ON FINITE GROUPS OF COMPONENT TYPE

BY

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In recent years great progress has been made toward the classification of finite simple groups in terms of local subgroups and in particular the centralizers of involutions. If this program is to be completed one must show that an arbitrary simple group G possesses an involution t for which $C_G(t)$ is isomorphic to a centralizer in a known simple group.

This paper concerns itself with that problem for simple groups of *component type*; that is groups G such that $E(C(t)/O(C(t))) \neq 1$ for some involution t in G. These include most of the Chevalley groups of odd characteristic, most of the alternating groups, and many of the sporadic simple groups. D. Gorenstein has conjectured that in a group of component type, the centralizer of some involution is usually in a "standard form." A proof is supplied here of a portion of that conjecture.

To be more precise, define a subgroup K of a finite group G to be *tightly embedded* in G if K has even order while $K \cap K^g$ has odd order for each $g \in G\text{-}N(K)$. Define a quasisimple subgroup A of G to be *standard* in G if $[A, A^g] \neq 1$ for each $g \in G$, $K = C_G(A)$ is tightly embedded in G, and $N(A) = N(K)$.

Let G be a finite simple group of component type in which $O_{2',E}(C(t)) = O(C(t))E(C(t))$ for each involution t in G. Let A be a "large component." Then it is shown, modulo a certain special case where A has 2-rank 1, that A is standard in G in the sense defined above.

Other theorems establish properties of tightly embedded subgroups. They show that, under the hypothesis of the last paragraph, the centralizer of each involution centralizing A contains at most one component distinct from A, and that component must have 2-rank 1 if it exists. Further, it can be shown that the 2-rank of the centralizer of A is bounded by a function of A, which seems to be 1 or 2 if A is not of even characteristic.

Proofs of the various theorems utilize properties of the Generalized Fitting Subgroup $F^*(G)$ of a group G, developed by Gorenstein and Walter. These properties appear in Section 2. Also important to the proof is the classification of groups with dihedral Sylow 2-groups, Alperin's fusion theorem, the recent result on 2-fusion due to Goldschmidt, and Theorem 3.3 in Section 3, which extends Bender's classification of groups with a strongly embedded subgroup.

Statements of the major theorems appear in Section 1, along with a brief explanation of notation.

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Gorenstein [7] and Walter [9] have proved results similar to Theorem 1.

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1. Notation, terminology, and statements of the principal theorems

We recall some terminology due to Gorenstein and Walter. A group X is *quasisimple* if $X = [X, X]$ and $X/Z(X)$ is a nonabelian simple group. X is *semisimple* if X is the central product of quasisimple groups (or if $X = 1$), in which case these factors are uniquely determined as the normal quasisimple subgroups of X and are called the *components* of X (e.g. Lemma 2.1). Denote by $F(X)$ the Fitting subgroup of X. Define $E(X)$ to the largest normal semisimple subgroup of X. Set $F^*(X) = F(X)E(X)$. Let $O_{2',E}(X)$ be the preimage in $X/O(X)$ of $E(X/O(X))$.

Let G be a group. If U and X are subgroups of G with U acting on X define

$$\Gamma_{1,U}(X) = \langle N_X(V): 1 \neq V \leq U\rangle.$$

E_n denotes an elementary 2-group of order 2^n.

Denote by $\mathscr{L}$ the set of all components of the subgroups $E(C(t))$ as t ranges over all involutions in G. Define a relation $<^*$ on $\mathscr{L}$ by $L <^* K$ if there exists an involution t with $L \trianglelefteq E(C(t))$, $K = [K, t]$ and $L \leq K$. Extend this relation to a partial order $\ll$ on $\mathscr{L}$ by defining $L \ll K$ if there exists a sequence $\{L_i\} \subseteq \mathscr{L}$ with $L = L_1$, $K = L_n$, and for each i, either $L_i = L_{i+1}$ or $L_i <^* L_{i+1}$. Let $\mathscr{L}^*$ be the set of maximal elements of $\mathscr{L}$ under this partial order.

For $X \leq G$ and $L \in \mathscr{L}$ define $\Delta_L(X) = \langle L^g: L^g \trianglelefteq E(X)\rangle$.

