MORE ON HIGH-ORDER NON-LOCAL UNIFORM ALGEBRAS

BY

S. J. SIDNEY[1]

1. Introduction

Consider the process of taking the uniform closure of the functions belonging locally to a uniform algebra. After how many iterations of this process does the resulting (transfinite) sequence of uniform algebras stabilize? The possibilities turn out to be the finite and countable ordinal numbers, for each of which an example with five generators is constructed, and the first uncountable ordinal number, for which each example must be non-separable. A similar result, with four generators instead of five, is true for the process of taking the uniform closure of the functions locally approximable by a uniform algebra. These theorems improve previous results of the author [6], both by extending (and determining) the number of iterations possible, and, for the second process, by introducing control over the number of generators of the algebras involved.

In §2, certain notions are recalled and the main results of the paper are stated. The proofs are given in §3 and §4. For many details, the reader will be referred to [6]. Finally, §5 is devoted to indicating a number of open problems concerning non-local algebras.

2. The main results

If (A, X) is a uniform algebra with spectrum X, we denote by $L(A)$ and $H(A)$ the respective closures of the functions locally belonging to A and the A-holomorphic functions (that is, the functions locally (uniformly) approximable by A). A is said to be local or non-local according as $L(A) = A$ or $L(A) \neq A$, and holomorphically closed or non-holomorphically closed according as $H(A) = A$ or $H(A) \neq A$. We can inductively define $L^\sigma(A)$ and $H^\sigma(A)$ for all ordinal numbers σ by the rules $L^0(A) = H^0(A) = A, L^{\sigma+1}(A) = L(L^\sigma(A))$ and $H^{\sigma+1}(A) = H(H^\sigma(A))$, and, if σ is a limit ordinal, $L^\sigma(A)$ is the uniform closure of

$$\bigcup\{L^{\sigma'}(A) : 0 \leq \sigma' < \sigma\}$$

and $H^\sigma(A)$ is the uniform closure of

$$\bigcup\{H^{\sigma'}(A) : 0 \leq \sigma' < \sigma\}.$$

Recall that for all σ, $(L^\sigma(A), X)$ and $(H^\sigma(A), X)$ are uniform algebras with spectrum X (see [6], §2). Let σ^* denote the first uncountable ordinal number.

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We remark that, using the algebras whose existence is guaranteed by Theorems 1 and 2 below, the reader will have no difficulty verifying that it need not always be the case that $L(L^\sigma(A)) = L^\sigma(L(A))$ or $H(H^\sigma(A)) = H^\sigma(H(A))$.

THEOREM 1. *If σ is any finite or countable ordinal number, there is an antisymmetric uniform algebra (A, X), generated as a Banach algebra (with unit) by five or fewer elements, such that $L^\sigma(A)$ is local but $L^{\sigma'}(A)$ is non-local for $0 \leq \sigma' < \sigma$.*

THEOREM 2. *If σ is any finite or countable ordinal number, there is an antisymmetric uniform algebra (A, X), generated as a Banach algebra (with unit) by four or fewer elements, such that $H^\sigma(A)$ is holomorphically closed but $H^{\sigma'}(A)$ is non-holomorphically closed for $0 \leq \sigma' < \sigma$.*

The proofs of these theorems occupy §3 and §4 respectively.

COROLLARY 3. *There is an antisymmetric uniform algebra (A, X) such that $L^{\sigma'}(A)$ is non-local for $0 \leq \sigma' < \sigma^*$.*

COROLLARY 4. *There is an antisymmetric uniform algebra (A, X) such that $H^{\sigma'}(A)$ is non-holomorphically closed for $0 \leq \sigma' < \sigma^*$.*

The corollaries are obtained from the corresponding theorems by taking the tensor products of the algebras constructed for $\sigma < \sigma^*$; the details are left to the reader.

The results above are essentially best possible, as the following proposition shows.

PROPOSITION 5. *Let (A, X) be a uniform algebra with spectrum X. Then:*

(a) *$L^{\sigma^*}(A)$ is local and $H^{\sigma^*}(A)$ is holomorphically closed.*

(b) *If $L^{\sigma'}(A)$ is non-local for all $\sigma' < \sigma^*$, or if $H^{\sigma'}(A)$ is nonholomorphically closed for all $\sigma' < \sigma^*$, it follows that A cannot be separable (equivalently, that X cannot be metrizable).*

Proof. (a) One verifies that

$$\bigcup\{L^\sigma(A) : 0 \leq \sigma < \sigma^*\} \quad \text{and} \quad \bigcup\{H^\sigma(A) : 0 \leq \sigma < \sigma^*\}$$

are uniformly closed, so

$$L^{\sigma^*}(A) = \bigcup\{L^\sigma(A) : 0 \leq \sigma < \sigma^*\} \quad \text{and} \quad H^{\sigma^*}(A) = \bigcup\{H^\sigma(A) : 0 \leq \sigma < \sigma^*\};$$

for instance, if

$$\{f_n\}_{1 \leqq n < \infty} \subset \bigcup\{L^\sigma(A) : 0 \leq \sigma < \sigma^*\}, \qquad f \in \mathcal{C}(X),$$

and $\| f_n - f \|_X \to 0$, then $f_n \in L^{\sigma_n}(A)$ for some $\sigma_n < \sigma^*$, there is $\sigma^- < \sigma^*$ such that $\sigma_n \leq \sigma^-$ for all n, $f_n \in L^{\sigma^-}(A)$ for all n, and so

$$f \in L^{\sigma^-}(A) \subset \bigcup\{L^\sigma(A) : 0 \leq \sigma < \sigma^*\}.$$

It now follows immediately that $L^{\sigma^*}(A)$ is local. On the other hand, suppose $f \in \mathcal{C}(X)$ is $H^{\sigma^*}(A)$-holomorphic. Then there is a finite open covering of X, say $U_1, \cdots, U_m$, and there are m sequences $\{f_{kn}\}_{1 \leq n < \infty} \subset H^{\sigma^*}(A)$, $1 \leq k \leq m$, such that $\lim_{n\to\infty} \| f_{kn} - f \|_{U_k} = 0, 1 \leq k \leq m$. There is $\sigma_{kn} < \sigma^*$ such that $f_{kn} \in H^{\sigma_{kn}}(A)$, and there is $\sigma^- < \sigma^*$ such that $\sigma_{kn} \leq \sigma^-$ for all k, n. Thus $f_{kn} \in H^{\sigma^-}(A)$ for all k, n, whence f is $H^{\sigma^-}(A)$-holomorphic, and so a member of $H^{\sigma^-+1}(A) \subset H^{\sigma^*}(A)$.

