SPACES OF H-STRUCTURES

BY

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Here we initiate the study of the homotopy groups in all dimenions of the space of H-structures on a given CW-complex Y. Calculations are offered for some of these groups in case Y is a finite Postnikov system or a sphere, and a representation of these groups is introduced into

$$\text{Hom } (\pi_* Y \otimes \pi_* Y \rightarrow \pi_* Y)$$

which in several senses generalizes the Samelson product.

In §1, we examine three sets of functions which are candidates for "the" set of H-structures on a complex Y, each of which may be endowed with the c-o or the k topology; it is shown that these six function spaces are all weakly homotopy equivalent. Four of these spaces are nonempty even when Y is not an H-space, yet weak equivalences persist among these four in that case.

In §2, a lemma is established which describes the set of components of the mapping space $\{X \wedge Y \rightarrow Z\}$ for certain complexes X, Y, Z. This lemma is immediately used to count the number of H-structures on Y when $\pi_i Y = 0$ unless $1 \leq n \leq i \leq 2n$ for some interger n: there is an isomorphism

$$\Phi : \pi_0\{Y \wedge Y \rightarrow Y\} \cong \text{Hom } (\pi_n Y \otimes \pi_n Y \rightarrow \pi_{2n} Y).$$

In §3 we argue that this function Φ may be considered as a homomorphism

$$\Phi : \pi_q\{Y \wedge Y \rightarrow Y\} \rightarrow \text{Hom } (\pi_r Y \otimes \pi_s Y \rightarrow \pi_{q+r+s} Y),$$

defined for all spaces Y and all integers q, r and s, which includes Samelson products. A generalization of James' separation element is defined in order to express Φ in terms of the space of functions

$$\{Y \times Y \xrightarrow{f} Y : f \mid Y \vee Y = 1 \vee 1\};$$

in this setting Φ includes a homotopy precursor of the binary homology operarions over H_n-spaces of W. Browder. Calculations are made for Φ when Y is a sphere; if Y has a finite Postnikov system then $\{Y \wedge Y \rightarrow Y\}$ is shown also to have bounded homotopy and its highest-dimensional nontrivial homotopy group is given. We conclude with a comment on the additional structure carried by Φ.

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1. The function spaces

We shall work in the category of pointed CW-complexes, with the obvious exceptions of functions spaces. All direct products and function spaces will

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be given the "convenient" k topology of R. Brown and Steenrod; this means that, in forming $X \times Y$ or $\{X \to Y\}(= Y^X)$, the direct product or compact-open topology is enlarged to its compactly-generated topology [4], [11]. This change of topology does not change the class of compact subsets of a space, and therefore does not affect its weak homotopy type. However, with the compactly-generated topology,

$$X \times Y \quad \text{and} \quad X \wedge Y = X \times Y/X \vee Y$$

are always CW-complexes, without further assumptions about X or Y. Similarly, the k topology on a function space always renders the evaluation map continuous, and the exponential law always holds; for our purposes this is merely an expository convenience, since our constructions can be shown to result in continuous functions even if the c-o topology is used.

Let Y be a space (i.e., CW-complex); the "folding" map $\varphi : Y \vee Y \to Y$ is defined by $\varphi(x, *) = x = \varphi(*, x)$, where $*$ is the distinguished point of Y; let κ denote the constant map $Y \vee Y \to Y$. We shall interrelate three function spaces:

$$\mathfrak{F}(Y) \text{ or } \mathfrak{F} = \{Y \wedge Y \xrightarrow{f} Y\},$$

$$\mathcal{G}(Y) \text{ or } \mathcal{G} = \{Y \times Y \xrightarrow{g} Y : g \mid Y \vee Y = \kappa\},$$

$$\mathcal{H}(Y) \text{ or } \mathcal{H} = \{Y \times Y \xrightarrow{h} Y : h \mid Y \vee Y = \varphi\};$$

here the base points for $\mathfrak{F}$ and $\mathcal{G}$ are constant maps; the base point for $\mathcal{H}$ is arbitrary. Thus $\mathcal{H}$ is the space of H-structures on Y (which have an exact identity); $\mathcal{H}$ may be empty, of course, but $\mathfrak{F}$ and $\mathcal{G}$ can hold interest even when Y is not an H-space (e.g., $\pi_i \mathfrak{F}(S^2) \cong \pi_{4+i}(S^2)$).

