A SCHWARZ INEQUALITY FOR POSITIVE LINEAR MAPS ON C^*-ALGEBRAS[1]

BY

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1. Introduction

Davis [5] has derived a Schwarz inequality for completely positive linear maps on C^*-algebras of operators. In this paper, we obtain the same inequality for positive linear maps, thus leading to better effect for 2-positive linear maps (in particular, for completely positive linear maps).

Herein, C^*-algebras possess an identity and are written in German type $\mathfrak{A}$, $\mathfrak{B}$. Capital letters A, B stand for operators, Greek letters Φ, Ψ, Ω for linear maps on C^*-algebras. $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded operators on the Hilbert space $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, we write $sp(T)$ for the spectrum of T, and $C^*(T)$ for the C^*-algebra generated by T. $C(\mathcal{S})$ stands for all continuous complex-valued functions defined on a compact Hausdorff space $\mathcal{S}$.

We denote by $\mathfrak{M}_n$ the collection of all $n \times n$ complex matrices. $\mathfrak{M}_n(\mathfrak{A}) = \mathfrak{A} \otimes \mathfrak{M}_n$ is the C^*-algebra of $n \times n$ matrices over $\mathfrak{A}$. A linear map $\Phi : \mathfrak{A} \rightarrow \mathfrak{B}$ is *positive* iff $\Phi(A)$ is positive for all positive A in $\mathfrak{A}$. We define

$$\Phi \otimes 1_n : \mathfrak{M}_n(\mathfrak{A}) \rightarrow \mathfrak{M}_n(\mathfrak{B})$$

by

$$\Phi \otimes 1_n((A_{jk})_{j,k}) = (\Phi(A_{jk}))_{j,k}.$$

Φ is *n-positive* iff $\Phi \otimes 1_n : \mathfrak{M}_n(\mathfrak{A}) \rightarrow \mathfrak{M}_n(\mathfrak{B})$ is positive; the set of such Φ is denoted by $\mathbf{P}_n[\mathfrak{A}, \mathfrak{B}]$. (The suffix 1 is deleted if $n = 1$.) Φ is *completely positive* iff Φ is n-positive for all positive integers n.

We presume that all linear maps on C^*-algebras preserve the identity.

In §2, a Schwarz inequality (Theorem 2.1) is derived: If $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$, then $\Phi(f(A)) \geq f(\Phi(A))$ for any operator-convex function f and Hermitian operator A provided $f(A)$ is defined. An immediate consequence is the well-known inequality due to Kadison [12]: $\Phi(A^2) \geq \Phi(A)^2$ for all Hermitian A. Another useful inequality (Corollary 2.3) is $\Phi(A^{-1}) \geq \Phi(A)^{-1}$ for all positive invertible A.

Stinespring [15] and Arveson [1], [2] have established that completely positive linear maps, rather than positive linear maps, are the natural generalizations of positive functionals. From [4], we know that $\mathbf{P}[\mathfrak{A}, \mathfrak{B}] = \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ iff $\mathfrak{A}$ or $\mathfrak{B}$ is commutative. Hence, it is desirable to investigate 2-positive linear maps with special attention to completely positive linear maps.

A more delicate inequality is derived in Corollary 2.8: If $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$, then $\Phi(A^*A) \geq \Phi(A^*)\Phi(A)$ for all A in $\mathfrak{A}$. As a consequence, every 2-positive linear map is 'locally' completely positive (Corollary 2.9).

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In §3, we relate any positive linear map with the 'multiplicative domain', an important subalgebra contained in the domain algebra.

Let $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$. The *multiplicative domain* of Φ, in notation, $\mathfrak{A}_\Phi$, is defined as $\{A \in \mathfrak{A} \mid \Phi(XA) = \Phi(X)\Phi(A) \text{ for all } X \in \mathfrak{A}\}$. The main theorem (Theorem 3.1) deduced from the Schwarz inequality says that if $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ then $\mathfrak{A}_\Phi$ has just the simple form

$$\{A \in \mathfrak{A} \mid \Phi(A^*A) = \Phi(A^*)\Phi(A)\}.$$

The extremal behavior of multiplicative domains really governs the effect of 2-positive maps. In particular, we see that maps in $\mathbf{P}[C(\mathfrak{S}), C(\mathfrak{I})]$ are decomposable canonically in terms of maps with trivial multiplicative domains (Remark 3.5).