(b) Under these conditions, we can find for each $\sigma < \sigma^*$ a function f_σ which belongs to $L^{\sigma+1}(A)$ (or to $H^{\sigma+1}(A)$) but is at distance ≥ 1 from $L^\sigma(A)$ (or $H^\sigma(A)$). Then $\{f_\sigma : 0 \leq \sigma < \sigma^*\}$ is an uncountable family in $\mathcal{C}(X)$, and the distance between any two distinct f_σ is at least 1, so $\mathcal{C}(X)$ cannot be separable, that is, X cannot be metrizable. ■

3. Proof of Theorem 1

Theorem 1 is proved essentially as is the corresponding theorem in [6], the main difference being a revised indexing procedure which enables us to treat large ordinal numbers. We retain the following notations, a bar denoting closure in $\mathbf{C}$ and ∂ denoting boundary in $\mathbf{C}$:

$$\Delta = \{z \in \mathbf{C} : |z| < 1\} \quad \text{and} \quad R = \{z \in \mathbf{C} : 1 < |z| < 2\};$$

$$\partial_1 R = \{z \in \mathbf{C} : |z| = 1\} \quad \text{and} \quad \partial_2 R = \{z \in \mathbf{C} : |z| = 2\};$$

$$B(\Delta) = \{f \in \mathcal{C}(\bar{\Delta}) : f \text{ is holomorphic on } \Delta\};$$

$$B(R) = \{f \in \mathcal{C}(\bar{R}) : f \text{ is holomorphic on } R\}.$$

Definition 6. The *radial algebra with data* $I_1, \cdots, I_n$ is the pair (A, X) where

$$X = (\bar{R} \times \{0\}) \cup (\partial R \times \bar{\Delta}) \cup (\textstyle\bigcup_{j=1}^n (I_j \times \bar{\Delta}_j)) \subset \mathbf{C}^2,$$

$$\begin{aligned} A = \{f \in \mathcal{C}(X) : \; & z \to f(z, 0) \text{ is holomorphic on } R; \\ & \omega \to f(z, \omega) \text{ is holomorphic on } \Delta, \quad \forall z \in \partial R; \\ & \omega \to f(z, \omega) \text{ is holomorphic on } \Delta_j, \quad \forall z \in I_j, \quad 1 \leq j \leq n; \\ & z \to (\partial f/\partial \omega)(z, 0) \text{ is in } B(R) \text{ on } \partial R \cup (\textstyle\bigcup_{j=1}^n I_j)\}. \end{aligned}$$

Here n is a non-negative integer, and for $1 \leq j \leq n$,

$$I_j = \{te^{i\theta_j} : 1 + \delta_j \leq t \leq 2\} \quad \text{and} \quad \Delta_j = \{\omega \in \mathbf{C} : |\omega| < \delta_j\},$$

where $0 < \delta_j < 1$ and $\theta_j \in \mathbf{R}$. The θ_j must satisfy $\theta_j \not\equiv \theta_k \pmod{2\pi}$ whenever $j \neq k$. We set $n(A, X) = n$ and

$$\text{size}_k(A, X) = \sup\{\delta_j : k < j \leq n\} \quad \text{if} \quad 0 \leq k < n,$$

$$\text{size}_k(A, X) = 0 \quad \text{if} \quad k \geq n.$$

Lemma 7. *Let (A, X) be the radial algebra with data $I_1, \cdots, I_n$. Then*

(A, X) is an antisymmetric uniform algebra with spectrum X. A is generated by the four functions z, $1/z$, ω and α where $\alpha : \bar{R} \times \bar{\Delta} \to \mathbf{C}$ is defined by

$$\alpha(z, \omega) = 0 \quad \text{for} \quad z \in \partial_1 R \quad \text{and} \quad \alpha(z, \omega) = \omega^2 \quad \text{for} \quad z \in \bar{R} \backslash \partial_1 R.$$

A is non-local, but

$$L(A) = H(A) = \{f \in \mathcal{C}(X) : z \to f(z, 0) \text{ is holomorphic on } R; \quad \omega \to f(z, \omega) \text{ is holomorphic on } \Delta, \quad \forall z \in \partial R; \quad \omega \to f(z, \omega) \text{ is holomorphic on } \Delta_j, \quad \forall z \in I_j, \quad 1 \leq j \leq n\},$$

and this last algebra is antisymmetric and holomorphically closed (and so local). If $f \in \mathcal{C}(X)$ belongs locally to A, then f belongs to A on

$$\{(z, \omega) \in X : z \notin \partial_1 R\} \quad \text{and on} \quad \{(z, \omega) \in X : z \notin \partial_2 R \cup (\textstyle\bigcup_{j=1}^n I_j)\}.$$

If in addition f belongs to A on an open subset of X which meets both

$$\{(z, \omega) \in X : z \in \partial_1 R\} \quad \text{and} \quad \{(z, \omega) \in X : z \in \partial_2 R \cup (\textstyle\bigcup_{j=1}^n I_j)\},$$

then $f \in A$.

Proof. See §3 of [6]. ■

Remark. $L(A)$ is generated by the four functions z, $1/z$, ω, and α_1 where $\alpha_1 : \bar{R} \times \bar{\Delta} \to \mathbf{C}$ is defined by $\alpha_1(z, \omega) = 0$ for $z \in \partial_1 R$ and $\alpha_1(z, \omega) = \omega$ for $z \in \bar{R} \backslash \partial_1 R$.

DEFINITION 8. Let (A, X) be the radial algebra with data $I_1, \cdots, I_n$ and let η be a positive number. A *satellite of radial algebras of size $\leq \eta$ which terminates in (A, X) is a pair $(\tilde{A}, \tilde{X})$ where*

$$\tilde{X} = \bigcup\{X_k \times \{\tau_k\} : 0 \leq k \leq \infty\} \subset \mathbf{C}^3,$$

$$\tilde{A} = \{f \in \mathcal{C}(\tilde{X}) : (z, \omega) \to f(z, \omega, \tau_k) \text{ is in } A_k, 0 \leq k \leq \infty\}.$$

Here:

(1) (A_k, X_k) is a radial algebra with data $I_1, \cdots, I_n, I_{n+1}^{(k)}, \cdots, I_{n(k)}^{(k)}$ where $n(k) = n(A_k, X_k) \geq n$ and $\text{size}_n (A_k, X_k) \leq \eta$.

(2) $(A_\infty, X_\infty) = (A, X)$.

(3) The τ_k are distinct real numbers such that $\{\tau_k : 0 \leq k \leq \infty\}$ is compact, $\tau_\infty \leq \tau_k \leq \tau_0$ for all k, and $\tau_0 - \tau_\infty \leq \eta$.

(4) For $0 \leq k < \infty$, $\sup\{\tau_j : \tau_j < \tau_k\} < \tau_k$.

(5) For $0 < k \leq \infty$, either

$$\inf\{\tau_j : \tau_j > \tau_k\} > \tau_k$$

or

$$\lim_{\tau_j \to \tau_k^+} \inf n(j) \geq n(k) \quad \text{and} \quad \lim_{\tau_j \to \tau_k^+} \text{size}_{n(k)} (A_j, X_j) = 0.$$

LEMMA 9. *Let $(\tilde{A}, \tilde{X})$ be a satellite of radial algebras. Then $(\tilde{A}, \tilde{X})$ is a* *uniform algebra with spectrum* $\tilde{X}$. $\tilde{A}$ *is generated by the five functions* z, $1/z$, ω, ζ *(the third complex variable), and* α' *where* $\alpha' : \bar{R} \times \bar{\Delta} \times \mathbf{C} \to \mathbf{C}$ *is defined by*

$$\alpha'(z, \omega, \zeta) = \alpha(z, \omega).$$

The maximal sets of antisymmetry for $\tilde{A}$ *are the sets* $X_k \times \{\tau_k\}$ *and*

$$\tilde{A} \mid (X_k \times \{\tau_k\}) = \{(z, \omega, \tau_k) \to f(z, \omega) : f \in A_k\}, \qquad 0 \leq k \leq \infty.$$

Proof. Conditions (3)–(5) of Definition 8 ensure that $\tilde{X}$ be compact and that $\alpha' \mid \tilde{X}$ be continuous. For the rest, see Lemma 7 of [6], and its proof. ∎

LEMMA 10. *Let* $(\tilde{A}, \tilde{X})$ *be a satellite of radial algebras. Then for every ordinal number* σ,

$$L^{\sigma}(\tilde{A}) \subset \{f \in \mathcal{C}(\tilde{X}) : (z, \omega) \to f(z, \omega, \tau_k) \text{ is in } L(A_k), 0 \leq k \leq \infty\},$$

which is holomorphically closed (and so local). The maximal sets of antisymmetry for each $L^{\sigma}(\tilde{A})$ *are the sets* $X_k \times \{\tau_k\}$.

