UNIFORM SPLINE INTERPOLATION OPERATORS IN L_2

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1. Let $m \geq 1$ and $n \geq 2$ be positive integers. Define the class $\mathcal{S}^{2m}$ of cardinal splines of degree $2m - 1$ to be those functions S satisfying

(i) S is a polynomial of degree at most $2m - 1$ on each of the intervals

$$[i, i + 1], i = 0, \pm 1, \pm 2, \cdots \tag{1}$$

(ii) $S \in C^{2m-2}(-\infty, \infty)$

If in addition

$$S(x + n) = S(x) \quad \text{all} \quad x, \tag{2}$$

we say S is a periodic spline and denote this class by $\mathcal{S}_n^{2m}$. Let $l_p(n)$ be the space of real n-tuples possessing the norm

$$\begin{aligned} \| y \|_p &= (\textstyle\sum_{i=1}^{n} | y_i |^p)^{1/p}, \qquad 1 \leq p < \infty \\ &= \max_{1 \leq i \leq n} | y_i |, \qquad p = \infty. \end{aligned}$$

Then the periodic spline interpolation operator $\mathcal{L}_n^{2m} : l_p(n) \to L_p[0, n]$ is defined by letting $\mathcal{L}_n^{2m} y$ be that unique element of $\mathcal{S}_n^{2m}$ satisfying

$$\mathcal{L}_n^{2m} y(i) = y_i, \qquad i = 1, 2, \cdots, n.$$

Similarly if $(y_i)_{i=-\infty}^{\infty}$ is in l_p, the class of doubly infinite real p – summable sequences, then the cardinal spline interpolation operator

$$\mathcal{L}^{2m} : l_p \to L_p(-\infty, \infty)$$

is defined by letting $\mathcal{L}^{2m} y$ be that unique element of $\mathcal{S}^{2m} \cap L_p(-\infty, \infty)$ which satisfies

$$\mathcal{L}^{2m} y(i) = y_i, \qquad i = 0, \pm 1, \pm 2, \cdots.$$

The problem of calculating the norms of these operators for $p = \infty$ was first posed by Schurer and Cheney [6], who obtained

THEOREM 1 (Schurer and Cheney). *Let* $\beta = 2 + \sqrt{3}$. *Then*

$$\begin{aligned} \| \mathcal{L}_n^4 \|_\infty &= 1 + \tfrac{3}{2}(\beta^k - \beta)(\beta^k + 1)^{-1}(\beta - 1)^{-1}, \qquad n = 2k \\ &= 1 + \tfrac{3}{2}(\beta^k - \beta)(\beta^k + \beta)(\beta^{2k} + \beta)^{-1}(\beta - 1)^{-1}, \quad n = 2k - 1, \end{aligned} \tag{3}$$

$$\| \mathcal{L}^4 \|_\infty = (1 + 3\sqrt{3})/4.$$

Solutions were later obtained by Schurer [5] for $m = 3$ and Richards [1] for

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arbitrary m. The case $p = 1$ was investigated by the author [2], who established the following:

THEOREM 2. *Let* $\alpha = 2 - \sqrt{3}$. *Then*

$$\begin{aligned} \| \mathcal{L}_n^4 \|_1 &= \frac{3 + \sqrt{3}}{6} [\sqrt{3} - (9\sqrt{3} - 15)\alpha^{k-1} \\ &\qquad + (7\sqrt{3} - 12)\alpha^{2k-1}](1 - \alpha^{2k})^{-1} \qquad n = 2k, \\ &= \frac{3 + \sqrt{3}}{3} [\sqrt{3} - (7\sqrt{3} - 12)\alpha^{2k}] \\ &\qquad \cdot (1 + \alpha^{2k+1})^{-1}, \quad n = 2k + 1, \end{aligned} \tag{4}$$

$$\| \mathcal{L}^4 \|_1 = (1 + \sqrt{3})/2.$$

The case $p = 2$ will be solved in this paper. The contrast with the previous results is striking.

THEOREM 3.

$$\| \mathcal{L}_n^{2m} \|_2 = 1. \tag{5}$$

More precisely

$$\| \mathcal{L}_n^{2m} y \|_2 \le \| y \|_2, \qquad y \in l_2(n) \tag{6}$$

and equality holds in (6) *if and only if* $y_i \equiv$ *constant.*

THEOREM 4.