The quotient map $q : (Y \times Y, Y \vee Y) \to (Y, *)$ induces a map $q^\# : \mathfrak{F} \to \mathcal{G}$ which is a 1–1 correspondence. Conveniently, $q^\#$ is a homeomorphism between the k-topologicalized spaces $\mathfrak{F}$ and $\mathcal{G}$ [11, Lemma 5.10]. (This implies, of course, that $q^\#$ is a weak homotopy equivalence between those spaces endowed with their c-o-topologies; in fact, it is not difficult to show that $q^\#$ is a homotopy equivalence in this case.)

Suppose that Y is an H-space with multiplication m which we take as base point for $\mathcal{H}$. Then there is a map $m^\# : \mathcal{G} \to \mathcal{H}$,

$$m^\#(g)(y, z) = m(m[y, z], g[y, z]).$$

If (Y, m) were a topological group, with $m^{-1} : Y \to Y$ the exact inversion for m, then an inverse map $m_\flat$ to $m^\#$ would be given by

$$m_\flat(h)(y, z) = m(m[m^{-1}(z), m^{-1}(y)], h[y, z)],$$

and $m^\#$ would be a homeomorphism. It is still possible to define a map $m_\# : \mathcal{H} \to \mathcal{G}$ when m is merely an H-structure: James [7] has shown that m has a left homotopy inverse m^{-1}, and in the above definition of $m_\flat$, its values $m_\flat(h)$ are, when restricted to $Y \vee Y$, all equal and homotopic, via H_m, to the constant map. The AHEP of $Y \vee Y$ in $Y \times Y$ is equivalent to the existence of a retraction

$$r : Y \times Y \times I \to Y \times Y \times 0 \cup (Y \vee Y) \times I$$

and we have a map on the range of r; let

$$m_\#(h) = ([m_\flat(h) \times 0] \cup H_m) \circ r \circ (1_{Y\times Y} \times 1).$$

We conjecture that $m_\#$ is a homotopy inverse to $m^\#$; an attack on this problem might proceed by strengthening James' theorem, that two maps $g, g' : K \to Y$ into an H-space, Y, which are homotopic and which agree on a retractile subcomplex L are homotopic rel L [7], to assert that the homotopies rel L may be chosen continuously as g' varies in $\{K \to Y\}$. However, we content ourselves with a weaker result (compare [1], [2], [9]); our proof uses a weak trick with the theorem of James which is quoted above.

PROPOSITION 1. *If (Y, m) is an H-space then $m^\# : \mathcal{G} \to \mathcal{H}$ is a weak homotopy equivalence.*

Proof. Let L be a retractile subcomplex of K and let $L \xrightarrow{i} K \xrightarrow{j} (K, L)$ be the inclusions; then the induced sequence of loops

$$0 \to [K, L \to Y, *] \to [K \to Y] \to [L \to Y]$$

is exact (our bracket notation $[K \to Y]$ means $\pi_0\{K \to Y\}$ as usual): at $[K, L \to Y, *]$, exactness is just the property which motivated James' definition of retractile subcomplexes; at $[K \to Y]$, exactness is the homotopy extension property. Furthermore, it is a standard fact about loops that if $\psi : K \to Y$ then left multiplication by ψ in $[K \to Y]$ defines a bijection between $\operatorname{Im}(j*) = \operatorname{Ker}(i^*)$ and $i^{*-1}(\psi \mid L)$; hence $i^{*-1}(\psi \mid L)$ is in bijective correspondence with $[K, L \to Y, *]$.

Now notice that the inclusion

$$k : [K \xrightarrow{\theta} Y : \theta \mid L = \psi \mid L] \to i^{*-1}(\psi \mid L)$$

is 1-1 since L is retractile in K, and is onto by the AHEP. Our proposition follows, then, if we take K to be $S^n \times Y \times Y$, let

$$L = * \times Y \times Y \cup S^n \times * \times Y \cup S^n \times Y \times *,$$

and define $\psi : K \to Y$ by $\psi(x, y, z) = m(y, z)$.□

We remark that (in either the k or the c-o topology) the homotopy type of $\mathcal{F}$ and $\mathcal{G}$ is a homotopy invariant of Y for standard reasons; thus the weak type of $\mathcal{H}$ is an invariant of the type of Y.