Proof. This is evident, given that each $L(A_k)$ is antisymmetric and holomorphically closed (lemma 7). ∎

PROPOSITION 11. *Let* (A, X) *be a radial algebra, let* $\eta > 0$, *and let* σ *be a finite or countable successor ordinal number. Then there exists* $(\tilde{A}, \tilde{X})$ *a satellite of radial algebras of size* $\leq \eta$ *which terminates in* (A, X) *and which has the following additional properties*:

(1) *For* $0 \leq \sigma' \leq \sigma$, *there is a partition* $\{k : 0 \leq k \leq \infty\} = S_{\sigma'} \cup T_{\sigma'}$ *such that*

$$L^{\sigma'}(\tilde{A}) = \{f \in \mathcal{C}(\tilde{X}) : (z, \omega) \to f(z, \omega, \tau_k) \text{ is in } A_k, \forall k \in S_{\sigma'};$$
$$(z, \omega) \to f(z, \omega, \tau_k) \text{ is in } L(A_k), \forall k \in T_{\sigma'}\}$$

and for all $k \in T_{\sigma'}$,

$$L^{\sigma'}(\tilde{A}) \mid (X_k \times \{\tau_k\}) = \{(z, \omega, \tau_k) \to f(z, \omega) : f \in L(A_k)\}.$$

(2) $S_{\sigma} = \emptyset$ *(and so* $L^{\sigma}(\tilde{A})$ *is holomorphically closed).*

(3) $S_{\sigma'} \neq \emptyset$ *for* $0 \leq \sigma' < \sigma$ *(and so* $L^{\sigma'}(\tilde{A})$ *is non-local for* $0 \leq \sigma' < \sigma$).

Proof. The proof is by induction on σ. To facilitate the induction, we replace (3) by the stronger condition:

(4) If $0 \leq \sigma' < \sigma$, then $\infty \in S_{\sigma'}$.

For $\sigma = 1$, set $\tau_k = 2^{-k}\eta$ and $(A_k, X_k) = (A, X), 0 \leq k \leq \infty$.

Now suppose that $\sigma_0 = \sigma_1 + 1$ where $\sigma_1 \geq 1$ is a finite or countable ordinal number, and that we have proved the proposition for all successor ordinal numbers $\sigma \leq \sigma_1$.

Suppose that (A, X) has data $I_1, \cdots, I_n$. Choose $\theta_{n+1} \in \mathbf{R}$ such that $\theta_{n+1} \not\equiv \theta_j \pmod{2\pi}$, $1 \leq j \leq n$. For $1 \leq s < \infty$ set $\delta^s_{n+1} = 2^{-s} \min(1, \eta)$,

$$I^s_{n+1} = \{te^{i\theta_{n+1}} : 1 + \delta^s_{n+1} \leq t \leq 2\}, \qquad \Delta^s_{n+1} = \{\omega \in \mathbf{C} : |\omega| < \delta^s_{n+1}\},$$

and let (A^s, X^s) be the radial algebra with data $I_1, \cdots, I_n, I^s_{n+1}$. Two cases now present themselves.

If σ_1 is a successor ordinal number, we apply our induction assumption to find for $1 \leq s < \infty$, $(\tilde{A}^s, \tilde{X}^s)$ a satellite of radial algebras of size $\leq 2^{-s}\eta$ which terminates in (A^s, X^s) and satisfies (1)–(4) for $\sigma = \sigma_1$.

If σ_1 is a limit ordinal number, let $\{\sigma^s\}_{1 \leq s < \infty}$ be an enumeration of the successor ordinal numbers $\sigma < \sigma_1$. We apply our induction assumption to find for $1 \leq s < \infty$, $(\tilde{A}^s, \tilde{X}^s)$ a satellite of radial algebras of size $\leq 2^{-s}\eta$ which terminates in (A^s, X^s) and satisfies (1)–(4) for $\sigma = \sigma^s$.

In either case we may demand in addition that $2^{-s}\eta < \zeta < 2^{-s+1}\eta$ for all $(z, \omega, \zeta) \in \tilde{X}^s$. Set

$$\tilde{X} = (X \times \{0\}) \cup (\bigcup \{\tilde{X}^s : 1 \leq s < \infty\}),$$

$$\tilde{A} = \{f \in \mathcal{C}(\tilde{X}) : (x, \omega) \rightarrow f(z, \omega, 0) \text{ is in } A \text{ on } X;$$
$$f \mid \tilde{X}^s \in \tilde{A}^s, 1 \leq s < \infty\}.$$

Then $(\tilde{A}, \tilde{X})$ is a satellite of radial algebras of size $\leq \eta$ which terminates in (A, X), and evidently $L^{\sigma'}(\tilde{A}) \mid \tilde{X}^s = L^{\sigma'}(\tilde{A}^s)$ for $1 \leq s < \infty$ and any ordinal number σ'. From the usual antisymmetry argument (see [6]), it remains only to show that

$$L^{\sigma'}(\tilde{A}) \mid (X \times \{0\}) \subset \{(z, \omega, 0) \rightarrow f(z, \omega) : f \in A\} \quad \text{if} \quad 0 \leq \sigma' \leq \sigma_1,$$

and that $\{(z, \omega) \rightarrow f(z, \omega, 0) : f \in L^{\sigma_0}(\tilde{A})\}$ contains a dense subset of $L(A)$. These are achieved as in the proofs of (1) and (2) of lemma 9 in [6]. ■

We can now prove the theorem. Once we obtain the proper analogue of proposition 11 for limit ordinal numbers, we can achieve antisymmetry for all σ, $1 \leq \sigma < \sigma^*$, by adding an "analytic rectangle" as in §6 of [6]. Thus we need only find the aforementioned analogue.

So let σ_0 be a countable limit ordinal number. Let $\{\sigma^s\}_{1 \leq s < \infty}$ be an enumeration of the successor ordinal numbers $\sigma < \sigma_0$. For $1 \leq s < \infty$, let $(\tilde{A}^s, \tilde{X}^s)$ be a satellite of radial algebras such that $L^{\sigma^s}(\tilde{A}^s)$ is local but $L^{\sigma'}(\tilde{A}^s)$ is non-local for $0 \leq \sigma' < \sigma^s$. We may suppose that $2^{-s} < \zeta < 2^{-s+1}$ for $(z, \omega, \zeta) \in \tilde{X}^s$. We set

$$\tilde{X}^s_+ = \{(z, 2^{-s}\omega, \zeta) : (z, \omega, \zeta) \in \tilde{X}^s\}$$

$$\tilde{A}^s_+ = \{f \in \mathcal{C}(\tilde{X}^s_+) : (z, \omega, \zeta) \rightarrow f(z, 2^{-s}\omega, \zeta) \text{ is in } \tilde{A}^s\}$$
$$= \{(z, \omega, \zeta) \rightarrow f(z, 2^s\omega, \zeta) : f \in \tilde{A}^s\}.$$

Thus $(\tilde{A}^s_+, \tilde{X}^s_+)$ is a uniform algebra with spectrum $\tilde{X}^s_+$, $\tilde{A}^s_+$ has the usual five generators, $L^{\sigma^s}(\tilde{A}^s_+)$ is local, and $L^{\sigma'}(\tilde{A}^s_+)$ is non-local for $0 \leq \sigma' < \sigma^s$. Finally, set

$$\tilde{X}_+ = (\bar{R} \times \{0\} \times \{0\}) \cup (\bigcup \{\tilde{X}^s_+ : 1 \leq s < \infty\}),$$

$$\tilde{A}_+ = \{f \in \mathcal{C}(\tilde{X}_+) : z \rightarrow f(z, 0, 0) \text{ is holomorphic on } R; f \mid \tilde{X}^s_+ \in \tilde{A}^s_+, 1 \leq s < \infty\}.$$

The usual antisymetry argument (see [6]) allows us to see that $(\tilde{A}_+, \tilde{X}_+)$ is a uniform algebra with spectrum $\tilde{X}_+$, that $\tilde{A}_+$ has the usual five generators, and that for all ordinal numbers σ,

$$L^\sigma(\tilde{A}_+) = \{f \in \mathcal{C}(\tilde{X}_+) : z \to f(z, 0, 0) \text{ is holomorphic on } R; \quad f \mid \tilde{X}_+^s \in L^\sigma(\tilde{A}_+^s),\ 1 \leq s < \infty\},$$

$$L^\sigma(\tilde{A}_+) \mid (\bar{R} \times \{0\} \times \{0\}) = \{(z, 0, 0) \to f(z) : f \in B(R)\},$$

and

$$L^\sigma(\tilde{A}_+) \mid \tilde{X}_+^s = \tilde{A}_+^s, \qquad 1 \leq s < \infty.$$

This establishes the appropriate analogue, and completes the proof of Theorem 1.