From time to time we will consider quasisimple groups satisfying the following hypothesis:

HYPOTHESIS I. *If α is an automorphism of L of order 2, then either $m(L) = 1$, or $C_L(\alpha) - Z(L)$ contains a 2-element.*

The remainder of our notation is standard and can be found in [6]. Now the statements of the major results:

THEOREM 1. *Let G be a finite group, let t be an involution in G, and let $A \trianglelefteq E(C(t))$, $A \in \mathscr{L}$. Assume for each involution $a \in G$ that $O_{2',E}(C(a)) = O(C(a))E(C(a))$. Assume further that if $K \in \mathscr{L}$ and A is a homomorphic image of K, then $K \in \mathscr{L}^*$. Then one of the following hold:*

(1) *$A = \Delta_A(N(A)) = \Delta_A(C(a))$ for each involution $a \in C(A)$. $[A, A^g] \neq 1$ for each $g \in G$.*

(2) *$D = \Delta_A(N(A)) = AA^x$, $x \in C(O_2(A))$. $m(A) = 1$. $\Delta_A(C(a)) \trianglelefteq D$ for each involution $a \in C(A)$. If $[A, A^g] = 1$ then $A^g = A^x$.*

(3) *There exists $K \in \mathscr{L}^*$ such that $K \neq \Delta_K(N(K)) = KK^t$, $A = C_{[K,t]}(t)'$, and K satisfies (2). A has dihedral Sylow 2-groups and $[A, A^g] \neq 1$ for $g \in G$.*

(4) *$A \trianglelefteq E(G)$, or $A = C_{[K,t]}(t)'$ for some component $K \neq K^t$ of $E(G)$.*

Cases (2) and (3) occur in $PSp_4(q)$, q odd, among other examples.

Notice in (1) that $Q = C(A)$ is tightly embedded in G, and then A is standard. The remaining theorems in this paper give information about tightly embedded subgroups, hence restricting $C(A)$ and $C(t)$.

THEOREM 2. *Let Q be a tightly embedded subgroup of the finite group G. Let $H = N_G(Q)$, $g \in G - H$, $T \in \mathrm{Syl}_2(Q^g \cap H)$, and $T \leq S \in \mathrm{Syl}_2(QT)$. Then:*

(1) *If $T \neq 1$ then $N_S(T) = Tx(N_S(T) \cap Q) \cong T \times T$ and $T \cap Q = 1$.*

(2) *If $T = 1$ for each $g \in G - H$ then either $Q \trianglelefteq G$ or $\langle Q^G \rangle \cap H$ is strongly embedded in $\langle Q^G \rangle$.*

THEOREM 3. *Let $T \neq 1$ be a 2-group acting on a group Q with $T \cap Q = 1$. Let $T \in \mathrm{Syl}_2(P)$, $P \leq QT$, and assume P is tightly embedded in QT. Let*

$$T \leq S \in \mathrm{Syl}_2(QT) \quad \textit{with } N_S(T) \in \mathrm{Syl}_2(N_{QT}(T)).$$

Assume $|T| \geq |N_S(T): T|$ and $O(Q) = 1$. Let W be the weak closure of T in S with respect to Q. Then one of the following holds:

(1) *T is cyclic, $S \cap Q$ is cyclic, quaternion, or dihedral, and S is dihedral, semidihedral, or wreathed.*

(2) *$T \trianglelefteq QT$.*

(3) *$O^{2'}(TQ) = O_2(QT) \times O^{2'}(Q) \cong E_n \times L_2(2^n)$.*

(4) *$W = T \times T^x = N_S(T) \trianglelefteq QT$. T is abelian and if $m(S/W) > 1$ then $T = \Omega_1(T)$.*

(5) *$W \trianglelefteq QT$, $|W| = |T|^3$, $Z(W) = W' = \Phi(W) \cong T = \Omega_1(T)$. $|S: W| = 2$.*

THEOREM 4. *Let Q be tightly embedded in G and assume $K = O_{2',E}(Q) \neq O(Q)$. Then one of the following hold:*

(1) *$K \trianglelefteq O_{2',E}(G)$.*

(2) *$m(K) = 1$ and if $m(Q) > 1$ then $N(Q)$ contains a Sylow 2-group of Q^g whenever $Q^g \cap N(Q)$ is of even order.*

(3) *$O^{2'}(QT)/O(Q) \cong E_n \times L_2(2^n)$ for some $g \in G - N(Q)$ and $T \in \mathrm{Syl}_2(Q^g)$.*

Theorem 4 implies that in Theorem 1, case (1), that if G is simple then $E(C(t))$ has at most one component distinct from A, and that component must have 2-rank 1 if it exists.

Our last theorem is crucial to the proof of Theorem 1 and Theorem 4.

