INTEGRATION AND ω-INTEGRATION IN A BANACH SPACE

BY
LARRY F. BENNETT

1. Introduction

It is known for real-valued functions f, g, and h defined on an interval $[a, b]$ that if f is bounded on $[a, b]$ and if $k(x) = \int_a^x g[dh]$, then $\int_a^b f[dk]$ exists if and only if $\int_a^b fg[dh]$ exists, and if the two integrals exist, then they are equal (see Hildebrandt [4, p. 53]). However, the situation for the integrals of functions in a Banach space is somewhat different. Therefore, theorems such as 2.5, 2.6, and 2.7 presented by MacNerney [6] appear. This paper is devoted to obtaining substitution theorems for a special form of integral in a Banach space. A theory of integration of point-interval functions with values in a Banach space is given which contains as special cases the integrals presented by Gowurin [3], MacNerney [6], Bochner and Taylor [1], and Hille and Phillips [5, pp. 62–67]. A substitution theorem, Theorem 3.5, is presented which contains all of the substitution theorems given by MacNerney [6] as special cases. In order to develop further substitution theorems, the concept of ω-integration (not to be confused with the ω-property developed in [3]) is introduced to generate additional substitution theorems as well as a special integration by parts theorem involving three functions.

2. Notation and terminology

The notation introduced in this section will be employed without further comment throughout the remainder of this paper. The symbol J will denote the closed interval $[a, b]$, while $X_1, X_2, \cdots, X_m$, and X will be Banach spaces. The norm of an element x in $X_1, X_2, \cdots, X_m$, or X will be designated by writing $\| x \|$. If D is any nonempty set and f is a function on D into X, then f is *bounded* on D if

$$\| f \|_D = \sup \{ \| f(x) \| : x \in D\} < +\infty .$$

When it is convenient, the symbol $\| f \|_D$ will be replaced by $\| f \|$. If D is a topological space and f is a function on D into X, then the continuity of f on D is defined in the usual manner. If f is a function from J into X, then f is of *bounded variation* on J if

$$V(f; J) = V(f; a, b) = \sup \{ \textstyle\sum_{i=1}^{n} \| f(t_i) - f(t_{i-1}) \| \} < +\infty,$$

where the indicated supremum is to be taken over all sets of the form

$$\{a = t_0 < t_1 \cdots < t_n = b\}.$$

A *product operator* P from $X_{i=1}^{m} X_i$ into X is any multilinear function having the

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property that $\| P(x_1, x_2, \cdots, x_n) \| \leqq \prod_{i=1}^{m} \| x_i \|$. Such a function P is clearly continuous.

3. Integration in a Banach space and substitution theorems

A *point-interval* in J is any ordered pair of the form (τ, T), where T is a closed subinterval of J and $\tau \epsilon T$. Following McShane [7], *a partition* of J is any collection $\pi = \{(\chi_i, X_i)\}_{i=1}^{n}$ of point-intervals $(\chi_i, X_i) = (\chi_i, [x_{i-1}, x_i])$ in J such that $a = x_0$, $b = x_n$, and $x_i \leqq x_{i+1}$ for $i = 0, 1, \cdots, n-1$. Each of the points χ_i is termed an *intermediate* point of π and the points x_i are *cut points* of π. The *norm* of π is the number

$$\| \pi \| = \max_{i=1}^{n} (x_i - x_{i-1}).$$

In denoting partitions, we will accept the convention that $A_i = [a_{i-1}, a_i]$, $B_i = [b_{i-1}, b_i]$, etc. Let P be the set of all partitions of J, and let Pt be the set of all point-intervals in J. Define two relations, $\geq$ and $\geqq$, on P by requiring $\pi_1 \geq \pi_2$ if $\| \pi_1 \| \leqq \| \pi_2 \|$ and $\pi_1 \geqq \pi_2$ if the set of cut points of π_1 contains the set of cut points of π_2. The ordered pair $(P, \geq)$ is a directed set.

A *point-interval function* on J into X is any function on Pt into X which has the property that $f(\tau, T) = 0$ if T is degenerate. A function S on P into X is defined by setting $S(\pi) = S(f; \pi) = \sum_{I \epsilon \pi} f(I)$, where the indicated sum is taken over all point-intervals in π. The point-interval function f is *integrable* over J or $\int_a^b f$ exists of $\lim_{\pi \epsilon P} S(\pi)$ exists as a limit with respect to the directed set $(P, \geq)$, and in this case, the *integral* of f over J is the vector $\int_a^b f = \lim_{\pi \epsilon P} S(\pi)$. If $a = b$, it follows from the definition that $\int_a^a f = 0$. We define $\int_b^a f = -\int_a^b f$. It is easy to show that $\int_a^b f$ exists if and only if S is Cauchy; that is, for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\pi_\alpha \epsilon P (\alpha = 1, 2)$, $\| \pi_1 \| < \delta$ and $\| \pi_2 \| < \delta$ implies $\| S(\pi_1) - S(\pi_2) \| < \varepsilon$. However, the following modification of the Cauchy condition is more useful.