Remark. For the algebras we have constructed, $L^{\sigma^*}(A)$ is generated by z, $1/z$, ω, ζ, and α_1' where $\alpha_1'(z, \omega, \zeta) = \alpha_1(z, \omega)$.

4. Proof of Theorem 2

The proof of Theorem 2 uses substantially the same induction procedure as we used in §3. However, there is considerably more work and somewhat less opportunity to refer to [6] for details. This is because the "building blocks" introduced in Definition 12 below are different from those used in the proof of Theorem 2 in [6], since we wish to add control over the number of generators. Furthermore, these new "building blocks" are not initially defined on their spectra, and we shall never completely identify their spectra, a fact which forces us to introduce certain intermediate steps in our constructions.

DEFINITION 12. The *annular algebra with data* $R_1, \cdots, R_n$ is the pair (A, X) where

$$X = (\bar{\Delta} \times \{0\}) \cup (\bar{R}_0 \times \bar{\Delta}) \cup (\textstyle\bigcup_{j=1}^n \bar{R}_j \times \bar{\Delta}_j) \subset \mathbf{C}^2,$$

$$\begin{aligned} A = \{f \in \mathcal{C}(X) : {} & z \to f(z, 0) \text{ is holomorphic on } \Delta; \\ & \omega \to f(z, \omega) \text{ is holomorphic on } \Delta,\ \forall z \in \bar{R}_0; \\ & \omega \to f(z, \omega) \text{ is holomorphic on } \Delta_j,\ \forall z \in \bar{R}_j,\ 1 \leq j \leq n; \\ & z \to (\partial f/\partial\omega)(z, 0) \text{ is in } B(\Delta) \text{ on } \bar{R}_n; \\ & z \to (\partial^k f/\partial\omega^k)(z, 0) \text{ is holomorphic on } R_n,\ k \geq 2\}. \end{aligned}$$

Here n is a non-negative integer, $R_0 = \{z \in \mathbf{C} : \delta_0 < |z| < 1\}$, and for $1 \leq j \leq n$,

$$R_j = \{z \in \mathbf{C} : \delta_j < |z| < 1\} \quad \text{and} \quad \Delta_j = \{\omega \in \mathbf{C} : |\omega| < \exp(-1/\delta_j)\},$$

where $0 < \delta_n < \cdots < \delta_1 < \delta_0 = 3/4$. We set $n(A, X) = n$ and

$$\text{size}_k(A, X) = \sup\{\delta_j : k < j \leq n\} \quad \text{if} \quad 0 \leq k < n,$$

$$\text{size}_k(A, X) = 0 \quad \text{if} \quad k \geq n.$$

LEMMA 13. *Let (A, X) be the annular algebra with data $R_1, \cdots, R_n$. Then (A, X) is an antisymmetric uniform algebra. If $\Sigma(A)$ is the spectrum of A, let*

$$\Phi : \Sigma(A) \to \mathbf{C}^2 \quad \text{be} \quad \Phi(\varphi) = (\varphi(z), \varphi(\omega)).$$

Then Φ is a homeomorphism of $\Sigma(A)$ onto a subset $\hat{X}$ of $\bar{\Delta} \times \bar{\Delta}$ which contains X and is such that:

(1) $(z, \omega) \in \hat{X}, |z| < \delta_n$ *imply* $\omega = 0$.

(2) *There is a (necessarily upper semi-continuous) function*

$$h : [\delta_n, 1] \to (0, 1]$$

such that for $z \in \bar{R}_n$,

$$\{\omega \in \mathbf{C} : (z, \omega) \in \hat{X}\} = \{\omega \in \mathbf{C} : |\omega| \leq h(|z|)\}.$$

If $\hat{g}$ denotes the gelfand transform of $g \in A$, then

$$\begin{aligned}\hat{A} \circ \Phi^{-1} = \{f \in \mathcal{C}(\hat{X}) : z \to f(z, 0) \text{ is holomorphic on } \Delta;\\ \omega \to f(z, \omega) \text{ is holomorphic on } \{\omega \in \mathbf{C} : |\omega| < h(|z|)\},\\ \forall z \in \bar{R}_n;\\ z \to (\partial f/\partial\omega)(z, 0) \text{ is in } B(\Delta) \text{ on } \bar{R}_n;\\ z \to (\partial^k f/\partial\omega^k)(z, 0) \text{ is holomorphic on } R_n, k \geq 2\},\end{aligned}$$

and $(f \mid X)^{\wedge} = f \circ \Phi$ for all $f \in \hat{A} \circ \Phi^{-1}$. A and $\hat{A} \circ \Phi^{-1}$ are generated by the functions z, ω, and β_k, $1 \leq k < \infty$, where $\beta_k : \bar{\Delta} \times \bar{\Delta} \to \mathbf{C}$ is defined by

$$\beta_k(z, \omega) = 0 \quad \text{if} \quad z = 0 \quad \text{and} \quad \beta_k(z, \omega) = z^{-k}\omega^2 \quad \text{if} \quad z \neq 0.$$

Proof. Clearly (A, X) is an antisymmetric uniform algebra which contains the β_k. Note that $\| \beta_k \|_X \leq \delta_n^{-k}$.

Let $f \in A$. Then f has a double Laurent expansion

$$\begin{aligned}&\sum_{k=0}^{\infty} a_k z^k + \sum_{k=0}^{\infty} b_k z^k \omega + \sum_{r=2}^{\infty} \sum_{k=-\infty}^{\infty} c_{kr} z^k \omega^r\\ &= \sum_{k=0}^{\infty} a_k z^k + \sum_{k=0}^{\infty} b_k z^k \omega + \sum_{r=2}^{\infty} \sum_{k=0}^{\infty} c_{kr} z^k \omega^r + \sum_{r=2}^{\infty} \sum_{k=-\infty}^{-1} c_{kr} \omega^{r-2} \beta_{-k}\end{aligned}$$

whose Cesaro means converge uniformly to f. Thus z, ω, and the β_k generate A.

Suppose $\varphi \in \Sigma(A)$. If $|\varphi(z)| < \delta_n$, then from $\omega^2 = z^k \beta_k$ we have

$$|\varphi(\omega)|^2 \leq |\varphi(z)|^k |\varphi(\beta_k)| \leq |\varphi(z)|^k \| \beta_k \|_X \leq (|\varphi(z)| / \delta_n)^k$$

and letting $k \to \infty$, we see that $\hat{X} = \Phi(\Sigma(A))$ satisfies (1).