THEOREM 5. *Let $D \leq G$ and set $\bar{D} = D/O(D)$. Assume $\bar{D}$ is the central product of semisimple groups $\bar{A}_i$, $1 \leq i \leq r$, permuted by $H = N_G(D)$ under conjugation. Assume further that:*

(i) *If t and t^g are 2-elements centralizing $\bar{A}_i$ and $\bar{A}_j$, respectively, and if $t \notin Z(G)$, then $g \in H$.*

(ii) *If $X \leq H^g$ and $A_i \leq XO(D)$ then $g \in H$.*

Then if $H \neq G$ either:

(1) *$D = A_1$, or*

(2) $D = A_1A_2$, $m(A_i) = 1$, *and if* a_i, $i = 1, 2$, *are commuting involutions with* $\bar{a}_i \in Z(\bar{A}_i)$, *then either* $a_1 = a_2$ *or* $a_1a_2 \in Z(G)$.

2. The generalized Fitting subgroup

LEMMA 2.1. *Let H be a group. Then*

(1) *$E(H)$ is the central product of all the quasisimple subnormal subgroups of H, known as the components of $E(H)$.*

(2) *If L is a component of $E(H)$ and $X \leq H$ then $L \leq [L, X]$ or $[L, X] = 1$. Further $[E(H), X]$ is the product of those components not centralized by X. If $L \leq N(X)$ then $L \trianglelefteq E(X)$ or $[L, X] = 1$.*

Proof. See 2.1 of [5].

LEMMA 2.2 *Let $X \trianglelefteq\trianglelefteq F^*(H)$ and $C_{F^*(H)}(X) \leq X$. Then*

$$X = E(H)(X \cap F(H)) \quad \textit{and} \quad C_{F(H)}(F(H) \cap X) \leq X.$$

Moreover $C_H(F^*(H)) \leq F^*(H)$.

Proof. See 2.2 of [5].

LEMMA 2.3 (Thompson $A \times B$ lemma). *Let A be a p-group, $B = O^p(B) \leq C(A)$ and assume AB acts on a p-group P with $C_P(A) \leq C_P(B)$. Then $[P, B] = 1$.*

Proof. See 5.3.4 in [6].

LEMMA 2.4. *Let L be a perfect group and X a group with $[X, L, L] = 1$. Then $[X, L] = 1$.*

Proof. By the 3-subgroup lemma, $[L, L, X] = 1$.

LEMMA 2.5. *Let a be an involution in H and L a component of $E(H)$ with $M = LL^a \neq L$. Then:*

(1) *$J = \{xx^a : x \in L\}$ is a homomorphic image of L contained in $C(a)$ with $Z(J) \leq Z(M)$.*

(2) *If $X \leq N(M)$ centralizes J then $[M, O^2(X)] = 1$.*

(3) *If $x \in M - JZ(M)$ then $M = \langle x, J\rangle$.*

(4) *If Y is a solvable subgroup of H normalized by J then $[J, Y] = 1$.*

Proof. (1) and (2) are Lemmas 2.1 and 2.2 from [8], respectively. Choose x as in (3). Then $x = uv$, $u \in L$, $v \in L^a$ and $x^{-1}uu^a = v^{-1}u^a = y \in L^a$. As $x \notin JZ(M)$, $y \notin Z(M)$. Therefore as L^a is quasisimple, $L^a = \langle y^M\rangle = \langle y^{LJ}\rangle = \langle y^J\rangle$, so $M = JL^a = \langle J, y\rangle = \langle J, x\rangle$.

Next let Y be as in (4). Then $W = [Y, J, J] \leq Y \cap M$ and W is normalized by J. Suppose $W \not\leq JZ(M)$. Then by (3), $M = \langle W, J\rangle \leq N(W)$, so $W \trianglelefteq M$. Thus $W \leq Z(M)$. So in any event, $W \trianglelefteq JZ(M)$, and then, by (1), $W \leq Z(M)$. Thus $[Y, J, J, J] = [W, J] = 1$, and two applications of 2.4 imply $[Y, J] = 1$.

LEMMA 2.6 *Let a be an involution in H and X a $C(a)$-invariant subgroup of H with $X = O(X)E(X)$. Let K be a component of $E(X)$ and L a component of $E(H)$. Then:*

(1) $X \leq N(L)$.

(2) $[O^2(C_X(a)), O_2(H)] = 1$.

(3) *If* $[L, a] = 1$ *then either* $[L, X] = 1$ *or* $L \trianglelefteq E(X)$.

(4) *Assume* $M = LL^a \neq L$ *and define J as in 2.5. Then* $[O(X), L] = 1$ *and either* $[K, L] = 1$, $K \trianglelefteq M$, *or* $K = J$.

Proof. Let $Y = O(X)$ and $Q = O_2(H)$. $[C_Q(a), X] \leq Q$. Then by 2.1.2, $[C_Q(a), E(X)] = 1$, while $[C_Q(a), Y] \leq Q \cap Y = 1$. Now the Thompson lemma implies (2).