THEOREM 3.1. *If f is a point-interval function on J into X, then $\int_a^b f$ exists if and only if for each $\varepsilon > 0$ there is a $\delta = \delta_\varepsilon > 0$ such that $\pi_1 \epsilon P$, $\| \pi_1 \| < \delta$, $\pi_2 \epsilon P$, and $\pi_2 \geqq \pi_1$ implies $\| S(\pi_1) - S(\pi_2) \| < \varepsilon$.*

Without making any additional assumptions, one can prove a great number of results for integration of point-interval functions with values in a Banach space which hold for integrals of so-called real-valued "interval functions". Burkill [2] first considered the possibility of taking integrals of real-valued interval functions, and a thorough treatment of this subject may be found in Hildebrandt [4]. Corresponding theorems for integrals of point-interval functions in a Banach space will be used in many cases without further comment.

If f is a function on J into X, then two point-interval functions are generated in a natural way by the function f. This is accomplished by thinking of $f(\tau, T) = f(\tau)$ and $df(\tau, T) = f(t) - f(s)$ for each point-interval $(\tau, T) = (\tau, [s, t])$ in J. If g is an integrable point-interval function from J into X, then $\int g$ will denote the point-interval function on J into X defined by setting

$(\int g)(I) = \int_s^t g$ for each point-interval $I = (\tau, [s, t])$ in J. For simplicity, $\int_I g$ will be used to denote $(\int g)(I)$. If P is a product operator on $X_{i=1}^m X_i$ into X and f_i is a point-interval function on J into X_i $(i = 1, 2, \cdots, m)$, then $P(f_1, \cdots, f_m)$ will be used to denote the point-interval function on J into X which is defined by setting

$$P(f_1, \cdots, f_m)(I) = P(f_1(I), \cdots, f_m(I))$$

for each point-interval I in J.

The following theorem is a generalization of Theorem 2.2 in [6].

THEOREM 3.2. *Suppose that P is a product operator on $X_{i=1}^m X_i$ into X, f_i is a bounded point-interval function on J into X_i $(1 \leqq i \leqq m - 1)$, h is a function of bounded variation on J into X_m, and $F = P(f_1, \cdots, f_{m-1}, dh)$. If $\int_a^b F$ exists and $c = \prod_{i=1}^{m-1} \| f_i \|$, then*

(a) *$\| \int_a^b F \| \leqq cV(h; J)$ and the function $\int_a^x F$ is of bounded variation on J, and*

(b) *the function $\int_a^x F$ is continuous if h is continuous.*

An easily established integration by parts theorem is given below.

THEOREM 3.3. *Suppose that f and g are functions on J into X_1 and X_2 respectively and P is a product operator on $X_1 \times X_2$ into X. Then $P(f, dg)$ is integrable if and only if $P(df, g)$ is integrable, and if the two integrals exist, then*

$$\int_a^b [dP(f, g) - P(df,g)] = \int_a^b P(f, dg).$$

An existence theorem is given below. If $m > 2$, then it appears that the continuity requirement for f_m in part (b) may not be dropped in general. However, for $m = 2$, Theorems 3.3 and 3.4(a) show that $\int_a^b P(df_1, f_2)$ exists if f_1 is continuous and f_2 is of bounded variation on J.

THEOREM 3.4. *Suppose that P is a product operator on $X_{i=1}^m X_i$ into X and f_i is a function on J into X_i $(1 \leqq i \leqq m)$. Let $F = P(f_1, \cdots, f_{m-1}, df_m)$. Then $\int_a^b F$ exists if either*

(a) *f_i is continuous for $i = 1, 2, \cdots, m - 1$ and f_m is of bounded variation, or*

(b) *f_i is of bounded variation for $i = 1, 2, \cdots, m - 1$ and f_m is continuous and of bounded variation.*

