THE LEVI PROBLEM IN CERTAIN INFINITE DIMENSIONAL VECTOR SPACES

BY

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1. Introduction

Let E be a complex vector space with a locally connected Hausdorff topology T which is at least as coarse as the finite topology T_0 on E (composed of those sets whose intersection with every finite dimensional subspace is an open set in the Euclidean topology). A complex valued function f defined on an open subset D of (E, T) is *Gâteaux differentiable* if for all $a, b \in E, f(a + ub)$ is holomorphic as a function of u in D. If in addition, f is continuous for the topology T, then f is said to be *holomorphic* in D. A *domain of holomorphy* is an open set D of (E, T) such that for every boundary point $b \in \partial D$, there exists a function f_b holomorphic on D which cannot be continued (locally) as a holomorphic function to any open neighborhood of b. If f_b can be chosen the same for every $b \in \partial D$, then D is said to be a *domain of existence*.

A function g which takes on real values in the range $[-\infty, +\infty)$ is said to be *plurisubharmonic* in an open set D of (E, T) if g is upper semi-continuous and if for all $a, b \in E$, $g(a + ub)$ is either identically $-\infty$ or a subharmonic function of u in D. We say that an open set D is *pseudoconvex* if for all $a \in E$, $-\log\ d_a(z)$ is plurisubharmonic, where

$$d_a(z) = \sup \{\tau : z + \lambda a \in D \text{ for all } \lambda, |\lambda| \leq \tau\}.$$

There have been attempts to characterize domains of holomorphy in infinite dimensions in terms of the characterizations given in finite dimensions. A characterization of the type Cartan-Thullen can be found for certain infinite dimensional space in [6], [7], [8]. We shall give a characterization here in terms of pseudoconvexity.

For finite dimensional spaces, the following properties are equivalent: (cf. [1], [4], [5])

(i) D is a domain of existence.
(ii) D is a domain of holomorphy.
(iii) D is pseudoconvex.

(It was Levi who first conjectured this result, which was proved by Oka.) We investigate two infinite dimensional cases for which this is still valid:

(1) for any complex vector space equipped with the finite topology T_0;
(2) for any separable Hilbert space

It was shown in [2] (for case (1)) and [3] (for case (2)) that every domain

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of holomorphy is pseudoconvex. Thus, we shall be concerned with proving the converse.

2. The finite topology and the Levi problem

We now prove that for any complex vector space equipped with the finite topology, a pseudoconvex is a domain of holomorphy. The argument will be a combination of transfinite and finite induction. But first we prove a lemma about extending holomorphic functions from subspaces.

DEFINITION 2.1. Let Q be an open set in C^n and let $\omega_i = \{z \in Q : z_i = 0\}$. Let f_i be a holomorphic function defined on ω_i (as an $(n-1)$-dimensional manifold). The f_i are said to be *compatible* if $f_i = f_j$ on $\omega_i \cap \omega_j$.

LEMMA 2.2 *Let Q be a pseudoconvex domain in C^n and f_i compatible holomorphic functions defined on ω_i. Then there exists a holomorphic function $g(z)$ in Q such that $g = f_i$ on ω_i and such that g cannot be extended as a holomorphic function to an open neighborhood of any boundary point.*

Proof. Let $\nu = (\nu_1, \cdots, \nu_n)$ be an n dimensional multi-index composed of zeros and ones. We say that $\mu \leq \nu$ if $\mu_i \leq \nu_i$ for all i; otherwise, $\mu \nleq \nu$. Let

$$\omega_\nu = \bigcap_{\nu_i = 1} \omega_{\nu_i} \quad \text{and} \quad \Omega_\nu = \omega_\nu \backslash \bigcup_{\mu \nleq \nu} \omega_\mu .$$

Let $z^i = (z_1, \cdots, z_{i-1}, z_{i+1}, \cdots, z_n)$, an $(n-1)$ tuple.