If $L \leq N(X)$ then by 2.1, either $[L, X] = 1$, or $L \trianglelefteq E(X)$. As $C(a) \leq N(X)$ we get (3). Further in either case (1) and (4) are clear, so we may assume $L \nleq N(X)$.

Assume $M = LL^a \neq L$. By 2.1, either $[K, C_M(a)] = 1$ or $K \leq [K, C_M(a)] \leq E(H) \cap X$. In the first case K normalizes $\langle C_M(a)^{E(H)} \rangle = M$ and then by 2.5, $[K, L] = 1$. In the second case clearly K normalizes $L \trianglelefteq E(H)$. Thus $K \leq M \cap X$. As $L \nleq N(X)$, 2.4 implies $M \cap X \leq Z(M)J$. Thus $K \leq N(J)$, and $K \leq [K, J] \leq J$. As $J \leq N(X)$, $K \trianglelefteq J$, so as K and J are quasisimple, $K = J$. This establishes the last part of (4).

As $J \leq N(Y)$, 2.5 implies $[Y, J] = 1$. So by 2.5, $Y = O^2(Y) \leq C(M)$. This completes (4).

Thus we may take $L = [L, a]$ and it remains to show $X \leq N(L)$. Set $W = [X, a]$. If (L_1, L_2) is a cycle of length 2 under a in L^W then by (4), W fixes L_1, so $L^W = \{L_1\} = \{L\}$. On the other hand if a fixes L^W pointwise then so does $W = [W, a]$. So in any event $W \leq N(L)$. Thus with 2.1.2 we may take $[X, a] = 1$.

Then by (2), $[Q, X] = 1$. Set $R = Q\langle a \rangle$. Then $X \trianglelefteq\trianglelefteq O(C(R))E(C(R))$. Let $S \in \mathrm{Syl}_2(N(R))$. Then $S \nleq Z(L)$, so $L = \langle S^{E(H)} \rangle$. By 2.1, either $[K, S] = 1$ or $K \leq [K, S] \leq E(H)$. In the first case K normalizes $\langle S^{E(H)} \rangle = L$. In the second case K normalizes $L \trianglelefteq E(H)$. Finally $[Y, S]$ has odd order, so $Y \leq N(L)$.

LEMMA 2.7 *Let a be an involution in H, K a component of $E(C(a))$ and L a component of $E(H)$. Then:*

(1) $E(C(a)) \leq E(H)C(E(H))$.

(2) $K = L$ *or* $[K, L] = 1$ *or* $L \neq L^a$ *and* $K = C_{[L,a]}(a)'$ *or* $L = [L, a] \geq K$.

(3) *If* $[K, O(F(H))] = 1$ *then* $K \leq E(H)$.

(4) *If* $K \trianglelefteq E(C(u))$ *for each involution u in some 4-group* $U \leq H$ *then* $K \leq E(H)$.

Proof. By 2.6, $X = E(C(a)) \leq N(L)$ and if $L \neq [L, a]$ then one of the alternatives of (2) holds. So assume $L = [L, a]$. Let S be a maximal $\langle a \rangle$ K-invariant 2-group of L. By 2.6.2, $[S, K] = 1$, so $K \trianglelefteq E(C(S\langle a \rangle))$. We may take $H = LK\langle a\rangle$, so maximality of S implies $K = E(C(S\langle a\rangle))$. Let $T \in \mathrm{Syl}_2\,(N_L(S\langle a\rangle))$. Then $[T, K] \leq L \cap K \trianglelefteq K$, so either $K = [T, K] \leq L$ or $[T, K] \leq Z(K)$ and hence by 2.4, $[T, K] = 1$. We wish to show $K \leq L$ or $K \leq C(L)$, so we may assume $[T, K] = 1$. Then maximality of S implies $S = T \in \mathrm{Syl}_2\,(L)$. But now a result of Glauberman [3] implies $K \leq C(L)$. This yields (2).

Suppose (1) is false and let M be maximal subject to $M \trianglelefteq E(H)$ and $X \leq MC(M)$. Choose $L \nleq M$. Then

$$\begin{aligned} X \leq LC(L) \cap MC(M) &= LM(LC(L) \cap C(M)) \\ &= LM(C(L) \cap C(M)) \end{aligned}$$

by modularity. But this contradicts the maximality of M and establishes (1).

Assume $[K, O(F(H)] = 1$. Then by 2.6.2, $[K, F(H)] = 1$. So

$$K \leq E(H)C(E(H)) \cap C(F(H)) = E(H)C(F^*(H)) \leq F^*(H)$$

by modularity and 2.2. This yields (3).