Since the proof of part (a) is not difficult and is similar to the usual proof given for the case $m = 2$, only the proof for part (b) is presented here. For each $i = 1, 2, \cdots, m - 1$, set

$$c_i = \prod \{ \| f_j \| : 1 \leqq j \leqq m - 1, j \neq i\} \quad \text{and} \quad c = \max \{c_i : 1 \leqq i \leqq m - 1\}.$$

Let $\varepsilon > 0$ be given and choose $\varepsilon^* > 0$ such that $\varepsilon^* c\{ \sum_{k=1}^{m-1} V(f_k; J)\} < \varepsilon$. Choose $\delta = \delta_\varepsilon > 0$ in such a way that $(s, t) \in [a, b]^2$, $| s - t | < \varepsilon$ implies $V(f_m; s, t) < \varepsilon^*$. If $\pi_1 = \{ (\tau_i, T_i)\}_{i=1}^n$ and $\pi_2 = \{ (\sigma_j; S_j)\}_{j=1}^m$ are two partitions of J such that $\| \pi_1 \| < \delta$, $\pi_2 \geqq \pi_1$, and for each i, $\{ S_{j_i}, S_{j_i+1}, \cdots, S_{j_i'}\}$

is the set of S's contained in T_i, then

$$\begin{aligned} \| S(\pi_1; F) - S(\pi_2; F) \| &\leqq \sum_{i=1}^{n} \sum_{j=j_i}^{j_i'} \sum_{k=1}^{m-1} cV(f_k; T_i)V(f_m; S_j) \\ &= c \sum_{i=1}^{n} \sum_{k=1}^{m-1} V(f_k; T_i)V(f_m; T_i) \\ &< c\varepsilon^* \left(\sum_{k=1}^{m-1} V(f_k; J)\right) \\ &< \varepsilon. \end{aligned}$$

Hence, by Theorem 3.1, $\int_a^b F$ exists.

Part (a) of the substitution theorem given below contains Theorems 2.5, 2.6, and 2.7 of MacNerney [6] as special cases, while part (b) is essentially new.

THEOREM 3.5. *Suppose that P_1 is a product operator on $X_{i=1}^{n_1} X_i$ into X, P_2 is a product operator on $X_{i=n_1+1}^{m} X_i$ into X_{n_1}, f_i is a function on J into X_i $(1 \leqq i \leqq n_1 - 1)$, g_i is a bounded point-interval function on J into X_i $(n_1 + 1 \leqq i \leqq m - 1)$, and h is a function of bounded variation on J into X_m. Suppose that*

$$T = P_2(g_{n_1+1} \cdots, g_{m-1}, dh)$$

is integrable on J. Set $F = P_1(f_1, \cdots, f_{n_1-1}, T)$ and $G = P_1(f_1, \cdots, f_{n_1-1}, \int T)$. If $\int_a^b F$ exists and either

(a) *each f_i is continuous, or*

(b) *each f_i is of bounded variation and h is continuous, then G is integrable on J and $\int_a^b F = \int_a^b G$.*

The proof for part (b) will be given, while the proof for part (a) is similar. By Theorem 3.2, the function $\int_a^x T$ is continuous and of bounded variation on J and hence $\int_a^b G$ exists by Theorem 3.4 (b).
Set

$$c = \left(\prod_{j=n_1+1}^{m-1} \| g_i \| \right) \left(\max_{i=1}^{n_1-1} \{\prod \| f_j \| : j \neq i\}\right).$$

Let $\varepsilon > 0$ be given, and choose $\varepsilon^* > 0$ such that $\varepsilon^* c\{\sum_{i=1}^{n_1-1} V(f_i; J)\} < \varepsilon$. Since h is continuous on J there is a $\delta > 0$ such that $|t - s| < \delta$ implies $V(h; s, t) < \varepsilon^*$. Let $\pi = \{(t_i, T_i)\}_{i=1}^{n}$ be any partition of J with $\| \pi \| < \delta$, and for $i = 1, 2, \cdots, n$, let K_i be the point-interval function on T_i into X defined by setting

$$K_i(I) = P_1(f_1(t_i), \cdots, f_{n_1-1}(t_i), T(I))$$

for each point-interval I in T_i. Then,

$$\begin{aligned} \int_a^b F = \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} F &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (F - K_i) + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} K_i \\ &= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (F - K_i) + S(G; \pi), \end{aligned}$$

and

$$\left\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (F - K_i) \right\| < c\varepsilon^* \{\sum_{i=1}^{n_1-1} V(f_i; J)\} < \varepsilon.$$

Thus, the result follows.

