space homeomorphic to $D \times [-1, 1]$ where $D$ corresponds to $D \times \{0\}$.

It is well known that any compact, orientable 3-manifold with nonempty connected boundary can be represented as $H \cup N(D_1) \cup \cdots \cup N(D_k)$ where $H$ is a solid torus, $D_i$ is a 2-cell for each $i$, $N(D_i) \cap N(D_j) = \emptyset$ if $i \neq j$ and $N(D_i) \cap H = \partial N(D_i) \cap \partial H$ corresponds to $\partial D_i \times [-1, 1]$ in $N(D_i)$. This will be called a Heegaard splitting (or H-splitting) for $M$, and $N(D_i)$ is called a handle of index 2. The genus of the splitting is the genus of $H$ and the smallest possible genus of an H-splitting of $M$ will be denoted $HG(M)$.

Downing [1] has shown that $M$ may also be represented as $H_1 \cup H_2$ where $H_1$ and $H_2$ are solid tori of the same genus and $H_1 \cap H_2 = \partial H_1 \cap \partial H_2$. This may always be done so that $\partial H_j \cap \partial M$ is a disk with holes such that $\pi_1(\partial H_j \cap \partial M)$ injects naturally onto a free factor of $\pi_1(H_j)$ for $j = 1, 2$. In this case, we call this an SD-splitting of $M$ and denote the minimal genus of such a splitting for $M$ by $SD(M)$. If we require only that $\pi_1(\partial H_j \cap \partial M)$ injects naturally into $\pi_1(H_j)$, $j = 1, 2$, we call this a D-splitting and the minimal genus of any D-splitting for $M$ is denoted $DG(M)$. If $X$ is a subspace of $Y$, $N_Y(X)$ will denote a regular neighborhood of $X$ in $Y$ taken to be “small” with respect to all previously chosen objects in a given argument. The closure of any set $A$ will be denoted $\text{Cl}(A)$.

If $F$ is a compact orientable surface of genus $g$ with $k$ boundary components, then $\chi(F) = 2 - 2g - k$ and $\pi_1(F)$ is free of rank $2g + k - 1$.

**Theorem 1.** Let $M$ be a compact, orientable 3-manifold with connected nonempty boundary of genus $k$. Let $M = H_1 \cup H_2$ be an SD-splitting of $M$ of genus $n$. Then $M$ has an H-splitting of genus $n$.

**Proof.** Let $K_i = \partial H_i \cap \partial M$ ($i = 1, 2$). Then each $K_i$ is a disk with $k$ holes and $\mu*(\pi_1(K_i))$ is a free factor of $\pi_1(H_i)$ where $\mu* : \pi_1(K_i) \to \pi_1(H_i)$ is induced by inclusion. Now choose simple closed curves $\alpha_1, \ldots, \alpha_k$ in

Received September 27, 1971.
int \left(K_1\right) which meet only in the base point and which generate \( \pi_1(K_1) \). This may be done so that the closure of each component of \( K_1 - N_{K_1}(\bigcup_{i=1}^l \alpha_i) \) is an annulus one of whose boundary components is a component of \( \partial K_1 \). Then [6] there exist properly embedded disks \( D_1, \ldots, D_k \) in \( \partial K_1 \) so that

\[
\text{Cl} \left( H_1 - \bigcup_{i=1}^k N_{\partial H_i}(D_i) \right)
\]

is a solid torus of genus \( (n - k) \), \( D_i \cap \alpha_i \) is a point for each \( i \), and \( \partial D_i \cap \alpha_i = \emptyset \) if \( i \neq j \). Then, by an isotopy if necessary, \( D_j \cap K_1 = \partial D_j \cap K_1 \) may be taken to be a single simple arc properly embedded in \( K_1 \).

For \( j = 1, \ldots, k \), let \( \beta_j = \text{Cl} \left( \partial D_j - K_1 \right) \). Then \( \beta_j \) is a simple arc in \( \partial D_j \cap \partial H_2 \). Now we find pairwise disjoint, properly embedded disks \( D_{k+1}, \ldots, D_n \) in \( H_1 \) so that \( \text{Cl} \left( H_1 - \bigcup_{i=1}^n N_{\partial \partial H_i}(D_i) \right) \) is a 3-cell. Since

\[
\text{Cl} \left( K_1 - \bigcup_{i=1}^k N_{\partial H_i}(D_i) \right)
\]

is a disk, we may assume \( D_j \cap K_{k+1} = \emptyset \) for \( j = k + 1, \ldots, n \).