Finally if $K \trianglelefteq E(C(u))$ for each involution u in some 4-group U, then by 2.1,

$$[K, C(u) \cap O(F(H))] = 1 \quad \text{for each } u \in U^\#.$$

Hence $O(F(H)) = \langle C(u) \cap O(F(H)) \colon u \in U^\# \rangle \leq C(K)$, and (3) implies (4).

LEMMA 2.8. *Let $U = \langle u, v\rangle$ be a 4-group in H and L a component of $E(H)$ with $L^u \neq L$. Then $L \leq \Gamma_{1,U}(L)$ and if $L^v = L$, $L \leq \langle C_{[L,u]}(u), C_{[L,u]}(uv)\rangle$ or $[L, v] = 1$.*

Proof. As $U \nleq N(L)$ there exists $u \in U^\#$ with $L \neq L^u$. Set $J = C_{[L,u]}(u)$, and let $v \in U - \langle u\rangle$. Suppose $N_U(L) = 1$ and let x be an element of odd order in $L - Z(L)$. Then xx^u, $x^{-u}x^{-uv}$, $x^{uv}x$ are all in $K = \Gamma_{1,U}(H)$, so their product x^2 is also in K. Then by 2.5, $L \leq \langle J, x^2\rangle \leq K$.

So take $v \in N(L)$. Let $\bar{M} = LL^u/Z(LL^u)$ and $I = C_M(vu)$. If $\bar{I} = \bar{J}$ then for each $x \in L$ there exists $y \in L$ with $\bar{x}\bar{x}^u = \bar{y}\bar{y}^{vu}$. Then $\bar{y}^{-1}\bar{x} \in \bar{L} \cap \bar{L}^u = 1$, so $[\bar{L}, v] = 1$. Hence by 2.1, $L \leq C(v) \leq K$. On the other hand if $\bar{I} \neq \bar{J}$ then by 2.5, $M = \langle I, J\rangle \leq K$.

LEMMA 2.9. *Let L be quasisimple with $m(Z(L)) \leq 1$. Then L satisfies hypothesis* I.

Proof. Let α be an automorphism of L of order 2 and assume $Z(L) = Z$ contains a Sylow 2-group of $C_L(\alpha)$. We may assume $O(L) = 1$, so Z is a 2-group. By hypothesis, Z is cyclic. Z is of index 2 in $Z\langle\alpha\rangle = W$, so W is abelian, dihedral, semidihedral, or modular (e.g. 5.4.4 in [6]). Thus there are at most two Z-classes of involutions in W-Z.

Set $\bar{L} = L/Z$, and let $\bar{T} \in \mathrm{Syl}_2\,(C_{\bar{L}}(\alpha))$. For $t \in T$, if $\alpha^t \in \alpha^Z$, then by a Frattini argument $t \in ZC_T(\alpha) = Z$. Hence as $W - Z$ has at most two Z-classes of involutions, $|\bar{T}| \leq 2$. So by the well known lemma of Suzuki, $\langle\alpha\rangle\bar{L}$ has dihedral or semidihedral Sylow 2-groups. As $\bar{L}$ is simple of index two in $\bar{L}\langle\alpha\rangle$, $\bar{L}$ has dihedral Sylow 2-groups. Thus $m(L) = 1$.

3. Fusion and generation

LEMMA 3.1. *Let V be a vector space of dimension n over $GF(2)$, $X = O^2(X)$ a group of automorphisms of V, and $Y \neq 1$ a cyclic subgroup of odd order in X, which acts transitively on $[Y, V]^\#$. Assume $U = C_V(Y)$ has dimension $k \geq 1$, $U \cap U^x = 0$ for $x \notin N_X(U)$, and $Y \trianglelefteq N_X(U)$. Then one of the following holds:*

(1) *$k = 1$ and $n = 3$.*

(2) *X normalizes $[Y, V]$ and U.*

(3) *X normalizes $[Y, V]$, Y acts regularly on $U^X - U$, and $V - [V, Y] = (\bigcup_X U^x)^\#$.*

Proof. This is essentially 2.11 of [5] with the hypothesis that X acts irreducibly on V removed. Irreducibility of X is used in the proof only to show $X \trianglerighteq Y$ and to show Y does not act regularly on $U^X - U$ if X is irreducible.

So assume Y is regular on $U^X - U$, and set $C = \{c \in V^\#: c \notin U^x, x \in X\}$. Then C has order $2^{n-k} - 1$ and C is X-invariant. As Y acts semiregularly on $V - U$, and $|C| = |Y|$, C is an orbit under Y. One can now check that $[V, Y] = \langle c + d: c, d \in C\rangle$, so $[V, Y]$ is normalized by X. Then $U^x \cap [V, Y] = 0$ for each $x \in X$, so if X does not normalize U then a counting argument yields (3).