4. ω-integration in a Banach space

Let f be a point-interval function on J into X. The *sum oscillation* of f on J will be defined as the extended real number

$$\omega(Sf;\, J) = \sup_{(\pi_1, \pi_2)} \| S(f, \pi_1) - S(f, \pi_2) \|,$$

where the supremum is taken over all partitions π_1 and π_2 of J. If $\omega(Sf;\, J)$ is real, then a point-interval function $\omega(Sf)$ on J into R may be defined by setting $\omega(Sf)(\tau, T) = \omega(Sf;\, T)$ for each point-interval (τ, T) in J. The point-interval function f will be termed *ω-integrable* on J if $\omega(Sf;\, J)$ is finite and $\int_a^b \omega(Sf) = 0$. It is easy to show that if f is ω-integrable on J, then f is integrable on J and f is ω-integrable on each closed subinterval of J. If X is the set R of real numbers with its usual topology, then f is integrable if and only if f is ω-integrable (see [4], p. 30). Using this fact, it can be demonstrated that a similar theorem holds if X is a finite dimensional inner product space. The author conjectures that integrability and ω-integrability are not equivalent in general Banach spaces, but leaves this as an open question.

The following theorem is proven in essentially the same way as the case when X is equal to R.

THEOREM 4.1. *Suppose that f is a point-interval function on J into X. If f is ω-integrable on J, then for each $\varepsilon > 0$ there is a $\delta = \delta_\varepsilon > 0$ such that for any partition $\pi = \{I_i\}_{i=1}^n$* of J, *$\| \pi \| < \delta$ implies*

$$\sum_{i=1}^n \left\| \int_{I_i} f - f(I_i) \right\| < \varepsilon.$$

In particular, if for each $I = (\tau, [s, t]) \in Pt$, we set $l(I) = t - s$, it follows that

$$\lim_{l(I) \to 0} \left\| \int_I f - f(I) \right\| = 0.$$

In order to establish the existence of a large class of functions which are ω-integrable, the next result is included. The proof is omitted since it is fairly simple.

THEOREM 4.2. *If P is a product operator on $X_1 \times X_2$ into X, f is a continuous function on J into X_1, and g is a function of bounded variation on J into X_2, then $P(f, dg)$ is ω-integrable on J.*

A larger class of ω-integrable functions is obtained by means of the next theorem and its corollary.

THEOREM 4.3. *If P is a product operator on $X_1 \times X_2$ into X and f_i is a function on J into X_i $(i = 1, 2)$, then $P(f_1, df_2)$ is ω-integrable on J if and only if $P(df_1, f_2)$ is ω-integrable on J.*

If $I = [c, d] \subseteq [a, b]$ and $\pi_1 = \{(\tau_i, T_i)\}_{i=1}^{n}$ and $\pi_2 = \{(\sigma_j, S_j)\}_{j=1}^{m}$ are any two partitions of $[c, d]$, then set $x_0 = c$, $x_{n+1} = d$, $\chi_{n+1} = t_n$, $y_0 = c$, $y_{m+1} = d$, $\gamma_{m+1} = d$, $x_i = \tau_i$ and $\chi_i = t_{i-1}$ $(1 \leqq i \leqq n)$, and $y_j = \sigma_j$ and $\gamma_j = s_{j-1}$ $(1 \leqq j \leqq m)$. Now form partitions $\pi_1' = \{(\chi_i, X_i)\}_{i=1}^{n+1}$ and $\pi_2' = \{(\gamma_j, Y_j)\}_{j=1}^{m+1}$ of $[c, d]$. Then

$$\| S(P(df_1, f_2); \pi_1) - S(P(df_1, f_2); \pi_2) \|$$
$$= \| S(P(f_1, df_2); \pi_1') - S(P(f_1, df_2); \pi_2') \| \leqq \omega(SP(f_1, df_2); I),$$

so that $\omega(SP(df_1, f_2; I) \leqq \omega(SP(f_1, df_2); I)$. A similar argument shows that the reverse inequality holds and therefore equality must hold. The conclusion of the theorem follows directly from this.

COROLLARY. *If P is a product operator on $X_1 \times X_2$ into X, if f_1 is a continuous function on J into X_1, and if f_2 is a function of bounded variation on J into X_2, then $P(df_1, f_2)$ is ω-integrable on J.*

The corollary stated above is a direct consequence of Theorems 4.2 and 4.3.