2. Multiplications on short Postnikov systems

D. W. Kahn [8] has given necessary and sufficient conditions that Y be an H-space, at least for countable and 1-connected complexes, in terms of H-structures on the stages in a Postnikov decomposition of Y and its k-invariants. And it is a folk theorem that the multiplication on an Eilenberg-MacLane space is unique, $\pi_0 \mathcal{H}(K[\pi, n]) = 0$. Copeland [5] has extended this to show, for associative and inversive H-spaces Y which have two nontrivial homotopy groups in dimensions n and m, $1 < n < m$, that $\pi_0 \mathcal{H}Y$ is in 1-1 correspondence with $H^m(Y \wedge Y, \pi_m Y)$. This latter group is, in general, difficult to calculate, although Curjel [6] computes $\pi_0 \mathcal{H}(S^1 \times K[Z, 2])$ to be the integers Z; our next theorem agrees with his result.

THEOREM 2. *Let Y be a space with $\pi_i Y = 0$ unless $1 \leq n \leq i \leq 2n$ for some integer n; then there exists a bijection*

$$\Phi : \pi_0 \mathfrak{F}Y \to \text{Hom}\,(\pi_n Y \otimes \pi_n Y \to \pi_{2n} Y).$$

This correspondence is homomorphic if Y is an H-space.

(Here and throughout this note, π_1 refers to the fundamental group made abelian.) The proof of this theorem is immediate to the following lemma.

LEMMA 3. *Let X be $(p\text{-}1)$-connected and Y be $(q\text{-}1)$-connected, and let Z be a space such that $\pi_i Z = 0$ for $i > p + q$. Then there exists a bijection*

$$\Phi : [X \wedge Y \to Z] \cong \text{Hom}\,(\pi_p X \otimes \pi_q Y \to \pi_{p+q} Z)$$

which is a homomorphism if its domain has the natural group structure defined by a suspension structure for X or by an H-structure on Z.

Proof. We shall define a function Φ taking each map

$$a : X \wedge Y \to Z$$

to an appropriate homomorphism; our notation will confuse a map with its homotopy class: if $b : S^p \to X$ and $c : S^q \to Y$, we define $\Phi(a)(b \otimes c)$ to be $a \circ (b \wedge c)$. Clearly Φa is bilinear since the smash product is bilinear and composition is linear on the right; its domain is S^{p+q}, via a fixed homeomorphism from S^{p+q} to $S^p \wedge S^q$. Furthermore, if Z has an H-structure then pointwise operations in the domain of Φ correspond to the group operation among its values. Thus Φ is also linear in a when $X = SX'$ is a suspension because

$$[SX' \wedge Y \to Z] \cong [X' \wedge Y \to \Omega Z]$$

and ΩZ has an H-structure.

Let $\alpha \in \text{Hom}\,(\pi_p X \otimes \pi_q Y \to \pi_{p+q} Z)$; we wish to construct a function Θ, an inverse to Φ, so that $\Theta\alpha = a \in [X \wedge Y \to Z]$. For each of the spaces X, Y and Z we choose a cell structure using E. Brown's representation of the functors π^X, π^Y and π^Z, so that, for example, the $(p\text{-}1)$-skeleton $X^{(p-1)} = *$, $X^{(p)}$ is the wedge of p-spheres e_b^p corresponding to generators $b \in \pi_p X$, the $(p + 1)$-cells are either spheres e_d^{p+1} corresponding to generators $d \in \pi_{p+1} X$ or else cells of the form e_r^{p+1}, attached by maps on their boundaries which realize generating relationships r in the kernel of $i_* : \pi_p[X^{(p)}] \to \pi_p X$, where $i : X^{(p)} \subset X$, and so on (see [10, pp. 406–410] for details). These cell structures on X and Y in turn visit a cell structure on $X \times Y$, $X \vee Y$, and so $X \wedge Y$; the cells of least positive dimension in $X \wedge Y$ are those of $X \times Y$ which are not in $X \vee Y$. That is, a cell of smallest positive dimension in $X \wedge Y$ must be of the form $e_b^p \wedge e_c^q$, where $b \in \pi_p X$ and $c \in \pi_q Y$; its dimension is $p + q$. We define $\Theta\alpha = a$ inductively: each cell $e_b^p \wedge e_c^q$ of $(X \wedge Y)^{(p+q)}$ is attached by a constant map, and so is a $(p + q)$ sphere; our map a is chosen to be a map of degree 1 from $e_b^p \wedge e_c^q$ to $e_{\alpha(b\otimes c)}^{p+q}$, where $\alpha(b \otimes c) \in \pi_{p+q} Z$ (here we may assume that $\alpha(b \otimes c)$ is in the generating set for $\pi_{p+q} Z$ which was used to build Z).