If $\varphi(z) = 0$, from $\beta_k^2 = \omega^2 \beta_{2k}$ and $\varphi(\omega) = 0$ we obtain $\varphi(\beta_k) = 0 = \beta_k(\Phi(\varphi))$. If $\varphi(z) \neq 0$, from $\omega^2 = z^k \beta_k$ we obtain $\varphi(\beta_k) = \varphi(z)^{-k}\varphi(\omega)^2 = \beta_k(\Phi(\varphi))$. Thus for all $\varphi \in \Sigma(A)$, $\varphi(\beta_k) = \beta_k(\Phi(\varphi))$ for all k and, since z, ω, and the β_k generate A, Φ is injective, thus a homeomorphism.

It is clear that $X \subset \hat{X} \subset \bar{\Delta} \times \bar{\Delta}$. In view of the preceding paragraph, $\hat{X}$ consists of those points $(z_0, \omega_0) \in \bar{\Delta} \times \bar{\Delta}$ such that for all polynomials P in

z, ω, and the β_k, one has $|P(z_0, \omega_0)| \leq \|P\|_X$. We shall show that if (z_0, ω_0) has this property then so has $(z_0, \lambda\omega_0)$ for all $\lambda \in \bar{\Delta}$, whence $\hat{X}$ will satisfy (2).

First consider the case $\lambda \in \partial\Delta$. Given P, let Q be obtained from P by using $\lambda\omega$ and $\lambda^2\beta_k$ in place of ω and β_k. Then

$$Q(X) = P(X) \quad \text{and} \quad Q(z_0, \omega_0) = P(z_0, \lambda\omega_0),$$

so $|P(z_0, \lambda\omega_0)| = |Q(z_0, \omega_0)| \leq \|Q\|_X = \|P\|_X$.

Finally, given P one notes that $\lambda \to P(z_0, \lambda\omega_0)$ is continuous on $\bar{\Delta}$ and holomorphic on Δ, so that

$$\sup\{|P(z_0, \lambda\omega_0)| : \lambda \in \bar{\Delta}\} = \sup\{|P(z_0, \lambda\omega_0)| : \lambda \in \partial\Delta\} \leq \|P\|_X.$$

Let B denote the object which is alleged to equal $\hat{A} \circ \Phi^{-1}$. Since z, ω, and the β_k generate A, also $\hat{z} \circ \Phi^{-1} = z$, $\hat{\omega} \circ \Phi^{-1} = \omega$, and the $\hat{\beta}_k \circ \Phi^{-1} = \beta_k$ generate $\hat{A} \circ \Phi^{-1}$. Since these functions all lie in B, $\hat{A} \circ \Phi^{-1} \subset B$. On the other hand, if $f \in B$, its double Laurent expansion shows as before that f is in the closed algebra generated by z, ω, and the β_k, hence in $\hat{A} \circ \Phi^{-1}$.

It is clear that $(f \mid X)^\wedge = f \circ \Phi$ for all $f \in \hat{A} \circ \Phi^{-1}$. ∎

We shall hereafter identify $\Sigma(A)$ with $\hat{X}$, and $\hat{A}$ with the algebra called B in the preceding proof. Thus $(f \mid X)^\wedge = f$ for all $f \in \hat{A}$, and each $f \in A$ has a unique extension $\hat{f} \in \hat{A}$, which is such that $\|\hat{f}\|_{\hat{X}} = \|f\|_X$.

COROLLARY 14. *Let (A, X) be an annular algebra. Then*

$$\|\beta_k\|_{\hat{X}} \leq [\max(4/3, k/e)]^k \leq (2k)^k, \qquad 1 \leq k < \infty.$$

The function $\gamma = \sum_{k=1}^\infty \exp(-k!)\beta_k$ belongs to $\hat{A}$, and A and $\hat{A}$ are generated by the three functions z, ω, and γ.

Proof. From $\sup\{\delta^{-k}\exp(-1/\delta) : 0 < \delta \leq 1\} = (k/e)^k$ one obtains

$$\|\beta_k\|_X \leq [\max(4/3, k/e)]^k,$$

and the same estimate follows for $\|\beta_k\|_{\hat{X}}$. Thus $\sum_{k=1}^\infty \exp(-k!)\,\|\beta_k\|_{\hat{X}} < \infty$, whence $\gamma \in \hat{A}$. Let B denote the closed subalgebra of $\mathcal{C}(\hat{X})$ generated by z, ω, and γ. It remains to prove that $\beta_k \in B$, $1 \leq k < \infty$.

The proof is by induction on k. Suppose $K \geq 1$ and we know that $\beta_k \in B$, $1 \leq k < K$. For $1 \leq m < \infty$ we have that

$$h_m = \exp((K+m)!)[z^m\gamma - \textstyle\sum_{k=1}^{m-1}\exp(-k!)z^{m-k}\omega^2$$
$$- \exp(-m!)\omega^2 - \textstyle\sum_{k=m+1}^{K+m-1}\exp(-k!)\beta_{k-m}]$$

belongs to B. But $h_m = \beta_K + \sum_{k=K+m+1}^\infty \exp((K+m)! - k!)\beta_{k-m}$, so

$$\|h_m - \beta_K\|_{\hat{X}} \leq \textstyle\sum_{k=K+m+1}^\infty \exp((K+m)! - k!)\,\|\beta_{k-m}\|_{\hat{X}}$$
$$\leq \textstyle\sum_{k=K+m+1}^\infty \exp(-k!/2)(2(k-m))^{k-m},$$

which tends to 0 as $m \to \infty$. ∎

LEMMA 15. *Let (A, X) be the annular algebra with data $R_1, \cdots, R_n$. Then $\hat{A}$ is local, whereas*

$$H(\hat{A}) = \{f \in \mathcal{C}(\hat{X}) : z \rightarrow f(z, 0) \text{ is holomorphic on } \Delta;$$
$$\omega \rightarrow f(z, \omega) \text{ is holomorphic on } \{\omega \in \mathbf{C} : |\omega| < h(|z|)\}, \quad \forall z \in \bar{R}_n;$$
$$z \rightarrow (\partial^k f/\partial\omega^k)(z, 0) \text{ is holomorphic on } R_n, k \geq 1\}.$$

This last algebra is antisymmetric and holomorphically closed, and is generated by the functions z, ω, and ψ_k, $1 \leq k < \infty$, where $\psi_k : \bar{\Delta} \times \bar{\Delta} \rightarrow \mathbf{C}$ is defined by

$$\psi_k(z, \omega) = 0 \quad \text{if} \quad z = 0$$

and

$$\psi_k(z, \omega) = z^{-k}\omega \quad \text{if} \quad z \neq 0.$$

Proof. It is clear that $\hat{A}$ is local and that the bracketed expression, which we shall call B for the moment, is an antisymmetric holomorphically closed (provided its spectrum is $\hat{X}$) uniform algebra which contains $\hat{A}$, hence $H(\hat{A})$. If the ψ_k belong to $H(\hat{A})$, then the Laurent expansion argument will complete the proof. To show that $\psi_k \in H(\hat{A})$, it suffices to find for each $z_0 \in \bar{\Delta}\backslash\{0\}$ a neighborhood $N(z_0)$ in $\mathbf{C}$ such that ψ_k is uniformly approximable by $\hat{A}$ on $\hat{X} \cap (N(z_0) \times \mathbf{C})$. Let

$$N(z_0) = \{z \in \mathbf{C} : |z - z_0| \leq |z_0|/2\}.$$

$N(z_0)$ being a closed disc, there is a sequence $\{p_m\}_{1 \leq m < \infty}$ of polynomials in one variable such that

$$\| p_m - z^k \|_{N(z_0)} \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty.$$

Then $h_m(z, \omega) = p_m(z)\omega$ belongs to $\hat{A}$, and

$$\| h_m - \psi_k \|_{\hat{X} \cap (N(z_0) \times \mathbf{C})} \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty. \qquad \blacksquare$$

Remark. $H(\hat{A})$ is in fact generated by z, ω, and $\sum_{k=1}^{\infty} \exp(-k!)\psi_k$.