THEOREM 4.4. *Suppose that P is a product operator on $X_{i=1}^{m} X_i$ into X and f_i is a point-interval function on J into X_i $(i = 1, 2, \cdots, m)$. If f_i is bounded for $i = 1, 2, \cdots, m-1$ and if f_m is ω-integrable on J, then the point-interval function*

$$F = P(f_1, \cdots, f_{m-1}, f_m)$$

is integrable if and only if

$$G = P(f_1, \cdots, f_{m-1}, \textstyle\int f_m)$$

is integrable. If either integral exists, then the two integrals are equal.

If π is any partition of J, then

$$\| \textstyle\sum_{I\in\pi} F(I) - \sum_{I\in\pi} G(I) \| \leqq (\prod_{i=1}^{m-1} \| f_i \|) \left\{ \sum_{I\in\pi} \left\| f_m(I) - \int_I f_m \right\| \right\}.$$

Theorem 4.1 guarantees that the right hand member in the above tends to 0 as $\| \pi \| \to 0$.

THEOREM 4.5. *With the hypothesis of Theorem 4.4, F is ω-integrable if and only if G is ω-integrable. In this case, the integrals of F and G are equal.*

To prove this result, suppose that G is ω-integrable on J. Set $a = \prod_{i=1}^{m-1} \| f_i \|$. Let $\varepsilon > 0$ be given, and choose $\varepsilon^* > 0$ such that $(2a + 1)\varepsilon^* < \varepsilon$. By Theorem 4.1, there is a $\delta > 0$ such that $\| \pi \| < \delta$ implies

$$\textstyle\sum_{I\in\pi} \left\| f_m(I) - \int_I f_m \right\| < \varepsilon^*.$$

Choose such a δ small enough so that $\| \pi \| < \delta$ implies $S(\omega(SG); \pi) < \varepsilon^*$. If $I = (\tau, T) = (\tau, [s, t])$ is a point-interval in J with $| t - s | < \delta$, and if

π_1 and π_2 are partitions of T, then,

$$\| S(F; \pi_1) - S(F; \pi_2)\| \leqq a \textstyle\sum_{K\epsilon\pi_1} \left\| \int_K f_m - f(K) \right\| + \omega(SG; T) + a \textstyle\sum_{K\epsilon\pi_2} \left\| f(K) - \int_K f \right\| .$$

Let $\pi = \{ (\tau_i, T_i)\}_{i=1}^n = \{I_i\}_{i=1}^n$ be a partition of J with $\| \pi \| < \delta$, and let π_{i1} and π_{i2} be partitions of T_i $(1 \leqq i \leqq n)$. Then, using the inequality above, it follows that

$$\begin{aligned} &\textstyle\sum_{i=1}^n \| S(F; \pi_{i1}) - S(F; \pi_{i2}) \| \\ &\leqq a \left\{ \textstyle\sum_{i=1}^n \sum_{K\epsilon\pi_{i1}} \left\| f_m(K) - \int_K f_m \right\| + \sum_{i=1}^n \sum_{K\epsilon\pi_{i2}} \left\| f_m(K) - \int_K f_m \right\| \right\} \\ &+ S(\omega(SG); \pi) \\ &< \varepsilon^*(2a + 1) \\ &< \varepsilon, \end{aligned}$$

so that $\sum_{i=1}^n \omega(SF)(I_i) \leqq \varepsilon$. Since $\varepsilon > 0$ was arbitrary, F is ω-integrable on J. A similar proof may be employed to show that if F is ω-integrable on J, then so is G. The equality of the integrals follows from Theorem 4.4.

THEOREM 4.6. *Suppose that P_2 is a product operator on $X_2 \times X_3$ into X_4, P_1 is a product operator on $X_1 \times X_4$ into X, and f_i is a function on J into X_i $(i = 1, 2, 3)$. If*

(a) *f_1 and f_3 are of bounded variation on J and f_2 is continuous and of bounded variation on J, or*

(b) *f_1 and f_3 are continuous on J and f_2 is of bounded variation on J,*
then $P_1(f_1, P_2(df_2, f_3))$ and $P_1(f_1, \int P_2(df_2, f_3))$ are ω-integrable on J and

$$\int_a^b P_1(f_1, P_2(df_2, f_3)) = \int_a^b P_1\left(f_1, \int P_2(df_2, f_3)\right).$$

Only the proof of (a) will be provided since the proof of (b) is similar. Since f_2 is continuous and of bounded variation on J, it follows by Theorem 4.2 that $P_2(df_2, f_3)$ is ω-integrable on J and the function $\int_a^x P_2(df_2, f_3)$ is continuous on J. By Theorems 4.2 and 4.5,

$$P_1(f_1, \textstyle\int P_2(df_2, f_3)) \quad \text{and} \quad P_1(f_1, P_2(df_2, f_3))$$

are ω-integrable and the integrals of these two point-interval functions are equal.