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2. A Schwarz inequality

A real-valued measurable function f defined on an interval $(-a, a)$ may be considered as an operator-valued function defined on Hermitian operators with spectra contained in $(-a, a)$. Indeed, for a Hermitian operator A with spectral resolution E_λ, $f(A)$ will mean $\int_{-a}^{a} f(\lambda)\, dE_\lambda$. f is called an *operator-convex* function iff

$$\tfrac{1}{2}(f(A) + f(B)) \geq f(\tfrac{1}{2}(A + B))$$

for all Hermitian operators A, B with spectra contained in $(-a, a)$.

Now we utilize the operator-valued functions to derive a Schwarz inequality.

THEOREM 2.1. *If $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$ and f is an operator-convex function on $(-a, a)$; then $\Phi(f(A)) \geq f(\Phi(A))$ for all Hermitian $A \in \mathfrak{A}$ such that $sp(A) \subseteq (-a, a)$.*

Proof. We notice first that $f(\Phi(A))$ is well defined since

$$sp(A) \subseteq [-a + \varepsilon, a - \varepsilon]$$

for some positive ε and $-a + \varepsilon = \Phi((-a + \varepsilon)I) \leq \Phi(A) \leq \Phi((a - \varepsilon)I) = a - \varepsilon$. $\Phi(f(A))$ is defined because (f being continuous) $f(A)$ belongs to $C^*(A) \subseteq \mathfrak{A}$.

Now for Hermitian A, $C^*(A)$ is a commutative C^*-algebra. So Φ restricted to $C^*(A)$ is completely positive. By Davis's Theorem [5, p. 44], $\Phi(f(A)) \geq f(\Phi(A))$ as required. ∎

Bendat and Sherman [3, §3] (See also Davis [9, §4]) have shown that a real function is operator-convex iff it has an integral form

$$f(t) = \int_{-a}^{a} \frac{t^2}{a^2 - tx}\, d\mathrm{m}(x) + bt + c$$

where $\mathbf{m}$ is a regular Borel positive finite measure on $[-a, a]$. Hence, a lot of inequalities can be derived for Hermitian operators. Here we mention two important cases:

COROLLARY 2.2 (Kadison [12, p. 495]). *If* $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$, *then* $\Phi(A^2) \geq \Phi(A)^2$ *for all Hermitian* $A \in \mathfrak{A}$.

Proof. In Bendat and Sherman's formula, put $b = c = 0$, $\mathbf{m}$ = one point measure at the origin; then $f(t) = t^2$ is an operator-convex function. (In fact, it is straightforward to check by definition that $f(t) = t^2$ is operator-convex.) ∎

COROLLARY 2.3. *If* $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$, *then* $\Phi(A^{-1}) \geq \Phi(A)^{-1}$ *for all positive invertible* $A \in \mathfrak{A}$.

Proof. In Bendat and Sherman's formula, put $a = b = c = 1$, $\mathbf{m}$ = one point measure such that $\mathbf{m}(\{1\}) = 1$; so $f(t) = (1 - t)^{-1}$ is operator-convex on $(-1, 1)$. By Theorem 2.1, $\Phi((I - X)^{-1}) \geq (I - \Phi(X))^{-1} = (\Phi(I - X))^{-1}$ for Hermitian X with spectrum contained in $(-1, 1)$. Replacing $I - X$ by εA, we get $\Phi(A^{-1}) \geq \Phi(A)^{-1}$ for positive A such that $sp(A) \subseteq (0, \varepsilon^{-1})$, hence for all positive invertible A. ∎

The inequality in Corollary 2.3 gives some non-vacuous information about positive linear maps. Indeed if $A \geq \varepsilon > 0$, the naïve definition says that $\Phi(A) \geq \varepsilon$ while the derived inequality says that

$$\Phi(A) \geq \Phi(A^{-1})^{-1} \geq \varepsilon.$$

We remark that Corollary 2.3 is not true for an arbitrary invertible Hermitian operator. For example; let $\mathfrak{A}$ = the commutative C^*-algebra of ordered pairs

$$\{(\alpha, \beta) \mid \alpha, \beta \text{ are complex numbers}\};$$

Φ = the linear functional such that $\Phi(\alpha, \beta) = \frac{1}{2}(\alpha + \beta)$; $A = (1, -1)$. Then $A = A^{-1}$, and $\Phi(A) = \Phi(A^{-1}) = 0$, so the inequality $\Phi(A^{-1}) \geq \Phi(A)^{-1}$ does not hold.