Now \( H_2 \cup (\bigcup_{i=1}^k N_{\partial H_i}(D_i)) \cong H_2 \cup (\bigcup_{i=k+1}^n N_{\partial H_i}(D_i)) \) is a solid torus of genus \( n \) with \( (n - k) \) handles of index 2 attached and \( \text{Cl} \left( H_1 - \bigcup_{i=1}^n N_{\partial H_i}(D_i) \right) \) is a 3-cell meeting this in a 2-cell on their common boundary. Hence,

\[
M \cong H_2 \cup (\bigcup_{i=1}^n N_{\partial H_i}(D_i)) \cong H_2 \cup (\bigcup_{i=k+1}^n N_{H_i}(D_i)).
\]

**Corollary.** If \( M \) is a compact, orientable 3-manifold with connected, non-empty boundary, then \( HG(M) \leq SD(M) \).

**Theorem 2.** Let \( M \) be a compact, orientable 3-manifold with connected, non-empty boundary of genus \( k \). Suppose \( M = H \cup N(D_1) \cup \cdots \cup N(D_{n-k}) \) is an \( H \)-splitting for \( M \) of genus \( n \). Then \( M \) has a \( D \)-splitting of genus \( n \).

**Proof.** If \( n - k = 0 \), the result is trivial, so assume \( n - k \geq 1 \). Let

\[
S = \text{Cl} \left( \partial H - \bigcup_{i=1}^{n-k} N(D_i) \right).
\]

Then \( S \) is an orientable surface of genus \( k \) with \( 2(n - k) \) boundary components, say \( \alpha_1, \beta_1, \ldots, \alpha_{n-k}, \beta_{n-k} \) where \( \alpha_i \cup \beta_i \subseteq \partial N(D_i) \) for \( i = 1, \ldots, n - k \).

Now we choose simple, properly embedded, pairwise disjoint arcs \( \gamma_1, \ldots, \gamma_n \) in \( S \) so that each \( \gamma_i \) joins some \( \alpha_i \) to \( \beta_i \) and \( T' = \text{Cl} \left( S - \bigcup_{i=1}^n N_{\partial \partial H_i}(\gamma_i) \right) \) is connected. Now \( \chi(S) = 2 - 2n \) and \( \chi(T') = 2 - 2n + n = 2 - n \). This may be done so that \( T' \) has \( n \) boundary components and is a surface of genus \( 0 \). Now, as indicated in Figure 1, choose properly embedded, pairwise disjoint arcs \( \delta_1, \ldots, \delta_{n-k+1} \) in \( T' \) so that each \( \delta_i \) joins some \( \gamma_j \) to \( \gamma_r \) \((j \neq r)\) and \( T = \text{Cl} \left( T' - \bigcup_{i=1}^{n-k} N_{\partial \partial H_i}(\delta_i) \right) \) is connected. Then \( T \) is a disk with \( k \) holes and the inclusion induced homomorphism \( \mu \ast : \pi_1(T) \to \pi_1(S) \) is an injection.

Now we assume that the inclusion induced homomorphism \( \nu \ast : \pi_1(S) \to \pi_1(H) \) is an injection. Then \( \nu \ast \mu \ast : \pi_1(T) \to \pi_1(H) \) is an injection. Let \( H_1 = \left( \bigcup_{i=1}^{n-k} N(D_i) \right) \cup \left( \bigcup_{i=1}^n N_H(\gamma_i) \right) \cup \left( \bigcup_{i=1}^{n-k} N_H(\delta_i) \right) \)
where
\[
\left( \bigcup_{i=1}^{n} N_{H}(\gamma_{i}) \right) \cup \left( \bigcup_{i=1}^{n-k} N_{H}(\delta_{i}) \right) \cap S = \left( \bigcup_{i=1}^{n} N_{S}(\gamma_{i}) \right) \cup \left( \bigcup_{i=1}^{n-k} N_{T}(\delta_{i}) \right).
\]

Let \( H_{2} = \text{Cl}(H - H_{1}) \). Then \( H_{1} \) and \( H_{2} \) are solid tori of genus \( n \) and \( M = H_{1} \cup H_{2} \).