The map a is now extended to the $(p + q + 1)$ cells of $X \wedge Y$: let a be constant on each such cell of the form $e_d^{p+1} \wedge e_c^q$, $d \in \pi_{p+1} X$, or the form $e_d^p \wedge e_d^{q+1}$ $d \in \pi_{q+1} Y$ (these cells have constant attaching maps). The map a may now be extended to a $(p + q + 1)$ cell of the form $e_r^{p+1} \wedge e_c^q$ iff the previously defined map a on the $(p + q)$ skeleton has a composition with the attaching map

$$\partial(e_r^{p+1} \wedge e_c^q) \to (X \wedge Y)^{(p+q)}$$

which is nul-homotopic. Express r as $\sum r_i b_i$, a linear combination in the kernel of

$$i_* : \pi_p[X^{(p)}] \to \pi_p X;$$

since $\pi_p[X^{(p)}] \cong H_p[X^{(p)}]$ is a free group on the generating set for $\pi_p X$, we may assume that each b_i is in that generating set (of course, additional argument is needed if $p = 1$), and so $\sum r_i b_i$ is an element of $\pi_p X$, namely zero. Now

$$\partial(e_r^{p+1} \wedge e_c^q) = (\partial e_r^{p+1}) \wedge e_c^q \cup e_r^{p+1} \wedge * = (\partial e_r^{p+1}) \wedge e_c^q,$$

and the smash product is bilinear. The attaching map, composed with a, is thus

$$\begin{aligned} a \circ ([\textstyle\sum r_i b_i] \wedge c) &= a \circ (\textstyle\sum r_i [b_i \wedge c]) = \textstyle\sum r_i\, a \circ (b_i \wedge c) \\ &= \textstyle\sum r_i\, \alpha(b_i \otimes c) = \alpha([\textstyle\sum r_i b_i] \otimes c) = 0, \end{aligned}$$

and a has an extension to $e_r^{p+1} \wedge e_c^q$. An identical argument extends a to cells of the form $e_b^p \wedge e_s^{q+1}$, where s is a relation in $\pi_q[Y^{(q)}]$; hence a may be extended to the $(p + q + 1)$ skeleton of $X \wedge Y$. An extension to all of $X \wedge Y$ is now guaranteed, since cells of higher dimension have attaching maps which compose inessentially with an inductively defined map a for dimensional reasons: $\pi_i Z = 0$ if $i > p + q$.

It is clear from the construction of a that $\Phi a = \alpha$; that is, Φ is onto. But if $\Phi a = \Phi a'$ then the restrictions of a and a' to $(X \wedge Y)^{(p+q)}$ must be homotopic, say via H, since they are homotopic on each $(p + q)$ cell. This defines a map

$$a \times 0 \cup H \cup a' \times 1$$

on $(X \wedge Y) \times 0 \cup (X \wedge Y)^{(p+q)} \times I \cup (X \wedge Y) \times 1$ into Z, and this map has an extension to all of $(X \wedge Y) \times I$ for dimensional reasons. Therefore, Φ is 1-1; the proof of the lemma is complete. □

We remark that the above proof is a thinly disguised computation of the cohomology group $H^{p+q}(X \wedge Y, \pi_{p+q} Z)$; to see this, replace Z in Lemma 3 by the penultimate stage Z_{p+q-1} in a Postnikov system for Z : each of our maps $a : X \wedge Y \to Z$ has an inessential composition with $\pi_{p+q-1} : Z \to Z_{p+q-1}$, so each map a is homotopic to a map into the fiber $K(\pi_{p+q} Z, p + q)$ of π_{p+q-1}. This suggests a common generalization of our Theorem 2 and Copeland's result, cited above; we omit details.

J. F. Adams has pointed out to us a proof that

$$H^{p+q}(X \wedge Y, \pi_{p+q} Z) \cong \mathrm{Hom}\,(H_p X \otimes H_q Y \to \pi_{p+q} X)$$

based on the universal coefficient theorem and the Künneth formula. When used in the proof of Lemma 3, this isomorphism becomes the function Φ for which we have given an explicit construction.