Referring to Theorem 2.1, the inequality may become equality for all Hermitian A. We will see that such will happen only in the extraordinary cases: f is *affine* (i.e., f is of the form $f(t) = a_1 t + a_0$) or Φ is extreme. We recall that for $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$, Φ is a *C^*-homomorphism* iff $\Phi(A^2) = \Phi(A)^2$ for every Hermitian A in $\mathfrak{A}$, and Størmer [16, p. 242] has proved that every C^*-homomorphism is extreme. The following lemma gives an alternative characterization of a C^*-homomorphism.

LEMMA 2.4. *Let* $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$. *Then* Φ *is a C^*-homomorphism iff* $\Phi(A^{-1}) = \Phi(A)^{-1}$ *for all positive invertible* A *in* $\mathfrak{A}$.

Proof. Assume Φ preserves the inverse for every positive invertible operator. Then for any positive invertible A, we apply Kadison's inequality (Corollary 2.2) and get

$$\Phi(A^2)^{-1} = \Phi(A^{-2}) \geq \Phi(A^{-1})^2 = \Phi(A)^{-2}, \qquad \Phi(A^2) \leq \Phi(A)^2.$$

Applying Kadison's inequality again, $\Phi(A^2) = \Phi(A)^2$.

To extend this to an arbitrary Hermitian operator A, replace A by $A + nI$ for a sufficiently large n. Hence Φ is a C^*-homomorphism.

The converse follows from the fact a C^*-homomorphism restricted to $C^*(A)$, for any Hermitian A, is a *homomorphism. ∎

THEOREM 2.5. *Let $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$. If f is a non-affine operator-convex function on $(-a, a)$, and $\Phi(f(A)) = f(\Phi(A))$ for all Hermitian A in $\mathfrak{A}$ such that $sp(A) \subseteq (-a, a)$, then Φ is a C^*-homomorphism.*

Proof. By Bendat and Sherman's formula,

$$f(t) = \int_{-a}^{a} \frac{t^2}{a^2 - tx}\, d\mathbf{m}(x) + bt + c.$$

Since f is non-affine, the *carrier* of $\mathbf{m}$ (the smallest closed subset $\mathcal{S}$ of $[-a, a]$ such that $\mathbf{m}(\mathcal{S}) = \mathbf{m}([-a, a])$) is nonvoid.

Now suppose $\Phi(f(A)) = f(\Phi(A))$ for all Hermitian A such that $sp(A) \subseteq (-a, a)$. For each s in the carrier of $\mathbf{m}$, $g(t) = t^2/(a^2 - st)$ is operator-convex, hence $\Phi(g(A)) = g(\Phi(A))$ by virtue of Theorem 2.1. In case $s = 0$, we get immediately that $\Phi(A^2) = \Phi(A)^2$. In case $s \neq 0$,

$$g(t) = t^2/(a^2 - st) = (a^4(a^2 - st)^{-1} - st - a^2)/s^2;$$

by a transformation as in the proof of Corollary 2.3, we deduce that $\Phi(A^{-1}) = \Phi(A)^{-1}$ for all positive invertible A. Therefore, Φ is a C^*-homomorphism in both cases. ∎

Remark 2.6. An operator-convex function plays an essential role in the above results. The following example shows that Theorem 2.1 would be false if we replace an *operator-convex* function by a general *convex* function:

The function $f(t) = t^4$ is convex but not operator-convex. Let $\Phi : \mathfrak{M}_3 \to \mathfrak{M}_2$ be the compression map

$$\Phi((a_{jk})_{1\leq j,k\leq 3}) = (a_{jk})_{1\leq j,k\leq 2},$$

and

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then

$$\Phi(A)^4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \not\geq \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix} = \Phi(A^4).$$

Theorem 2.5 is of interest when referring to Theorem 2.1. However, certain facts reveal that a more general case may be true. We conjecture that

Theorem 2.5 remains true if we require f to be a general non-affine real continuous function instead of an operator-convex function.

The Schwarz inequality derived in Theorem 2.1 is in some respects unsatisfactory. For example, it does not govern non-Hermitian operators. We will achieve this effect for 2-positive linear maps. A function f on $(-a, a)$ is *even* iff $f(t) = f(-t)$ for all t. Indeed, every operator-convex function f induces an even operator-convex function $f(t) + f(-t)$. Following is a modified Schwarz inequality.