$$\| \mathcal{L}^{2m} \|_2 = 1, \tag{7}$$

and

$$\| \mathcal{L}^{2m} y \|_2 < \| y \|_2, \qquad y \in l_2. \tag{8}$$

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2. Before proceeding with the proofs of the theorems, we must first discuss some preliminary results found in [4]. Let

$$\begin{aligned} M_1(x) &= 1, \qquad |x| \le \tfrac{1}{2} \\ &= 0, \qquad |x| > \tfrac{1}{2} \end{aligned}$$

and define the central B-spline of order k, $M_k(x)$, to be the k-fold convolution of $M_1(x)$ with itself:

$$M_k(x) = M_1 * M_1 * \cdots * M_1(x) \quad (k \text{ times}). \tag{9}$$

$M_k(x)$ is a cardinal spline of degree $k - 1$ having support on $[-k/2, k/2]$ and is a symmetric function. In addition, the Fourier transform of $M_k(x)$ is easily computed to be

$$\int_{-\infty}^{\infty} M_k(x)e^{itx}\,dx = \psi_k(t), \tag{10}$$

where

$$\psi_k(t) = ((2/t)\sin\tfrac{1}{2}t)^k \tag{11}$$

Proof of Theorem 3. Define

$$\bar{M}_{2m}(x) = \sum_{\nu=-\infty}^{\infty} M_{2m}(x+\nu n). \tag{12}$$

Schoenberg [3] has shown that the functions $\bar{M}_{2m}(x-1), \bar{M}_{2m}(x-2), \cdots, \bar{M}_{2m}(x-n)$ form a basis for $\mathcal{S}_n^{2m}$. Hence if $\mathcal{L}_n^{2m}y = S$, there exist reals $c_1, c_2, \cdots, c_n$ such that

$$S(x) = \sum_{i=1}^{n} c_i \bar{M}_{2m}(x-i) \tag{13}$$

and thus

$$y_i = S(i) = \sum_{j=1}^{n} \bar{M}_{2m}(i-j)c_j, \qquad i = 1, 2, \cdots, n. \tag{14}$$

Inverting this non-singular system and using matrix notation we obtain

$$c = \Omega y \quad (\Omega^{-1} = \bar{M}_{2m}). \tag{15}$$

Upon squaring both sides of (13) and then integrating, we get

$$\| \mathcal{L}_n^{2mm}y \|_2^2 = \int_0^n (S(x))^2\,dx = \sum_{i=1}^{n}\sum_{j=1}^{n} \Lambda_{ij}c_i c_j = (c, \Lambda c), \tag{16}$$

where

$$\Lambda_{ij} = \int_0^n \bar{M}_{2m}(x-i)\bar{M}_{2m}(x-j)\,dx, \quad i, j = 1, 2, \cdots, n. \tag{17}$$

Since $\bar{M}_{2m}$ is symmetric, (15) and (16) imply

$$\| \mathcal{L}_n^{2m} \|_2^2 = \sup_{\|y\|_2=1} (y, \Omega\Lambda\Omega y). \tag{18}$$

But Λ, and therefore $\Omega\Lambda\Omega$, is also symmetric. Thus if $\rho(\Omega\Lambda\Omega)$ denotes the spectral radius of $\Omega\Lambda\Omega$, then

$$\| \mathcal{L}_n^{2m} \|_2^2 = \rho(\Omega\Lambda\Omega). \tag{19}$$

We shall now compute the eigenvalues and corresponding eigenvectors of the relevant matrices.

Because $\bar{M}_{2m}(x)$ has period n, $\bar{M}_{2m}(i-j)_{i,j=1}^{n}$ is the circulant matrix

$$\bar{M}_{2m} = C(\bar{M}_{2m}(0), \bar{M}_{2m}(1), \cdots, \bar{M}_{2m}(m-1), \\ \cdots \bar{M}_{2m}(n-m+1), \cdots, \bar{M}_{2m}(n-1)).$$

For the time being let us assume that

$$n \geq 4m \tag{20}$$

and thus each of the functions $M_{2m}(x + \nu n)$, $\nu = 0, \pm 1, \pm 2, \cdots$ has disjoint support. Then by recalling (12) and the fact that $M_{2m}(x)$ has support on $[-m, m]$, we see that

$$\begin{aligned} \bar{M}_{2m}(i) &= M_{2m}(i), && i = 0, 1, \cdots, m-1, \\ &= 0, && i = m, \cdots, n-m, \\ &= M_{2m}(i-n), && i = n-m+1, \cdots, n-1. \end{aligned}$$

Hence

$$\begin{aligned} \bar{M}_{2m} = C(&M_{2m}(0), M_{2m}(1), \cdots, M_{2m}(m-1), \\ &0, \cdots, 0, M_{2m}(-m+1), \cdots, M_{2m}(-1)) \end{aligned} \tag{21}$$

Thus if we define the function φ_k by

$$\varphi_k(\theta) = \textstyle\sum_{\nu=-m+1}^{m-1} M_k(\nu)e^{i\nu\theta} \tag{22}$$

and let

$$\theta_j = 2\pi j/n, \qquad j = 0, 1, \cdots, n-1,$$

then $\bar{M}_{2m}$ has eigenvalues $\varphi_{2m}(\theta_j)$ and corresponding eigenvectors

$$v_j = (1, \varepsilon_j, \varepsilon_j^2, \cdots, \varepsilon_j^{n-1}), \qquad j = 0, 1, \cdots, n-1, \tag{23}$$

where $\varepsilon_j = e^{i\theta_j}$ is an n-th root of unity. Therefore Ω has eigenvalues $(\varphi_{2m}(\theta_j))^{-1}$.