LEMMA 16. *Let (A, X) be the annular algebra with data $R_1, \cdots, R_n$. If $f \in \mathcal{C}(\hat{X})$ is locally approximable by $\hat{A}$, and if in addition f is uniformly approximable by $\hat{A}$ on an open subset U of $\hat{X}$ which contains $\{(z, 0) : |z| = \delta_n\}$, then $f \in \hat{A}$.*

Proof. Let $\{f_m\}_{1 \leq m < \infty}$ be a sequence in $\hat{A}$ such that $\| f_m - f \|_U \rightarrow 0$ as $m \rightarrow \infty$. For a certain $\varepsilon > 0$, $\{(z, \omega) : |z| = \delta_n, |\omega| \leq \varepsilon\} \subset U$, hence

$$\lim_{m \rightarrow \infty} \sup_{|z| = \delta_n} |(\partial f_m/\partial\omega)(z, 0) - (\partial f/\partial\omega)(z, 0)| = 0.$$

There is $h_m \in B(\Delta)$ such that $(\partial f_m/\partial\omega)(z, 0) = h_m(z)$ whenever $|z| = \delta_n$. Thus there is h continuous on $\{z \in \mathbf{C} : |z| \leq \delta_n\}$ and holomorphic on its interior such that $(\partial f/\partial\omega)(z, 0) = h(z)$ whenever $|z| = \delta_n$. Thus if we define $F : \bar{\Delta} \rightarrow \mathbf{C}$ by $F(z) = (\partial f/\partial\omega)(z, 0)$ if $|z| \geq \delta_n$ and $F(z) = h(z)$ otherwise, then F is continuous on $\bar{\Delta}$ and holomorphic on $\bar{\Delta}/\bar{R}_n$, while it is clear that F is holomorphic on R_n. It follows (by, say, Morera's theorem) that $F \in B(\Delta)$, that is, $f \in \hat{A}$. ■

DEFINITION 17. Let (A, X) be the annular algebra with data $R_1, \cdots, R_n$ and let η be a positive number. A *satellite of annular algebras of size* $\leq \eta$ *which terminates in* (A, X) is a pair (A_+, X_+) where

$$X_+ = \bigcup \{X_k \times \{\tau_k\} : 0 \leq k \leq \infty\} \subset \mathbf{C}^3,$$

$$A_+ = \{f \in \mathfrak{C}(X_+) : (z, \omega) \to f(z, \omega, \tau_k) \quad \text{is in} \quad A_k, \qquad 0 \leq k \leq \infty\}.$$

Here:

(1′) (A_k, X_k) is an annular algebra with data $R_1, \cdots, R_n, R_{n+1}^{(k)}, \cdots, R_{n(k)}^{(k)}$ where $n(k) = n(A_k, X_k) \geq n$ and $\text{size}_n(A_k, X_k) \leq \eta$.

(2′) The conditions (2)–(5) of Definition 8 hold.

LEMMA 18. *Let (A_+, X_+) be a satellite of annular algebras. Then (A_+, X_+) is a uniform algebra generated by the functions z, ω, ζ and β_k', $1 \leq k < \infty$, where*

$$\beta_k' : \bar{\Delta} \times \bar{\Delta} \times \mathbf{C} \to \mathbf{C}$$

is defined by

$$\beta_k'(z, \omega, \zeta) = \beta_k(z, \omega).$$

The maximal sets of antisymmetry for A_+ are the sets $X_k \times \{\tau_k\}$, and

$$A_+ \mid (X_k \times \{\tau_k\}) = \{(z, \omega, \tau_k) \to f(z, \omega) : f \in A_k\}, \quad 0 \leq k \leq \infty.$$

Proof. Once we check that X_+ is compact and that the β_k' belong to A_+, the usual antisymmetry argument [6, Lemma 7 and its proof] will complete the proof. But conditions (3)–(5) ensure that X_+ be compact and that β_k' be continuous on X_+, whence $\beta_k' \in A_+$. ■

LEMMA 19. *Let (A_+, X_+) be a satellite of annular algebras. If $\Sigma(A_+)$ is the spectrum of A_+, let $\tilde{\Phi} : \Sigma(A_+) \to \mathbf{C}^3$ be $\tilde{\Phi}(\varphi) = (\varphi(z), \varphi(\omega), \varphi(\zeta))$. Then $\tilde{\Phi}$ is a homeomorphism of $\Sigma(A_+)$ onto $\hat{X}_+ = \bigcup \{\hat{X}_k \times \{\tau_k\} : 0 \leq k \leq \infty\}$,*

$$\hat{A}_+ \circ \tilde{\Phi}^{-1} = \{f \in \mathbf{C}(\hat{X}_+) : (z, \omega) \to f(z, \omega, \tau_k) \quad \textit{is in} \quad \hat{A}_k, \qquad 0 \leq k \leq \infty\},$$

the maximal sets of antisymmetry for $\hat{A}_+ \circ \tilde{\Phi}^{-1}$ are the sets $\hat{X}_k \times \{\tau_k\}$,

$$(\hat{A}_+ \circ \tilde{\Phi}^{-1}) \mid (\hat{X}_k \times \{\tau_k\}) = \{(z, \omega, \tau_k) \to f(z, \omega) : f \in \hat{A}_k\}, \quad 0 \leq k \leq \infty,$$

and

$$(f \mid X_+)^\wedge = f \circ \tilde{\Phi} \quad \textit{for all} \quad f \in \hat{A}_+ \circ \tilde{\Phi}^{-1}.$$

$\hat{A}_+ \circ \tilde{\Phi}^{-1}$ is generated by the functions z, ω, ζ, and β_k', $1 \leq k < \infty$.

Proof. All this follows from properly interpreting Lemmas 13 and 18 in the light of some observations of I. Glicksberg [2, p. 419] about the antisymmetric decomposition. ■

We can hereafter identify $\Sigma(A_+)$ with $\hat{X}_+$ and $\hat{A}_+$ with

$$\{f \in \mathcal{C}(\hat{X}_+) : (z, \omega) \to f(z, \omega, \tau_k) \quad \text{is} \quad \text{in } \hat{A}_k, \qquad 0 \leq k \leq \infty\}.$$

Thus $(f \mid X_+)^\wedge = f$ for all $f \in \hat{A}_+$, and $f \in A_+$ has a unique extension $\hat{f}$ in $\hat{A}_+$, which is such that $\| \hat{f} \|_{\hat{X}_+} = \| f \|_{X_+}$.

Corollary 20. *Let (A_+, X_+) be a satellite of annular algebras. The function $\gamma' = \sum_{k=1}^{\infty} \exp(-k!)\beta_k'$ belongs to $\hat{A}_+$, and A_+ and $\hat{A}_+$ are generated by the functions z, ω, ζ, and γ'.*

Proof. Since the β_k' belong to $\hat{A}_+$ by lemma 19, it follows as in the proof of Corollary 14 that $\gamma' \in \hat{A}_+$. From corollary 14 and the usual antisymmetry argument, it follows that z, ω, ζ, and γ' generate $\hat{A}_+$. ∎

Remark. We have proceeded indirectly, first constructing (A_+, X_+) and then identifying $\Sigma(A_+)$ with $\bigcup\{\hat{X}_k \times \{\tau_k\}\}$, rather than directly taking this union and the algebra of continuous functions on it which "belong to $\hat{A}_k$" on $\hat{X}_k \times \{\tau_k\}$, because we had no way of knowing *a priori* that this union is compact, or that the β_k' are continuous on it.