THEOREM 2.7. *Let $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$. If f is an even operator-convex function on $(-a, a)$, then for every $A \in \mathfrak{A}$ with the norm less than a,*

$$\Phi(f(\mid A \mid)) \geq f(\mid \Phi(A) \mid).$$

(Here $\mid X \mid$ stands for $(X^*X)^{1/2}$.)

Proof. Applying Theorem 2.1 to $\Phi \otimes 1_2 \in \mathbf{P}[\mathfrak{M}_2(\mathfrak{A}), \mathfrak{M}_2(\mathfrak{B})]$, we get

$$(*) \qquad \Phi \otimes 1_2(f(T)) \geq f(\Phi \otimes 1_2(T))$$

for all Hermitian $T \in \mathfrak{M}_2(\mathfrak{A})$ with $sp(T) \subseteq (-a, a)$. Now, let

$$T = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix};$$

then

$$\mid T \mid = \begin{bmatrix} \mid A \mid & 0 \\ 0 & \mid A^* \mid \end{bmatrix}.$$

As f is even, $f(t) = f(\mid t \mid)$, so

$$f(T) = f(\mid T \mid) = \begin{bmatrix} f(\mid A \mid) & 0 \\ 0 & f(\mid A^* \mid) \end{bmatrix}.$$

Similarly,

$$f(\Phi \otimes 1_2(T)) = \begin{bmatrix} f(\mid \Phi(A) \mid) & 0 \\ 0 & f(\mid \Phi(A^*) \mid) \end{bmatrix}.$$

By $(*)$, we obtain the required inequality. ∎

Putting $f(t) = t^2$ in the above theorem, we get the important result:

COROLLARY 2.8. *If $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$, then $\Phi(A^*A) \geq \Phi(A^*)\Phi(A)$ for all A in $\mathfrak{A}$.* ∎

COROLLARY 2.9. *Every 2-positive linear map is locally completely positive. This means that, if $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$, then for any x in $\mathfrak{H}$ the underlying space of $\mathfrak{B}$, there exists a completely positive linear map $\Psi_x : \mathfrak{A} \to \mathfrak{B}(\mathfrak{H})$ (which need not preserve identity) with $\| \Psi_x \| \leq 1$, such that $\Phi(\cdot)x = \Psi_x(\cdot)x$.*

Proof. With the Schwarz inequality of Corollary 2.8 in hand, we are ready to refer to Størmer [16, p. 268] and obtain the required result.

For completeness, we sketch the short proof.

We may assume $\| x \| = 1$. Starting from the positive functional $\langle \Phi(\cdot)x, x\rangle$ on $\mathfrak{A}$, we construct the 'associated representation' Π of $\mathfrak{A}$ on a Hilbert space $\mathcal{K}$; i.e., Π is a cyclic representation with a cyclic vector $v \in \mathcal{K}$ such that

$$\langle \Pi(A)v, v\rangle = \langle \Phi(A)x, x\rangle \quad \text{for all} \quad A \in \mathfrak{A}.$$

Define $V : \mathcal{K} \to \mathcal{K}$ by $\Pi(A)v \mapsto \Phi(A)x$; the Schwarz inequality of Corollary 2.8 guarantees that V is well defined. Then $\Psi_x = V\Pi(\cdot)V^*$ is the required completely positive map for Φ at x. ∎

3. Multiplicative domains

In Corollary 2.8, we showed that if $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ then

$$\Phi(A^*A) \geq \Phi(A^*)\Phi(A) \quad \text{for all} \quad A \in \mathfrak{A};$$

now, we examine the subset of $\mathfrak{A}$ for which equality holds:

THEOREM 3.1. *If $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$, then the set $\{A \in \mathfrak{A} \mid \Phi(A^*A) = \Phi(A^*)\Phi(A)\}$ is a closed subalgebra of $\mathfrak{A}$. In fact, it is just the multiplicative domain,*

$$\mathfrak{A}_\Phi \equiv \{A \in \mathfrak{A} \mid \Phi(XA) = \Phi(X)\Phi(A) \quad \text{for all} \quad X \in \mathfrak{A}\}.$$

Proof. It is straightforward to see that $\mathfrak{A}_\Phi$ is a closed algebra. It remains to show that if $\Phi(A^*A) = \Phi(A^*)\Phi(A)$, then $\Phi(XA) = \Phi(X)\Phi(A)$ for all X in $\mathfrak{A}$.