Corollary 4 *For each abelian group G there exists an abelian topological group Y for which $\pi_0 \mathfrak{F} Y \cong \pi_0 \mathfrak{H} Y \cong G$.*

Proof. Apply Theorem 2 to $Y = S^1 \times K(G, 2)$. □

3. The homomorphism Φ

The values of the homomorphism Φ of Theorem 2 may look somewhat familiar. If Y is, for example, a topological group with product m (which we indicate by juxtaposition, etc.), $\bar{m}$ is the converse of m (so $\bar{m}(y, z) = m(z, y)$), and $m_\flat : \mathfrak{H} \to \mathcal{G}$ is the map defined above Proposition 1 let $f = (q^\#)^{-1}\bar{m}_\flat(m)$. This defines $f \circ q(y, z) = yzy^{-1}z^{-1}$, a commutator map which Φ carries to a homomorphism whose value at $b \otimes c \in \pi_n Y \otimes \pi_n Y$ is the Samelson product $\langle b, c\rangle$. Our definition of Φf, $\Phi(f)(b \otimes c) = f \circ (b \wedge c)$, readily extends to elements b, c of every dimension in $\pi_* Y$, and with this extension, the values of Φ include all Samelson products. Likewise, our definition of Φ need not be restricted to finite Posnikov systems Y; if it is applied to $Y = S^3$ it is easy to see that

$$\mathfrak{F}(S^3) = \{S^3 \wedge S^3 \to S^3\}$$

and thus the domain of Φ is $\pi_0\mathfrak{F}(S^3) = \pi_6(S^3) = Z_{12}$; if i_3 is a generator of $\pi_3(S^3)$ and $a \in \pi_6(S^3)$ then $\Phi(a)(i_3 \otimes i_3) = a$, so Φ is onto

$$\mathrm{Hom}\,(\pi_3[S^3] \otimes \pi_3[S^3] \to \pi_6[S^3]) = Z_{12}\,.$$

However, James has shown [7] that the Samelson products given by the set of H-structures on S^3 (or S^7) have values at $i_3 \otimes i_3$ (or $i_7 \otimes i_7$) which are the odd members only of Z_{12} or (Z_{120}). Hence the values of Φ give a proper generalization of the Samelson products as geometrically defined homomorphisms

$$\pi_q Y \otimes \pi_r Y \to \pi_{q+r} Y.$$

Phrased in terms of elements h of $\pi_0 \mathfrak{H}$, the picture is that of nul-homotopies defined by h for the Whitehead products $[b, c]$ over Y : the Samelson product compares these nul-homotopies for a given h and its converse, $\bar{h}$ while Φ offers a comparison of these nul-homotopies for any two elements h and h' of $\pi_0\mathfrak{H}$. Furthermore, $\pi_0 \mathfrak{F}$ may have a (pointwise) group structure which Φ respects, a concept impossible to phrase in terms of Samelson products.

To continue, we recall that the space Y of Theorem 2 was not required to be an H-space; obviously our function Φ works just as well if $\mathfrak{H}$ is empty. If, for instance, Y is the n-sphere S^n, the argument sketched above for S^3 shows that $\pi_0 \mathfrak{F}(S^n) = \pi_{2n}(S^n)$, and that Φ is faithful, since $\Phi(a)(i_n \otimes i_n) = a$. Thus to each element of $[Y \wedge Y \rightarrow Y]$ we associate a bilinear multiplication on $\pi_* Y$, just as the Samelson product does for H-spaces Y. There are more of these products even for S^3 and S^7, and they are nontrivial for other spheres.

We now point out that our definition of Φ, $\Phi(f)(b \otimes c) = f \circ (b \wedge c)$, can be restated as

$$\Phi(f)(b \otimes c) = \omega \circ (f \wedge b \wedge c),$$

where $\omega : y^{Y\wedge Y} \wedge Y \wedge Y \rightarrow Y$ is the evaluation map. But, in this form, the definition of Φ is seen to extend to all of $\pi_*\{Y \wedge Y \rightarrow Y\}$: if

$$a : S^q \rightarrow \{Y \wedge Y \rightarrow Y\}$$

then $\Phi(a)(b \otimes c) = \omega \circ (a \wedge b \wedge c)$. This yields a function Φ which is linear in a and whose values are bilinear in b and c, since this smash product is trilinear. If $b \in \pi_r(Y)$ and $c \in \pi_s(Y)$ then

$$\Phi(a)(b \otimes c) \in \pi_{q+r+s}(Y):$$

we shall say that $\Phi(a)$ is a product on $\pi_*(Y)$ of degree q. To formally describe the range of Φ, let us define the graded group $\mathfrak{A}(G)$ of products on a graded group G by

$$\mathfrak{A}_q = \textstyle\sum_{r,s} \operatorname{Hom}(G_r \otimes G_s \rightarrow G_{q+r+s}).$$