To handle Λ, we first note that the condition (20) ensures that any one "hump" of $\bar{M}_{2m}(x - i)$ will "hit" at most one other "hump" of $\bar{M}_{2m}(x - j)$. Then since (see [4, p. 177])

$$\int_{-\infty}^{\infty} M_{2m}(x-i)M_{2m}(x-j) = M_{4m}(i-j) \tag{24}$$

and using the periodicity of $\bar{M}_{2m}(x)$, it easily follows that

$$\Lambda_{ij} = \bar{M}_{4m}(i-j), \qquad i, j = 1, \cdots, n. \tag{25}$$

Thus Λ has eigenvalues $\varphi_{4m}(\theta_j)$.

These results show that $\Omega\Lambda\Omega$ has eigenvalues $\varphi_{4m}(\theta_j)/\varphi_{2m}^2(\theta_j)$ and corresponding eigenvectors v_j, $j = 0, 1, \cdots, n-1$.

It will now be shown that

$$\varphi_{4m}(\theta)/\varphi_{2m}^2(\theta) \leq 1 \tag{26}$$

with equality holding in (26) if and only if $\theta \equiv 0 \pmod{2\pi}$. This will establish Theorem 3, as the supremum in (18) will be attained only when y is some multiple of $v_0 = (1, 1, \cdots, 1)$.

Schoenberg [4] has shown that

$$\varphi_k(\theta) = \sum_{i=-\infty}^{\infty} \psi_k(\theta + 2\pi i). \tag{27}$$

Since $\psi_{2m}(\theta) \geq 0$, we have

$$\begin{aligned} \varphi_{2m}^2(\theta) &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \psi_{2m}(\theta + 2\pi i)\psi_{2m}(\theta + 2\pi j) \\ &\geq \sum_{i=-\infty}^{\infty} \psi_{2m}^2(\theta + 2\pi i) = \varphi_{4m}(\theta). \end{aligned} \tag{28}$$

This proves (26). Note that we get equality in (28) only if $\theta \equiv 0 \pmod{2\pi}$, since otherwise $\psi_{2m}(\theta + 2\pi i) > 0$ for all i.

We make the observation that the condition (20) may be discarded, since if $n < 4m$, the data $(y_i)_{i=1}^n$ and $\mathcal{L}_n^{2m} y$ may be extended periodically to $[0, rn]$, where r is some integer such that $rn \geq 4m$. Theorem 3 may now be applied on $[0, rn]$. ▩

Proof of Theorem 4. For $y \in l_2$, let $\mathcal{L}^{2m} y = S$. Then there exist reals c_i, $i = 0, \pm 1, \pm 2, \cdots$ such that

$$S(x) = \sum_{i=-\infty}^{\infty} c_i M_{2m}(x - i). \tag{29}$$

Proceeding as before we obtain

$$\| \mathcal{L}^{2m} y \|_2^2 = \int_{-\infty}^{\infty} (S(x))^2 \, dx = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \lambda_{ij} c_i c_j = (c, \lambda c), \tag{30}$$

where

$$\lambda_{ij} = \int_{-\infty}^{\infty} M_{2m}(x - i) M_{2m}(x - j) \, dx = M_{4m}(i - j), \qquad i, j = 0, \pm 1, \pm 2, \cdots \tag{31}$$

and

$$y_i = \sum_{j=-\infty}^{\infty} M_{2m}(i - j) c_i, \qquad i = 0, \pm 1, \pm 2, \cdots . \tag{32}$$

Since $\varphi_{2m}(\theta) > 0$, we may invert the sequence convolution transformation (32) to get

$$c_i = \sum_j \omega(i - j) y_j, \tag{33}$$

where

$$\sum_{\nu=-\infty}^{\infty} \omega(\nu) e^{i\nu\theta} = (\varphi_{2m}(\theta))^{-1} \tag{34}$$

and by the Wiener-Lévy theorem

$$\sum_\nu | \omega(\nu) | < \infty .$$

Thus

$$\| \mathcal{L}^{2m} y \|_2^2 = (y, \omega\lambda\omega y). \tag{35}$$

But then using the correspondence

$$(y_\nu)_{\nu=-\infty}^{\infty} \leftrightarrow \tilde{y}(\theta) = \textstyle\sum_{\nu=-\infty}^{\infty} y_\nu e^{i\nu\theta},$$

from Parseval's identity it follows that

$$\| \mathcal{L}^{2m} y \|_2^2 = \left(\tilde{y}, \frac{\varphi_{4m}}{\varphi_{2m}^2} \tilde{y}\right) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\varphi_{4m}(\theta)}{\varphi_{2m}^2(\theta)} \, | \tilde{y}(\theta) |^2 \, d\theta. \tag{36}$$

Theorem 4 is an immediate consequence of (26) and (36).

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