Since the pair \( (H_{2}, H_{2} \cap \partial M) \) is homeomorphic to \( (H, T) \), we have that \( \pi_{1}(H_{2} \cap \partial M) \) injects into \( \pi_{1}(H_{2}) \). Now

\[
H_{1} \cap \partial M = \left( \bigcup_{i=1}^{n-k} (D_{i} \times \{-1, 1\}) \right) \cup \left( \bigcup_{i=1}^{n} N_{S}(\gamma_{i}) \right) \cup \left( \bigcup_{i=1}^{n-k} N_{T}(\delta_{i}) \right)
\]
is connected, has \( k + 1 \) boundary components and \( \chi(H_{1} \cap \partial M) = 2 - (k + 1) \).

Hence, \( H_{1} \cap \partial M \) is a disk with \( k \) holes. By the construction of \( H_{1} \) we also have that the inclusion induced homomorphism \( \pi_{1}(H_{1} \cap \partial M) \rightarrow \pi_{1}(H_{1}) \) is injective. Hence, \( M \) has a \( D \)-splitting of genus \( n \).

If \( \nu^{*} : \pi_{1}(S) \rightarrow \pi_{1}(H) \) is not injective, we find by Dehn's lemma [5] and the loop theorem [4] a simple closed curve \( J \) in \( S \) that does not contract in \( S \) but bounds a disk \( E \) in \( H \). Cutting along \( E \), either we separate \( M \) into manifolds \( M_{1} \) and \( M_{2} \) with \( H \)-splittings of genuses \( n_{1}, n_{2} \) (both > 0) so that \( n_{1} + n_{2} = n \) or we remove a handle of index 1 from \( M \) to get a manifold \( M_{1} \) with an \( H \)-splitting of genus \( n - 1 \).
Now by [2], if $H_1 \cup H_2$ is a $D$-splitting for $M_i$, any disk or pair of disks in $\partial M_i$ can by an isotopy be assumed to meet $H_j \cap \partial M_i$ in a disk for $j = 1, 2$. Hence, by induction on $n$ and the fact that the theorem is trivial if $n = 1$, we are finished. □

**Corollary.** If $M$ is a compact, orientable 3-manifold with connected, non-empty boundary, then $DG(M) \leq HG(M)$.

We now give a partial converse to Theorem 1.

**Proposition 3.** Let $M$ be a compact, orientable 3-manifold with connected, nonempty boundary of genus $k$. Let $M = H \cup N(D_1) \cup \cdots \cup N(D_{n-k})$ be an $H$-splitting for $M$ of genus $n$. Suppose $K$ is a surface of genus 0 with $k + 1$ boundary components in $\partial H - \bigcup N(D_i)$. Further assume that the inclusion induced map $\pi_1(K) \rightarrow \pi_1(H)$ is an injection onto a free factor of $\pi_1(H)$ and that

$$
\partial H - (K \cup N(D_1) \cup \cdots \cup N(D_{n-k}))
$$

is connected. Then $M$ has an $SD$-splitting of genus $n$.

**Proof.** Let $H'$ be a solid torus of genus $n$ as in Figure 2. For each $i = 1, \ldots, n - k$, let $J_i$ be a simple closed curve in $N(D_i) \cap H$ so that $N(D_i) \cap H = N_{\partial H}(J_i)$. Then there is a homeomorphism $h : \partial H' - \text{Int } A' \rightarrow \partial H - \text{Int } K$ such that $h(\partial E_i) = J_i$ for $i = 1, \ldots, n - k$.

Let $M' = H \cup h H'$. Then this gives an $SD$-splitting of $M'$ of genus $n$. However, $H'$ collapses to $(\partial H' - \text{Int } A') \cup E_1 \cup \cdots \cup E_{n-k}$ and so $M'$ collapses to $H \cup E_1 \cup \cdots \cup E_{n-k}$. Hence $M'$ is homeomorphic to $M$. □

**Corollary.** Let $M = H \cup N(D)$ where $H$ is a solid torus of genus 2 and
\[ \partial M \text{ is connected. Suppose } K \text{ is a simple closed curve in } \partial H = N(D) \text{ which represents a primitive element for } \pi_1(H). \text{ Then } M \text{ has an } SD\text{-splitting of genus 2.} \]

**Proof.** If \( \partial H = (N(D) \cup K) \) is not connected, then \( K \) and one component of \( \partial N(D) \cap \partial H \) cobound an annulus. Hence, \( \partial N(D) \cap \partial H \) represents a primitive element of \( \pi_1(H) \) and we may choose a new curve \( K' \) which represents a complementary primitive element. Therefore we may assume that \( \partial H = (N(D) \cup K) \) is connected and Proposition 3 may be applied.

The author thanks Professor John Hempel for pointing out Proposition 3.

**References**


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