Lemma 21. *Let (A_+, X_+) be a satellite of annular algebras. Then for every ordinal number σ,*

$$H^\sigma(\hat{A}_+) \subset \{f \in \mathcal{C}(\hat{X}_+) : (z, \omega) \to f(z, \omega, \tau_k) \quad \textit{is in} \quad H(\hat{A}_k), \qquad 0 \leq k \leq \infty\},$$

which is holomorphically closed. The maximal sets of antisymmetry for each $H^\sigma(\hat{A}_+)$ are the sets $\hat{X}_k \times \{\tau_k\}$.

Proof. This is evident, given that each $H(\hat{A}_k)$ is antisymmetric and holomorphically closed (Lemma 15). ∎

Proposition 22. *Let (A, X) be an annular algebra, let $\eta > 0$, and let σ be a finite or countable successor ordinal number. Then there exists (A_+, X_+) a satellite of annular algebras of size $\leq \eta$ which terminates in (A, X) and which has the following additional properties*:

(1) *For $0 \leq \sigma' \leq \sigma$, there is a partition $\{k : 0 \leq k \leq \infty\} = S_{\sigma'} \cup T_{\sigma'}$ such that*

$$H^{\sigma'}(\hat{A}_+) = \{f \in \mathcal{C}(\hat{X}_+) : (z, \omega) \to f(z, \omega, \tau_k) \quad \textit{is in} \quad \hat{A}_k, \qquad \forall k \in S_{\sigma'};$$
$$(z, \omega) \to f(z, \omega, \tau_k) \quad \textit{is in} \quad H(\hat{A}_k), \qquad \forall k \in T_{\sigma'}\}$$

and for all $k \in T_{\sigma'}$,

$$H^{\sigma'}(\hat{A}_+) \mid (\hat{X}_k \times \{\tau_k\}) = \{(z, \omega, \tau_k) \to f(z, \omega) : f \in H(\hat{A}_k)\}.$$

(2) *$S_\sigma = \emptyset$ (and so $H^\sigma(\hat{A}_+)$ is holomorphically closed).*

(3) *$S_{\sigma'} \neq \emptyset$ for $0 \leq \sigma' < \sigma$ (and so $H^{\sigma'}(\hat{A}_+)$ is non holomorphically closed for $0 \leq \sigma' < \sigma$).*

Proof. The proof is essentially that of proposition 11. We again facilitate the induction by replacing (3) by:

(4) If $0 \leq \sigma' < \sigma$, then $\infty \in S_{\sigma'}$.

For $\sigma = 1$, again set $\tau_k = 2^{-k}\eta$ and $(A_k, X_k) = (A, X), 0 \leq k \leq \infty$.

Again, suppose that $\sigma_0 = \sigma_1 + 1$, $1 \leq \sigma_1 < \sigma^*$, and that we have proved the proposition for successor ordinals $\sigma \leq \sigma_1$. Suppose that (A, X) has data $R_1, \cdots, R_n$. For $1 < s \leq \infty$ set $\delta^s_{n+1} = 2^{-s} \min(\delta_n, \eta)$,

$$R^s_{n+1} = \{z \in \mathbf{C} : \delta^s_{n+1} < |z| < 1\}, \quad \Delta^s_{n+1} = \{\omega \in \mathbf{C} : |\omega| < \exp(-1/\delta^s_{n+1})$$

and let (A^s, X^s) be the annular algebra with data $R_1, \cdots, R_n, R^s_{n+1}$. Construct the (A^s_+, X^s_+) and (A_+, X_+) as $(\tilde{A}^s, \tilde{X}^s)$ and $(\tilde{A}, \tilde{X})$ were constructed in the proof of Proposition 11. Then (A_+, X_+) is a satellite of annular algebras of size $\leq \eta$ which terminates in (A, X). Evidently $H^{\sigma'}(\hat{A}_+) \mid \hat{X}^s_+ = H^{\sigma'}(\hat{A}^s_+)$ for $1 \leq s < \infty$ and any ordinal number σ'. From the usual antisymmetry argument, it remains to show that

(5) $H^{\sigma'}(\hat{A}_+) \mid (\hat{X} \times \{0\}) \subset \{(z, \omega, 0) \to f(f, \omega) : f \in \hat{A}\}$ if $0 \leq \sigma' \leq \sigma_1$,

and

(6) $\{(z, \omega) \to f(z, \omega, 0) : f \in H^{\sigma_0}(\hat{A}_+)\}$ contains a dense subset of $H(\hat{A})$.

One achieves (5) as in the proof of (4) of §7 of [6], with the aid of Lemmas 16 and 21. Together with antisymmetry, (5) implies that

(7) $$H^{\sigma'}(\hat{A}_+) = \{f \in \mathcal{C}(\hat{X}_+) : (z, \omega) \to f(z, \omega, 0) \text{ is in } \hat{A} \text{ on } \hat{X};\ f \mid \hat{X}^s_+ \in H^{\sigma'}(\hat{A}^s_+),\ 1 \leq s < \infty\},\quad 0 \leq \sigma' \leq \sigma_1.$$

The argument for (6) goes along the same lines as that for (5) of §7 of [6], but a few details are in order. By Lemma 15, it suffices to verify that there is $f_k \in H^{\sigma_0}(\hat{A}_+)$ such that $f_k(z, \omega, 0) = \psi_k(z, \omega)$ whenever $(z, \omega) \in \hat{X}$. Consider $\psi'_k : \bar{\Delta} \times \bar{\Delta} \times \mathbf{C} \to \mathbf{C}$ defined by $\psi'_k(z, \omega, \zeta) = \psi_k(z, \omega)$. We assert that ψ'_k is locally approximable by $H^{\sigma_1}(\hat{A}_+)$, whence we can take $f_k = \psi'_k$.

To begin with, $\psi'_k \mid \hat{X}_+$ is continuous except possibly at points of

$$Y = \{(z, \omega, \zeta) \in \hat{X}_+ : z = 0\},$$

and on this locus $\psi'_k = 0$. Now $(\psi'_k)^2 = \beta'_{2k}$ is known to be continuous on $\hat{X}_+$ (Lemma 19), so as $(z, \omega, \zeta) \in \hat{X}_+$ approaches Y, $\beta'_{2k}(z, \omega, \zeta)$, and so $\psi'_k(z, \omega, \zeta)$, approaches 0. Thus $\psi'_k \in \mathcal{C}(\hat{X}_+)$.

It follows from Lemma 15 and our induction assumption that

$$\psi'_k \mid \hat{X}^s_+ \in H^{\sigma_1}(\hat{A}_+) \mid \hat{X}^s_+, \quad 1 \leq s < \infty.$$

Further, the argument in the proof of Lemma 15 applies to show that ψ'_k is locally approximable by $\hat{A}_+$, and so by $H^{\sigma_1}(\hat{A}_+)$, at each point of $\hat{X}_+ \backslash Y$. It remains to consider the point (0, 0, 0). For $1 \leq m < \infty$, let $g_m : \hat{X}_+ \to \mathbf{C}$ be $g_m = \psi'_k$ on

$$\bigcup \{\hat{X}^s_+ : 1 \leq s \leq m\},$$

$g_m = 0$ on

$$(\hat{X} \times \{0\}) \cup (\bigcup \{\hat{X}^s_+ : m + 1 \leq s < \infty\}).$$

Then $g_m \in H^{\sigma_1}(\hat{A}_+)$ by (7) and

$$\| g_m - \psi'_k \|_U \to 0 \quad \text{as} \quad m \to \infty,$$

where $U = \{(z, \omega, \zeta) \in \hat{X}_+ : |z| < \delta_n\}$. The proof is complete. ■

The theorem can now be proved essentially as was Theorem 1. The addition of an "analytic rectangle" is carried out as in §7 rather than §6 of [6].