Let H be a Hermitian operator in $\mathfrak{A}$. By Kadison's inequality,

$$\Phi \otimes 1_2\left(\begin{bmatrix} 0 & A^* \\ A & H \end{bmatrix}^2\right) \geq \left(\Phi \otimes 1_2\begin{bmatrix} 0 & A^* \\ A & H \end{bmatrix}\right)^2,$$

i.e.,

$$\begin{bmatrix} \Phi(A^*A) & \Phi(A^*H) \\ \Phi(HA) & \Phi(AA^* + H^2) \end{bmatrix} \geq \begin{bmatrix} \Phi(A^*)\Phi(A) & \Phi(A^*)\Phi(H) \\ \Phi(H)\Phi(A) & \Phi(A)\Phi(A^*) + \Phi(H)^2 \end{bmatrix}.$$

That $\Phi(A^*A) = \Phi(A^*)\Phi(A)$ forces $\Phi(HA) = \Phi(H)\Phi(A)$. Now for arbitrary X in $\mathfrak{A}$, $X = \mathrm{re}X + i\,\mathrm{im}X$. Thus the desired result is immediate. ∎

The preceding theorem does not hold for a general positive linear map. For example, let Φ be the transpose map $\mathfrak{M}_n \to \mathfrak{M}_n$ $(n > 1)$. Then

$$\{A \in \mathfrak{M}_n \mid \Phi(A^*A) = \Phi(A^*)\Phi(A)\} = \{\text{normal matrices}\},$$

which is not an algebra; while $(\mathfrak{M}_n)_\Phi$ consists of scalars only.

COROLLARY 3.2. *Every 2-positive C^*-homomorphism is a *homomorphism.*

Proof. Let $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ be a C^*-homomorphism, i.e., for all Hermitian A in $\mathfrak{A}$, $\Phi(A^2) = \Phi(A)^2$. By Theorem 3.1, $\mathfrak{A}_\Phi$ is an algebra containing all Hermitian operators in $\mathfrak{A}$; so $\mathfrak{A}_\Phi = \mathfrak{A}$. Hence we conclude that Φ is a *homomorphism. ∎

An alternative proof of Corollary 3.2 without using Theorem 3.1 is to combine Corollary 2.8 with Størmer [16, Corollary 3.6, p. 446]. This, however, involves a much deeper structure theorem of C^*-homomorphisms.

If $\mathfrak{A}$, $\mathfrak{B}$ are commutative, and $\Phi \in \mathbf{P}[\mathfrak{A}, \mathfrak{B}]$, then $\mathfrak{A}_\Phi$ is a C^*-algebra and it represents the amount of 'extremeness' that Φ possesses, in fact, $\mathfrak{A}_\Phi = \mathfrak{A}$ iff Φ is extreme.

In the general case, Φ may be extreme while $\mathfrak{A}_\Phi \subsetneq \mathfrak{A}$. Nevertheless $\mathfrak{A}_\Phi$ has a great deal to do with the 'extremal behaviour' of Φ.

THEOREM 3.3. *If* $\Phi, \Psi, \Omega \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ *and* $\Phi = \frac{1}{2}(\Psi + \Omega)$, *then*

$$\mathfrak{A}_\Phi = \mathfrak{A}_\Psi \cap \mathfrak{A}_\Omega \cap \{A \in \mathfrak{A} \mid \Phi(A) = \Psi(A) = \Omega(A)\}.$$

Proof. For any A in $\mathfrak{A}$,

$$\begin{aligned}
\Phi(A^*A) &= \tfrac{1}{2}(\Psi(A^*A) + \Omega(A^*A)) \\
&\geq \tfrac{1}{2}(\Psi(A^*)\Psi(A) + \Omega(A^*)\Omega(A)) \\
&= \tfrac{1}{4}(\Psi(A^*) + \Omega(A^*))(\Psi(A) + \Omega(A)) \\
&\quad + \tfrac{1}{4}(\Psi(A^*) - \Omega(A^*))(\Psi(A) - \Omega(A)) \\
&\geq \tfrac{1}{4}(\Psi(A^*) + \Omega(A^*))(\Psi(A) + \Omega(A)) \\
&= \Phi(A^*)\Phi(A).
\end{aligned}$$