For every $z_0 \in \Omega_\nu$ for some ν, there exists an open polydisc Δ_{z_0} such that $\Delta_{z_0} \subset Q$ and $\Delta_{z_0} \cap \Omega_\mu = \emptyset$ for $\mu \nleq \nu$. We define a holomorphic function in Δ_{z_0} in the following way: let i be the smallest integer such that $\nu_i = 1$; we expand f_i to a holomorphic function in Δ_{z_0} by setting $g_1(z) = f_i(z^i)$; having defined $g_1(z)$, we choose the next smallest integer $j \geq i$ for which $\nu_j = 1$ and let $h_j = f_j - g_1$ be a holomorphic function defined on $\Delta_{z_0} \cap \omega_{\nu_j}$; note that $h_j \equiv 0$ on $\omega_i \cap \omega_j$, since $f_i = f_j$ there; we set $h_2(z) = h_j(z^j)$ in Δ_{z_0} and let $g_2(z) = g_1(z) + h_2(z)$; then $g_2 = f_i$ on ω_i and $g_2 = f_j$ on ω_j; we continue in this way, and after exhausting those indicies for which ν has a one, we arrive at a holomorphic function $G_{z_0}(z)$ defined in Δ_{z_0} such that $G_{z_0} = f_k$ on $\omega_k \cap \Delta_{z_0}$ for all k such that $\nu_k = 1$. By the construction of the polydiscs, for $z_0 \in \Omega\nu$, $z'_0 \in \Omega_\mu$ and $z \in \Delta_{z_0} \cap \Delta_{z'_0}$, $z \in \Omega_\lambda$ we have $\lambda_i \leq \min(\nu_i, \mu_i)$, and thus, since $G_{z_0} = f_k = G_{z'_0}$ in $\omega_k \cap \Delta_{z_0} \cap \Delta_{z'_0}$ for $\lambda_k = 1$, $G_{z_0} - G_{z'_0}$ is divisible by z_k in $\Delta_{z_0} \cap \Delta_{z'_0}$.

Let $g_{z_0}(z) = G_{z_0}/(z_1 \cdots z_n)$, which defines a meromorphic function in Δ_{z_0}. If $\Delta_{z_0} \cap \Delta_{z'_0} \neq \emptyset$, then

$$g_{z_0} - g_{z'_0} = (G_{z_0} - G_{z'_0})/(z_1 \cdots z_n)$$

defines a holomorphic function in $\Delta_{z_0} \cap \Delta_{z'_0}$, since $G_{z_0} - G_{z'_0}$ is divisible by z_i for all i such that $z_i = 0$ in $\Delta_{z_0} \cap \Delta_{z'_0}$. We set $g_0 \equiv 1$ in

$$\Gamma = Q \backslash \bigcup_{i=1}^n \omega_i .$$

Then the g's form a set of Cousin I data in Q, and since Q is pseudoconvex, there exists a meromorphic function $m(z)$ defined in Q such that $h_{z_0}(z) = m(z) - g_{z_0}(z)$ is holomorphic in Δ_{z_0} (and $m(z) - g_0(z)$ is holomorphic in Γ) [1]. Let $w(z) = z_1 \cdots z_n m(z)$. Then $w(z)$ is holomorphic in Q and $w(z) = G_{z_0}(z) = f_i(z)$ for $z \in \omega_i$.

Since Q is a pseudoconvex domain, by the solution of the Levi problem in finite dimensions, there exists a holomorphic function $s(z)$ which cannot be continued as a holomorphic function to any neighborhood of any boundary point of Q. Let

$$s_r(z) = rz_1 \cdots z_n s(z) + w(z).$$

Since the topology of Q has a countable base, there exists at most a countable number of r for which $s_r(z)$ can be continued to an open neighborhood of some boundary point (for each neighborhood, there is at most one r), and hence, there exists r_0 such that $s_{r_0}(z)$ cannot be thus extended. This function satisfies the conclusions of the Lemma. Q.E.D.

We have now done most of the work for the following:

THEOREM 2.3. *Let E be a complex vector space equipped with the finite topology T_0. Then an open set D is a domain of existence if (and only if) it is pseudoconvex.*

Proof. We assume, withoutout loss of generality, that the origin lies in D. We choose a Hamel basis $\{z_\sigma\}_{\sigma \in \Sigma}$ for E, where we assume the set Σ to be well ordered. Let σ_0 be the the smallest element of Σ in this ordering and let $D_0 = D \cap Q_0$, where Q_0 is the linear space spanned by z_{σ_0}. Then there exists a function f_{σ_0} on D_0 which cannot be continued to an open neighborhood of any boundary point of D_0. By a process of transfinite induction on Σ, we extend f_{σ_0} to a holomorphic function F on D which cannot be continued as a holomorphic function to an open neighborhood of any boundary point.

For a given σ', let X, be the subspace spanned by the set

$$Y' = \{z_\sigma : \sigma < \sigma'\}.$$

We equip X' with the finite topology, and we assume that there exists a holomorphic function f' defined on $D \cap X'$ such that if Q is any subspace spanned by a finite number of the z_σ, $f'|_{D \cap Q}$ cannot be continued as a holmorphic function to an open neighborhood of any boundary point of $D \cap Q$ (in the Euclidean topology). Let X'' be the subspace spanned by

$$Y'' = Y' \cup \{z_{\sigma'}\}.$$

We show that we can find a holomorphic function f'' defined on $D \cap X''$ (equipped with the finite topology) such that $f'' = f'$ on $D \cap X'$ and such that, if Q is any subspace spanned by a finite number of elements of Y'', f'' cannot be extended as a holomorphic function to an open neighborhood of any boundary point of $D \cap Q$.