THEOREM 3.2. *Let z be an involution in the center of a Sylow 2-subgroup of G, let $H < G$, and assume:*

(i) *$z \in H^g$ if and only if $g \in H$.*

(ii) *If u is an involution with $z \in C(u) \nleq H$ then $H \cap L$ is strongly embedded in $L = \langle z^G \cap C(u)\rangle$. Then $H \cap \langle z^G\rangle$ is strongly embedded in $\langle z^G\rangle$.*

Proof. See [1].

THEOREM 3.3. *Let z be an involution in G, $H < G$, and assume $z \in H^g$ if and only if $g \in H$. Then the following are equivalent:*

(1) *If $z \neq t \in z^G \cap C(z)$ then $(zt)^g \in H$ if and only if $g \in H$.*

(2) *z is in the center of a Sylow 2-subgroup of G, and if $z \neq t \in z^G \cap C(z)$ then $C(zt) \leq H$.*

(3) *$L = \langle z^G\rangle$ has 2-rank 1, or $\bar{L} = L/O(L) = \langle u\rangle \times \bar{L}'$, $|u| \leq 2$, $H \cap L'$ is strongly embedded in L', and $z^G \subseteq uL'$.*

Proof. This is an easy corollary of 3.2. As similar corollaries have appeared elsewhere we only sketch the proof.

Assume (1), let $m = |z^G \cap H|$, $Hx \neq H$ a coset of H in G, and $D = H \cap H^x$. Arguing as in 4.3 of [1], we find that $|z^G \cap Hx| = m$. Let $t \in z^G \cap Hx$ and $E = D\langle t\rangle$. Suppose $s \in E \cap z^G$ and st has even order, and let u be the involution in $\langle st\rangle$. Then $ut \in z^G$ so by hypothesis $u = (ut)t$ lies in the unique conjugate of H containing t and no other. But $u \in D \leq H$, a contradiction. Thus

$z^G \cap E = z^E$ is of odd order. That is m is odd. But $|z^G| = |G\colon H|\,|z^G \cap H| = |G\colon H|m$ with both $|G\colon H|$ and m odd. Thus (1) implies (2). Also it shows that (2) is inherited by subgroups. That (3) implies (1) is obvious.

The proof that (2) implies (3) is by induction on the order of G, so let G be a minimal counter example. Then $O(G) = 1$. By 3.2 there exists an involution u centralizing z such that $C(u) \nleq H$ and $H \cap L$ is not strongly embedded in $L = \langle z^G \cap C(u)\rangle$. Conjugates of z in distinct conjugates of H are conjugate in their join, so $z^G \cap C(u) = z^L$. Minimality of G implies either $\mathrm{L} = G$, or $L = \langle u\rangle \times L'$ with $z^L \subseteq uL'$, and $H \cap L'$ strongly embedded in L'.

Assume the latter, and suppose $v = u^g \in C(u)$. Let $K = \langle z^L \cap C(v)\rangle$. Then $K \nleq H$, we may take $z \in K$, and either $L'/O(L') \cong L_2(4)$ and v induces an outer automorphism on L', or $\langle u\rangle = O_2(K)$. In the first case vz is conjugate to vu under L. Now vz is the product of conjugates of z in $C(v)$, so by hypothesis $C(vz)$ and hence $C(uv)$ is contained in H. But $K \nleq H$. So $\langle u\rangle = O_2(K)$. But by symmetry $\langle v\rangle = O_2(K)$. So u is isolated, and by the Z^*-theorem $u \in Z^*(G) = Z(G)$.

So in any event $L = G$. Minimality of G implies that $\bar{G} = G/\langle u\rangle$ has 2-rank 1, or $\bar{G} = \langle \bar{v}\rangle \times \bar{G}'$ with $\bar{z}^G \subseteq \bar{v}\bar{G}'$, and $\bar{H} \cap \bar{G}'$ is strongly embedded in $\bar{G}'$. In the latter case $\bar{G}'$ is a Bender group so G' is simple, $SL_2(5)$, or $S\hat{z}(8)$. As z is a 2-central involution acting nontrivially on G', we conclude $G' \ncong SL_2(5)$ or $S\hat{z}(8)$. As $G = \langle z^G\rangle$ we conclude $G = \langle u\rangle \times G'$ with $z^G \subseteq uG'$.