If $A \in \mathfrak{A}_\Phi$, then $\Phi(A^*A) = \Phi(A^*)\Phi(A)$ and all of the above inequalities become equalities. Hence

$$\Psi(A^*A) = \Psi(A^*)\Psi(A), \qquad \Omega(A^*A) = \Omega(A^*)\Omega(A) \quad \text{and} \quad \Psi(A) = \Omega(A).$$

Thus we conclude that $\mathfrak{A}_\Phi \subseteq \mathfrak{A}_\Psi \cap \mathfrak{A}_\Omega \cap \{A \in \mathfrak{A} \mid \Phi(A) = \Psi(A) = \Omega(A)\}$.

The opposite inclusion is trivial. ∎

The set $\mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ is convex. If $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ is not extreme, then there is an open line-segment in $\mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ passing through Φ. Theorem 3.3 says that every map lying in the open segment has the same multiplicative domain and agrees with Φ on the multiplicative domain.

Remark 3.4. Let $\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$. The *left kernel* of Φ is the set

$$\{A \in \mathfrak{A} \mid \Phi(A^*A) = 0\}.$$

From the Schwarz inequality, $\Phi(A^*A) \geq \Phi(A^*)\Phi(A) \geq 0$, it follows that $\Phi(A^*A) = 0$ iff $\Phi(A^*A) = \Phi(A^*)\Phi(A)$ and $\Phi(A) = 0$; that is, the left kernel is the intersection of the kernel and the multiplicative domain. Alternatively, the left kernel is the largest left ideal contained in the kernel. Furthermore, Φ restricted to $\mathfrak{A}_\Phi$ is an algebraic homomorphism; the kernel of the restricted map is the left kernel of Φ.

$\Phi \in \mathbf{P}_2[\mathfrak{A}, \mathfrak{B}]$ is *faithful* iff the left kernel of Φ is trivial. Equivalently, Φ is faithful iff $\Phi|_{\mathfrak{A}_\Phi}$ is an algebraic isomorphism.

Remark 3.5. Multiplicative domains of $\mathbf{P}[C(\mathcal{S}), C(\mathfrak{T})]$. The significance of multiplicative domains can be best revealed by the tractable structure of positive linear maps on continuous functions. A thorough description is divided into four parts as follows.

(i) *Suppose* $\Phi \in \mathbf{P}[C(\mathcal{S}), C(\mathfrak{T})]$. *Then* $C(\mathcal{S})_\Phi$ *is a* C^**-algebra. So the left kernel is an ideal contained in* $C(\mathcal{S})_\Phi$. *By factoring out the left kernel, we may assume* Φ *is faithful* (Remark 3.4). (To be precise, we should say that there exists a faithful

$$\Phi_0 \in \mathbf{P}[C(\mathcal{S}_0), C(\mathfrak{T})],$$

where $\mathcal{S}_0$ is a closed subset of $\mathcal{S}$, such that $\Phi(f) = \Phi_0(f|_{\mathcal{S}_0})$ for all $f \in C(\mathcal{S})$.)

(ii) *Suppose* $\Phi \in \mathbf{P}[C(\mathcal{S}), C(\mathfrak{T})]$ *is faithful. Let* $g \in C(\mathcal{S})$. *We write* $\mathfrak{X} =$ *Range* g, *and* $\mathcal{S}_x = \{s \in \mathcal{S} \mid g(s) = x\}$, $\mathfrak{T}_x = \{t \in \mathfrak{T} \mid \Phi(g)(t) = x\}$, *for each* $x \in \mathfrak{X}$. *Then the following are equivalent*:

(a) $g \in C(\mathcal{S})_\Phi$.

(b) $\mathfrak{T} = \bigcup \mathfrak{T}_x$, *and there exist* $\Phi_x \in \mathbf{P}[C(\mathcal{S}_x), C(\mathfrak{T}_x)]$ *such that*

$$\Phi(f) = \oplus \Phi_x(f|_{\mathcal{S}_x}) \quad \textit{for all} \quad f \in C(\mathcal{S}).$$

(Roughly, we say that $\mathcal{S}$, $\mathfrak{T}$ are broken into the same number of slices, and Φ sends each slice of $\mathcal{S}$ to the corresponding slice of $\mathfrak{T}$.)