To do this, we apply finite induction to the dimension of Q. If $\dim Q = 1$

and Q is spanned by $z_\sigma (\sigma \neq \sigma')$, then $f'' = f'$; if Q is spanned by $z_{\sigma'}$, we need only find a holomorphic function $f_{\sigma'}$ defined in $D \cap Q$ with $f_{\sigma'}(0) = f'(0)$ and such that f cannot be continued as a holomorphic function to an open neighborhood of any boundary point of $D \cap Q$, which is always possible. We now assume that we have defined f'' on all subspaces for which $\dim Q \leq n - 1$ and let Q_0 be a subspace spanned by n elements of Y''. If $z_{\sigma'}$ is not one of these elements, we let $f'' = f'$ on $Q_0 \cap D$. If $z_{\sigma'}$ is one of these elements, the set is formed by $\{z_{\sigma'}, z_{\tau_i}\}$, $\tau_i = 1, \cdots, n-1$, $z_{\tau_i} \in Y'$. Let

$$\omega_{\sigma'} = Q_0 \cap D \cap \{z: z_{\sigma'} = 0\},\ \omega_{\tau_i} = Q_0 \cap D \cap \{z: z_{\tau_i} = 0\}.$$

Then $f_\sigma = f'$, which is defined on $\omega_{\sigma'}$ by the transfinite induction hypothesis, and $f_{\tau_i} = f''$, which is defined on ω_{τ_i} by the finite induction hypothesis, define a set of compatible functions in $Q_0 \cap D$, so by Lemma 2.2, we can extend f'' to $Q_0 \cap D$ as a holomorphic function which cannot be continued to an open neighborhood of any boundary point. This establishes the finite induction, so f'' is defined on $D \cap X''$ and holomorphic for the finite topology, since every finite dimensional subspace of X'' can be embedded as a hyperplane in one of these finite dimensional subspaces Q. This also completes the transfinite induction, so f_{σ_0} can be extended to a function F on D such that F cannot be continued as a holomorphic function to a neighborhood of any boundary point.

Q.E.D.

The main problem with the finite topology is that it allows too many holomorphic functions for most purposes. We turn our attention now to another topology.

3. Separable Hilbert spaces and the Levi problem

If E is a Banach space, the Hamel basis that we chose in Section 2 will in general have no relation to the norm topology and so the function F that we defined will not be continuous for that topology. But if E is a Hilbert space, we have a norm which is essentially the same as the norm of the underlying Euclidean subspaces. It is then possible to define a function through extension which converges to a holomorphic function in the domain of definition. We formulate the extension properties in the following lemma.

LEMMA 3.1. *Let D be a domain of holomorphy in C^n and let*

$$z' = (z_1, \cdots, z_{n-1})$$

be a coordinate of $Q = D \cap \{z_n = 0\}$, which we assume to be non-empty. Let $f(z')$ be a holomorphic function on Q (as an $(n-1)$-dimensional manifold). Let

$$d(z) = \inf \{\|z - z''\| : z'' \in \complement D\}.$$

Then given $A > 0$, there exists a holomorphic function $G_A(z)$ defined in D such that

$$|f(z', 0) - G_A(z', z_n)| \leq |z_n|^2$$

for

$z \in K_A = \{z \in D: -\log d(z) + \log |z_n| \leq -1, \|y\| \leq A, -\log d(z) \leq \log A\}.$

Furthermore, $G_A(z)$ can be chosen so that it cannot be continued as a holomorphic function to a neighborhood of any boundary point of D.

Proof. (The proof is an adaptation of an idea of Hörmander [6, p. 88].) The set

$$M = \{z: z \in D, (z_1, \cdots, z_{n-1}, 0) \notin Q\}$$

is disjoint from K_A, for $z \in M$ implies that $|z_n| \geq d(z)$. Thus M and $K_A \cup Q$ are two disjoint relatively closed sets in D (since K_A is compact in D), and so there exists a $\mathcal{C}^\infty$ function ϕ such that $\phi \equiv 1$ in a neighborhood of $K_A \cup Q$ and $\phi \equiv 0$ in a neighborhood of M. Let $D' = D \backslash \bar{M}$ and define

$$f'(z) = f(z_1, \cdots, z_{n-1}, 0),$$

which is holomorphic in D'. Let $f''(z) = \phi(z)$, which we extend to all of D by setting it equal to zero on $\bar{M}$. We let

$$F(z) = f''(z) - z_n^2 v(z),$$

where we determine $v(z)$ so as to make F a holomorphic function (i.e. such that $\bar{\partial} F = 0$, where $\bar{\partial} = \sum_{i=1}^n (\partial / \partial z_i) d\bar{z}_i$). This is equivalent to

$$\bar{\partial} v = (1/z_n^2) \bar{\partial} \phi \wedge f'';$$

the right hand side is $\mathcal{C}^\infty$ and $\bar{\partial}$ closed, so there exists a v satisfying the equation in D, and since $\bar{\partial}\phi = 0$ in a neighborhood of K_A, v is holomorphic in a neighborhood of K_A.