LEMMA 3.4. *Let $A_i \leq S \in \mathrm{Syl}_2(G)$ such that $[A_1, A_2] = 1$ and $m(A_i) > 1$. Let H be the subgroup of G generated by all $g \in G$ with $A_i^g \cap A_j \neq 1$, for some $i, j \in \{1, 2\}$. Assume*

(*) *$A_1 \cup A_2$ is strongly closed in $H \cap S$ with respect to H and in $N_S(A_i)$ with respect to G.*

Then one of the following hold:

(1) *$H = G$.*

(2) *$A_i \subseteq A_j$, some $i \neq j$, and $H \cap \langle A_1^G\rangle$ is strongly embedded in $\langle A_1^G\rangle$.*

(3) *A_1 is conjugate to A_2 under an element fused into A_1, A_1 is dihedral or semidihedral, and $|A_1 \cap A_2| = 2$.*

Proof. Let G be a minimal counterexample. Set $A = A_1$, let Ω be the set of cosets of H in G, and represent G on Ω. Then

(a) *$X \leq G$ fixes the unique point H of Ω exactly when $N(X) \leq H$ and $X^G \cap H = X^H$.*

Let $X \leq A$, $|X| > 2$, and $X^g \leq N_S(\{A, A_2\})$. Then $|X^g\colon X^g \cap N_S(A)| \leq 2$, so $X^g \cap N_S(A) = Y \neq 1$. Then by (*), $Y \leq A_1 \cup A_2$, so $g \in H$ and then (*) implies $X^g \subseteq A_1 \cup A_2$. This has several consequences. First

(b) $S = N_S(\{A, A_2\})$ *and* $|S\colon N_S(A)| \leq 2$.

Second, with remark (a) it follows that

(c) *if $X \leq A$ and $|X| > 2$, then X fixes a unique point of Ω.*

Now let u be an involution in AA_2 and $K = C_G(u)$. Suppose $m(C_{A_i}(u)) > 1$, $C_{A_i}(u) \not\leq A_j$, $i \neq j$, and if $C_A(u)$ is conjugate to $C_{A_2}(u)$, then $C_A(u)$ is not dihedral or semidihedral. By (c), $C_A(u)$ fixes a unique point of Ω, so $H \cap K$ contains a Sylow 2-group of K, which we may take to be $S \cap K$. As $C_{A_i}(u) \not\leq A_j$, $N_S(A_i \cap K) \leq N_S(A_i)$ by (b). Hence K satisfies our hypothesis, and by minimality of G,

(d) *either $K = G$ or $K \leq H$.*

Assume a is an involution in A and $t = a^g \in S$ with $g \notin H$. By (*), $A^t = A_2$. Let $B_i = A_i^g$, $t \in B_1 = B$, and $T = S^g$.

Assume first $A_1 \cap A_2 = 1$. If $t^x \in C(t) \leq H^g$ we may assume $t^x \in T$ and then as $[t, t^x] = 1$ and $t \in B - B_2$, even $t^x \in N_T(B)$. But then by (*), $t^x \in B \cup B_2$, so $x \in H^g$. Let v be an involution in A. Then $v^t \in A_2 \leq C(A)$, so $[t, t^v] = 1$. Thus $v \in H^g$. But then $\Omega_1(A) \leq H^g$, against (c).

So $A_1 \cap A_2 \neq 1$. By (c), $C_{A \cap A_2}(t) = \langle z \rangle$ is of order 2, so $D = (A \cap A_2)\langle t \rangle$ is dihedral or semidihedral. Let $v \in A - A_2$ with $v^2 \in A_2$. Then $x = vv^t = v^2 t^v t \in C(t)$ and $x^2 \in A \cap A_2$, so $x^2 \in \langle z \rangle$. We may assume $V = \langle x, z \rangle \leq T$, so as $B^z \neq B$, $x^2 \neq z$. Thus $x^2 = 1$ and t inverts v^2.

Suppose $\Omega_1(A) \leq A_2$. As D is dihedral or semidihedral and $A \cap A_2 \leq Z(A)$, $A \cap A_2$ is a 4-group. As t inverts v^2, $v^2 = z$. Also there is an involution $w \in A \cap A_2$ with $[w, t] = z$. Notice $(vw)(vw)^t = xz$. As V acts on $\{B, B_2\}$ and $B^z = B_2$, with the symmetry between x and xz we may choose $x \in N(B)$. Now $t^{vw} = tx \in N(B)$, so $vw \in H^g$, against (c).

So we may choose v to be an involution. Then $[t, t^v] = 1$ and as $v \notin H^g$, $t^v \notin N(B)$. Hence $x \notin N(B)$, so $xz \in N(B)$. By (c) $txz \notin B$. But $(txz)^v = tz$, and $txz \in N(B)$, so by (*), tz is not fused into A. Therefore as D is dihedral or semidihedral, $A \cap A_2 = \langle z \rangle$.