Proof. (b) $\Rightarrow$ (a). As g assumes scalar value x on $\mathcal{S}_x$, by the presumed decomposition formula $\Phi(f) = \oplus \Phi_x(f|_{\mathcal{S}_x})$, it is immediate that

$$\Phi(g^*g) = \Phi(g^*)\Phi(g) = \oplus \mid x \mid^2 I_x$$

where I_x is the identity of $C(\mathfrak{T}_x)$. Hence $g \in C(\mathcal{S})_\Phi$.

(a) $\Rightarrow$ (b). As $C^*(g) \subseteq C(\mathcal{S})_\Phi$ and Φ is faithful, so

$$\Phi(C^*(g)) \simeq C^*(g) \simeq C(\mathfrak{X});$$

they are related in such a manner that for any $x \in \mathfrak{X}$ and $g' \in C^*(g)$, both $g'|_{\mathcal{S}_x}$ and $\Phi(g')|_{\mathfrak{T}_x}$ assume a common constant value. Evidently, $\mathfrak{T} = \bigcup \mathfrak{T}_x$ is the disjoint union of a class of non-void closed subsets.

Now for each fixed $a \in \mathfrak{X}$, define $\Phi_a : C(\mathcal{S}_a) \to C(\mathfrak{T}_a)$ with

$$\Phi_a(f|_{\mathcal{S}_a}) = \Phi(f)|_{\mathfrak{T}_a} \quad \text{for all} \quad f \in C(\mathcal{S}).$$

It is well defined since if $f|_{\mathcal{S}_a} = 0$, then there exist $g_n \in C^*(g)$ such that $g_n|_{\mathcal{S}_a} = I$ and $\| fg_n \| \to 0$ (as $n \to \infty$); thus

$$\Phi(f)|_{\mathfrak{T}_a} = \Phi(f)\Phi(g_n)|_{\mathfrak{T}_a} = \Phi(fg_n)|_{\mathfrak{T}_a}$$

must be zero. (An example to construct g_n: First pick up $h_n \in C(\mathfrak{X})$ such that $\| h_n \| = 1$, $h_n(a) = 1$ and h_n restricted to

$$\{x \in \mathfrak{X} : \| f|_{\mathcal{S}_x} \| \geq 1/n\}$$

is zero. Then define $g_n(s) = h_n(x)$ whenever $s \in \mathcal{S}_x$.) Hence

$$\Phi(f) = \oplus \Phi_x(f|_{\mathcal{S}_x})$$

as required. ∎

(iii) *Let* $\Phi \in \mathbf{P}[C(\mathcal{S}), C(\mathfrak{J})]$ *be faithful. Suppose* $C(\mathcal{S})_\Phi$ *is equivalent to* $C(\mathfrak{X})$. *Then there exist continuous surjections*

$$\sigma : \mathcal{S} \to \mathfrak{X} \quad \textit{and} \quad \tau : \mathfrak{J} \to \mathfrak{X};$$

and for each $x \in \mathfrak{X}$, *there corresponds* $\Phi_x \in \mathbf{P}[C(\sigma^{-1}\{x\}), C(\tau^{-1}\{x\})]$ *such that* Φ_x *has the trivial multiplicative domain and*

$$\Phi(f) = \oplus \Phi_x(f|_{\sigma^{-1}\{x\}})$$

for all $f \in C(\mathcal{S})$.

Proof. As $\Phi(C(\mathcal{S})_\Phi) \simeq C(\mathcal{S})_\Phi \simeq C(\mathfrak{X})$, there exist continuous surjections

$$\sigma : \mathcal{S} \to \mathfrak{X} \quad \text{and} \quad \tau : \mathfrak{J} \to \mathfrak{X}$$

such that for each $x \in \mathfrak{X}$ and $g \in C(\mathcal{S})_\Phi$, $g|_{\sigma^{-1}\{x\}}$ and $\Phi(g)|_{\tau^{-1}\{x\}}$ assume a common constant value. The rest of the proof is similar to (ii). ∎

(iv) *Let* $\Phi \in \mathbf{P}[C(\mathcal{S}), C(\mathfrak{J})]$ *be faithful. Then* $C(\mathcal{S})_\Phi = \{$*scalars*$\}$ *iff* Φ *is 'indecomposable' in the sense that* $\mathcal{S}$, $\mathfrak{J}$ *cannot be further 'sliced'* (see (ii)).

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