By Theorem 4.3.2 [6], v can be uniformly approximated in K_A by functions holomorphic in D Thus, there exists $h(z)$ holomorphic in D such that $|v(z) - h(z)| \leq 1/2$ on K_A. Then $H(z) = F(z) - z_n^2 h(z)$ satisfies $|H(z) - f'(z)| \leq |z_n|^2/2$ on K_A.

Since D is a domain of holomorphy, there exists a function $s(z)$ holomorphic in D which cannot be continued as a holomorphic function to an open neighborhood of any boundary point. Since K_A is compact, there is a constant $k > 0$ such that $|s(z)| \leq k$ on K_A. For $F_c(z) = H(z) + cz_n^2 s(z)$, given any open neighborhood of any boundary point, there is at most one value of c for which $F_c(z)$ can be continued as a holomorphic function to that neighborhood. Since C^n has a countable base of open sets, there exists c_0, $|c_0| \leq 1/2k$, such that $G_A(z) = H(z) + c_0 z_n^2 s(z)$ satisfies the conclusions of the lemma. Q.E.D.

Lemma 3.1 now allows us to prove the following:

THEOREM 3.2. *Let D be a pseudoconvex domain in a separable Hilbert space E. Then D is a domain of existence.*

Proof. Let z_i, $i = 1, 2, \cdots$ be a countable set which is dense in the boundary of D. By the Gram-Schmidt orthogonalization process, we convert the z_i's into an orthonormal system which we expand to a complete orthonormal system ϕ_k in E. Let W_n be the subspace spanned by $(\phi_1, \cdots, \phi_n)$ and let $Q_n = D \cap W_n$, which is a domain of holomorphy in the finite dimensional Euclidean topology on W_n since it is pseudoconvex.

Let $A_n = n$. For Q_1, there exists a function G_1 which cannot be continued as a holomorphic function to an open neighborhood of any boundary point. Having defined G_n in Q_n, we define G_{n+1} in Q_{n+1} satisfying the conclusions of Lemma 3.1 for $K_{A_{n+1}}$. We show that the functions thus defined converge to a function holomorphic in D.

Let $\eta > 0$ be given and let $z \in D$. Then there exists δ, $1 > \delta > 0$, such that $z' \in D$ for $\|z' - z\| < \delta$. Let $n \geq \max\ (\|z\| + 1, 4/\delta)$ and let $m \geq n$ be so large that

$$\left(\sum_{k=m+1}^{\infty} | z_k |^2\right)^{1/2} < \min\ (\delta/4e, \eta/8).$$

For $k \geq m$, in Q_k, $z^k = (z_1, \cdots, z_k) \in K_{A_k}$, and $z'^k \in K_{A_k}$ for $\|z' - z\| < \delta/4e$. Since $G_m(z)$ is continuous, there exists $\zeta > 0$ such that

$$| G_m(z'^k) - G_m(z^k) | < \eta/4 \quad \text{for } \|z' - z\| < \zeta.$$

Let $\zeta = \min\ (\zeta, \delta/4e, \eta/8)$ and let $\|z' - z\| < \zeta$. Then

$$\left(\sum_{k=m+1}^{\infty} | z'_k |^2\right)^{1/2} \leq \zeta + \left(\sum_{k=m+1}^{\infty} | z_k |^2\right)^{1/2}.$$

Hence

$$\begin{aligned} | G(z') - G(z) | &< | G_m(z'^m) - G_m(z^m) | + \sum_{k=m+1}^{\infty} | z'_k |^2 + \sum_{k=m+1}^{\infty} | z_k |^2 \\ &< \eta/4 + \eta/4 + \eta/4 \end{aligned}$$

so G_n converges to a continuous function in D. Furthermore, $G(z + uh)$ is holomorphic for $\|uh\| < \zeta$, since it is the uniform limit of holomorphic functions. It is also clear that G cannot be extended to any larger domain, for then it could be extended beyond some boundary point, which is not possible by the construction of the function.

Q.E.D.

While proving the above theorem, we also proved:

THEOREM 3.3. *Let D be a domain of holomorphy in a separable Hilbert space E and let W be a finite dimensional subspace. Then for any function defined and holomorphic in $Q = W \cap D$, there exists a function G defined and holomorphic in D such that $G(z) = f(z)$ on Q. $G(z)$ can also be chosen to have D as its natural domain of definition.*

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