Then Φ is a homomorphism from $\pi_*\mathfrak{F}Y$ to $\mathfrak{A}\pi_*Y$. It is nontrivial: our previous argument generalizes to show that if

$$a \in \pi_{2n+q}(S^n) = \pi_q\{S^n \wedge S^n \rightarrow S^n\}$$

then $\Phi(a)(i_n \otimes i_n) = a$, so Φ is monic if $Y = S^n$. It is not difficult to prove, more generally, that $\Phi(a)(b \otimes c) = (-1)^{q(r+s)} a \circ S^q(b \wedge c)$ when $Y = S^n$ and S denotes the suspension functor. We can also calculate Φ partially for finite Postnikov systems.

THEOREM 5. *Let Y have only a finite number of nonzero homotopy groups, say $\pi_i Y = 0$ unless $1 \leq n \leq i \leq 2n + k$ for some integers n, k. Then*

$$\Phi : \pi_j \mathfrak{F}Y \cong \mathfrak{A}_j \pi_* Y$$

for every integer $j \geq k$. That is,

$$\Phi : \pi_k \mathfrak{F} Y \cong \text{Hom}\,(\pi_n Y \otimes \pi_n Y \to \pi_{2n+k} Y)$$

and $\pi_j \mathfrak{F} Y = 0$ *if* $j > k$.

Proof. Apply Lemma 3 to $X = S^j Y$ and $Z = Y$. □

These products of positive degree on $\pi_* Y$ remind one of the binary operations of degree n which Browder described [3] for the homology graded groups of the H_n-spaces of Araki-Kudo: it can be shown that the Hurewicz homomorphism carries our homotopy product defined by an H_n-structure (by use of our Proposition 1) to Browder's homology product via a commutative diagram, giving them a relationship like that of the Samelson and Pontrjagin products in degree zero. In fact, our homomorphism Φ can easily be seen to work for cubical homology as well as for homotopy; it may thus be used to generalize Browder's binary operations to non-H_n-spaces (although we have failed to obtain the Araki-Kudo operations of one variable for such spaces).

Let Y be an H-space and consider Φ to be defined on $\pi_q \mathfrak{H}$ as in the preceding paragraph; it is natural to ask for a geometric picture relating Φ to James' definition of the Samelson product in terms of his separation elements [7]. We view the separation element of two maps $f, g : I^n, \dot{I}^n \to X$ which agree on $\dot{I}^n$ as a construction applied to a 0-sphere of nul-homotopies of $f \mid \dot{I}^n = g \mid \dot{I}^n$ means the boundary of the n-cube I^n. (In the case of the Samelson product, $f \mid \dot{I}^n$ is the Whitehead product, with f and g the extensions to I^n given by an H-structure m and its converse $\bar{m}$ on X.) In general, let $\theta : \dot{I}^n \to X$ be given along with

$$a : \dot{I}^{q+1} \to \{I^n \xrightarrow{f} X : f \mid \dot{I}^n = \theta\};$$

our q-dimensional separation element is the element of $\pi_{q+n} X$ given by

$$I^{q+n} \to \dot{I}^{q+n+1} \cong \dot{I}^{q+1} \times I^n \cup I^{q+1} \times \dot{I}^n \to X;$$

here the first two maps are the usual (relative) homeomorphisms of degree one and the third map is $\hat{a} \cup \theta \circ p_2$, where $\hat{a}$ is the associate of a and p_2 is projection on the second factor. Clearly this specializes to the separation element if $q = 0$, and it describes the translation of Φ to $\mathfrak{H}$ in higher degrees; it may be useful elsewhere.

The function $\mathfrak{F}$ is a functor on an appropriate category, and it has a rich structure: a covering map $\rho : \tilde{Y} \to Y$ induces a map $\mathfrak{F} Y \to \mathfrak{F} \tilde{Y}$, and there are homomorphisms $\pi_i \mathfrak{F} Y \to \pi_{i+1} \mathfrak{F}\, \Omega\, Y$, $\pi_{i+1} \mathfrak{F} Y \to \pi_i \mathfrak{F} S Y$, and $\pi_i Y \to \pi_i \mathfrak{F} Y$ with good algebraic properties.

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