THEOREM 4.7. *Let P_1 be a product operator on $X_{i=1}^{m-2} X_i$ into X, let P_2 be a product operator on $X_{m-1} \times X_m$ into X_{m-2}, and let f_i be a bounded point-interval* function on J into X_i *for $i = 1, 2, \cdots, m - 3$. Let f be a function on J into X_{m-1} and let g be a function on J into X_m. If $P_2(f, dg)$ is ω-integrable and if*

$$P_1(f_1, \cdots, f_{m-3}, P_2(df, g)) \quad \text{and} \quad P_1(f_1, \cdots, f_{m-3}, dP_2(f, g))$$

are integrable on J, *then* $P_1(f_1, \cdots, f_{m-3}, P_2(f, dg))$ *is integrable on* J *and*

$$\int_a^b P_1(f_1, \cdots, f_{m-3}, dP_2(f, g)) = \int_a^b P_1(f_1, \cdots, f_{m-3}, P_2(df, g)) + \int_a^b P_1(f_1, \cdots, f_{m-3}, P_2(f, dg)).$$

Theorem 4.3 together with the fact that $P_2(f, dg)$ is ω-integrable shows that $P_2(df, g)$ is ω-integrable. Also, $dP_2(f, g)$ is clearly ω-integrable. Using Theorem 4.4, Theorem 3.3, and Theorem 4.4 in turn, it follows that

$$P_1(f_1, \cdots, f_{m-3}, P_2(f, dg))$$

is integrable and

$$\begin{aligned}\int_a^b P_1(f_1, \cdots, f_{m-3}, dP_2(f, g)) - \int_a^b P_1(f_1, \cdots, f_{m-3}, P_2(df, g))\\ &= \int_a^b P_1(f_1, \cdots, f_{m-3}, dP_2(f, g) - P_2(df, g))\\ &= \int_a^b P_1\left(f_1, \cdots, f_{m-3}, d_s\left[\int_a^s \left(dP_2(f, g) - P_2(df, g)\right]\right)\right.\\ &= \int_a^b P_1\left(f_1, \cdots, f_{m-3}, d_s\left[\int_a^s P_2(f, dg)\right]\right)\\ &= \int_a^b P_1(f_1, \cdots, f_{m-3}, P_2(f, dg)).\end{aligned}$$

THEOREM 4.8. *Let* P_1 *be a product operator on* $X_1 \times X_2$ *into* X *and let* P_2 *be a product operator on* $X_3 \times X_4$ *into* X_2. *Suppose that* f *is a bounded function on* J *into* X_1, g *is a function on* J *into* X_3, *and* h *is a function on* J *into* X_4. *If* $P_2(g, dh)$ *is* ω-*integrable on* J *and if* $F_1 = P_1(f, P_2(g, dh))$ *and* $F_2 = P_1(df, P_2(g, h))$ *are integrable on* J, *then* $F_3 = P_1(f, P_2(dg, h))$ *is integrable on* J *and*

$$P_1(f(t), P_2(g(t), h(t)))\Big|_{t=a}^{t=b} = \int_a^b F_1 + \int_a^b F_2 + \int_a^b F_3.$$

Applying Theorems 4.4 and 3.3, we have

$$\begin{aligned}\int_a^b F_1 &= \int_a^b P_1\left(f(s), d_s\left[\int_a^s P_2(g, dh)\right]\right)\\ &= \int_a^b P_1\left(f(s), d_s\left[P_2(g(s), h(s)) - \int_a^s P_2(dg, h)\right]\right).\end{aligned} \tag{4.1}$$

The existence of the above integral together with the existence of the integral

$$\int_a^b F_2 = \int_a^b P_1(df, P_2(g, h))$$

(and hence of $\int_a^b P_1(f, dP_2(g, h)))$ insures that

$$\int_a^b P_1\left(f(s), d_s\left[\int_a^s P_2(dg, h)\right]\right)$$

exists. But $P_2(g, dh)$ ω-integrable on J implies by Theorem 4.3 that $P_2(dg, h)$ is ω-integrable on J and hence by an application of Theorem 4.4, $\int_a^b F_3$ exists and equals $\int_a^b P_1(f(s), d_s[\int_a^s P_2(dg, h)])$. Then by (4.1) and Theorem 4.7, the result follows.

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South Dakota State University
Brookings, South Dakota