If σ_0 is a countable limit ordinal, we again let $\{\sigma^s\}_{1 \le s < \infty}$ enumerate the successor ordinals $\sigma < \sigma_0$, and we choose (A^s_+, X^s_+) a satellite of annular algebras such that $H^{\sigma^s}(\hat{A}^s_+)$ is holomorphically closed, $H^{\sigma'}(\hat{A}^s_+)$ is non-holomorphically closed for $0 \le \sigma' < \sigma^s$, and $2^{-s} < \zeta < 2^{-s+1}$ whenever $(z, \omega, \zeta) \in \hat{X}^s_+$. We set

$$\tilde{X}^s_+ = \{(z, 2^{-s}\omega, \zeta) : (z, \omega, \zeta) \in \hat{X}^s_+\},$$

$$\tilde{A}^s_+ = \{f \in \mathcal{C}(\tilde{X}^s_+) : (z, \omega, \zeta) \to f(z, 2^{-s}\omega, \zeta) \quad \text{is in} \quad \hat{A}^s_+\},$$

$$\tilde{X}_+ = (\bar{\Delta} \times \{0\} \times \{0\}) \cup (\bigcup \{\tilde{X}^s_+ : 1 \le s < \infty\}),$$

$$\tilde{A}_+ = \{f \in \mathcal{C}(\tilde{X}_+) : z \to f(z, 0, 0) \quad \text{is holomorphic on } \Delta;$$
$$f \mid \tilde{X}^s_+ \in \tilde{A}^s_+, \quad 1 \le s < \infty\}.$$

Clearly $\tilde{X}^s_+$ is the spectrum of $\tilde{A}^s_+$, $\tilde{A}^s_+$ is generated by z, ω, ζ, and the β'_k, $H^{\sigma^s}(\tilde{A}^s_+)$ is holomorphically closed, $H^{\sigma'}(\tilde{A}^s_+)$ is non-holomorphically closed for $0 \le \sigma' < \sigma^s$, and $\tilde{X}_+$ is compact, so $(\tilde{A}_+, \tilde{X}_+)$ is a uniform algebra. Observe that

$$\| \beta'_k \|_{\tilde{X}^s_+} = 4^{-s} \| \beta'_k \|_{\hat{X}^s_+} \le 4^{-s}(2k)^k$$

(Corollary 14), so β'_k is continuous on $\tilde{X}_+$, hence belongs to $\tilde{A}_+$. Then $\| \beta'_k \|_{\tilde{X}_+} \le (2k)^k$ implies that $\gamma' = \sum_{k=1}^{\infty} \exp(-k!)\beta'_k$ belongs to $\tilde{A}_+$. The usual antisymmetry argument now shows that $\tilde{X}_+$ is the spectrum of $\tilde{A}_+$, that $\tilde{A}_+$ is generated by z, ω, ζ, and γ' (or, alternatively, by z, ω, ζ, and the β'_k), and that for all ordinals σ,

$$H^{\sigma}(\tilde{A}_+) = \{f \in \mathcal{C}(\tilde{X}_+) : z \to f(z, 0, 0) \quad \text{is holomorphic on } \Delta;$$
$$f \mid \tilde{X}^s_+ \in H^{\sigma}(\tilde{A}^s_+), \quad 1 \le s < \infty\},$$

$$H^{\sigma}(\tilde{A}_+) \mid (\bar{\Delta} \times \{0\} \times \{0\}) = \{(z, 0, 0) \to f(z) : f \in B(\Delta)\},$$

and

$$H^{\sigma}(\tilde{A}_+) \mid \tilde{X}^s_+ = \tilde{A}^s_+, \quad 1 \le s < \infty.$$

This establishes the analogue of Proposition 22 for σ_0, and completes the proof of Theorem 2.

Remark. For the algebras we have constructed to satisfy the conditions of Theorem 2, $H^{\sigma^*}(A)$ is generated by z, ω, ζ, and $\sum_{k=1}^{\infty} \exp(-k!)\psi'_k$ (or, alternatively, by z, ω, ζ, and the ψ'_k).

5. Some Problems

Here we state and comment on a number of apparently open questions which concern non-local algebras and which are not mentioned in [6]. We shall let (A, X) denote a uniform algebra with spectrum X.

(1) (A. Bernard) Suppose that f belongs locally to A and does not take the value 0. Then, because $L(A)$ has spectrum X, $1/f \in L(A)$. Must $1/f$ in fact belong locally to A? It is easy to prove that the corresponding question for A-holomorphic functions has an affirmative answer.

This question leads to the problem of identifying the spectra of certain Banach algebras, which in turn has a relation to some questions about hulls. To see how this comes about in a simple case, let us suppose that X can be covered by two open sets U, V such that f belongs to A on U and on V. Let B consist of all functions on X which belong to A on U and on V. B is not generally a uniform algebra, but it is the quotient of one. Indeed,

$$\tilde{B} = \{(f, g) \in A \oplus A : f = g \quad \text{on} \quad U \cap V\}$$

is a uniform algebra with spectrum obtained by identifying two copies of X along hull (kernel $(U \cap V)$) [7], and there is a homomorphism of $\tilde{B}$ onto B obtained by taking (f, g) onto the function which is f on U and g on V. The kernel J of this homomorphism consists then of those (f, g) such that $f = 0$ on U and $g = 0$ on V, and the spectrum of B is the hull of J in the spectrum of $\tilde{B}$. This hull consists in turn of hull (kernel (U)) in the first copy of X, together with hull (kernel (V)) in the second. It follows that the hull of J will project injectively onto X if and only if

$$U \cup \text{hull (kernel } (U \cap V)) \quad \text{and} \quad V \cup \text{hull (kernel } (U \cap V))$$

are hulls. In this case, of course, B will have spectrum X, and so $1/f \in B$.

The reader will readily supply other problems in the symbolic calculus of the functions locally belonging to A, which are subject to a similar analysis. It is worth noting that none of these questions depends on the uniform algebra setting; they can be equally well formulated for commutative Banach algebras with unit.

(2) There exists a non-local uniform algebra with three generators [6, pp. 740–741]. On the other hand, from Mergelyan's theorem it follows that a singly-generated uniform algebra must be holomorphically closed. What about two generators? One can equally well try to reduce the number of generators in theorems 1 and 2 for $\sigma \geq 2$.

(3) Every non-local uniform algebra of which I am aware is based on the fundamental example of E. Kallin [3], which uses a cross-sectional derivative in a direction which "is not always there". Are there other ways to construct non-local algebras? The answers to some of our other questions may well depend on such methods.

The remaining questions ask whether certain features of A are necessarily retained by $L(A)$ or $H(A)$. Each can equally well be formulated for all $L^\sigma(A)$ or all $H^\sigma(A)$, and the two formulations are not in general (obviously) equivalent.

(4) Can $L(A)$ or $H(A)$ ever require more generators than A? Fewer generators? In such an example, at least one of A, $L(A)$, $H(A)$ must be finitely generated. If one is finitely-generated, must the others be?

(5) Evidently the antisymmetric decompositions of $L(A)$ and $H(A)$ refine that of A. Must they in fact be the same as that of A?

(6) Is it possible to have $A \neq \mathcal{C}(X)$ but $L(A) = \mathcal{C}(X)$ or $H(A) = \mathcal{C}(X)$?

(7) It is known that $L(A)$ and $H(A)$ have the same Šilov boundaries and the same spectra as A [8], [4]; see also [9, 14.9]; these are essentially formal consequences of H. Rossi's local maximum modulus theorem [5]. Must $L(A)$ and $H(A)$ have the same Choquet boundaries as A? This appears not to be a formal consequence of the local maximum modulus theorem, even in its strengthened forms [1]; see also [9, §9].

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University of Connecticut
Storrs, Connecticut
University Scientifique et Médicale de Grenoble
Grenoble, France