A is not dihedral or we are in (3). Also as $B \neq B^x$, x is not rooted in $C(t)$, so v is not rooted in A. Therefore as $\{aa^t : a \in A\} \cong A/\langle z \rangle$, we may choose $u \in A - A_2$ such that $U = \langle x, y \rangle$ is a 4-group, for $y = uu^t$. We may assume $y \in N(B)$, so as above, $u^2 \neq 1$. Then $u^2 = z$. If $C_A(u)$ is cyclic then A is dihedral or semidihedral (e.g. 5.4.8. in [6]), a contradiction. So y satisfies the hypothesis in (d) and hence either $C(y) \leq H$ or $y \in Z(G)$. In the first case by (c), $|C_B(y)| = 2$, so B is dihedral, a contradiction. Thus $y \in Z(G)$. Set $\bar{G} = G/\langle y \rangle$. Now if $\bar{A} \neq \bar{A}_2$ then $\bar{G}$ satisfies our hypothesis and minimality of G yields a contradiction. So $\bar{A} = \bar{A}_2$. Thus $|A| = |A \cap A_2|\,|A : A \cap A_2| = 4$, and A is dihedral, a contradiction. Hence we have shown

(e) *$A_1 \cup A_2$ is strongly closed in S with respect to G.*

Together with (a) this implies

(f) *Each element of A fixes a unique point of Ω.*

Let a_i, $i = 1, 2$, be involutions in A_i with $u = a_1 a_2 \neq 1$. Suppose $K = C(u) \not\leq H$. If $C_A(u) = C_A(a_1)$ is a 4-group then A is dihedral or semidihedral.

If $C_A(u)$ is nonabelian dihedral or semidihedral then $A = C_A(u)$. Therefore as (3) does not hold, the hypothesis of (d) are satisfied, and then $u \in Z(G)$. But (e) implies $G/\langle u \rangle$ satisfies our hypothesis, so again minimality of G leads to a contradiction. Therefore

(g) $C(u) \leq H$.

Suppose $t = u^g \in S, g \notin H$. Then $C(t) \leq H^g$, so by (f), $C_A(t) = 1$ and hence $A^t = A_2$ and $A \cap A_2 = 1$. So $U = C_{AA_2}(t) \cong A$. We may assume $U \leq S^g$. Then U has a subgroup V of index 2 normalizing A^g and $1 \neq C_{A^g}(V) \leq H$ by (g), contradicting (f). Thus $u^G \cap H = u^H$, so by (g) and (a) we conclude that u fixes a unique point of Ω. Now 3.3 yields the desired result.

LEMMA 3.5. *Let L be a quasisimple normal subgroup of H of 2-rank 1 and U a 4-group in H such that $L \nleq \Gamma_{1,U}(H)$. Then:*
(1) *$L \cong \hat{A}_7$ or $SL_2(q)$, $q = 5, 7$, or 9.*
(2) *$U \leq LC(L)$.*
(3) *$H/C(L)$ contains no quaternion group X with $m(C_{H/C(L)}(X)) > 1$.*
(4) *If A is an abelian subgroup of H of 2-rank 3 then $L \leq \Gamma_{1,A}(L)$.*

Proof. $m(L) = 1$, so $L \cong \hat{A}_7$ or $SL_2(q)$, q odd. Set $\bar{L} = L/Z(L)$. $m(\bar{L}) = 2$, so (2) implies (4). If L is as in (1), then Aut $(\bar{L})$ contains no quaternion subgroup Q unless $L \cong SL_2(q)$. Further in that case Q contains its centralizer. Therefore (1) implies (3).

Assume $L \cong \hat{A}_7$. We may assume $u \in U^\#$ induces an outer automorphism on $\bar{L}$ and then take $\bar{u} = (1, 2)$ or $\bar{u} = (1, 2)(3, 4)(5, 6)$, representing L on $\{1 \leq i \leq 7\} = \Omega$. Then $C_L(u)$ is isomorphic to $SL_2(5)$ or $SL_2(3)$, respectively, and in the latter case is transitive on $\Omega - \{7\}$. Let $v \in U^\#$ induce an inner automorphism on $\bar{L}$. Then $C_L(v)$ is cyclic of order 24 and has orbits of length 3 and 4 on Ω. Let $X = \langle C_L(v), C_L(u) \rangle$. If $\bar{u}$ is a transposition then the global stabilizer of $\{1, 2\}$ is the unique maximal subgroup of L containing $C_L(u)$, so as $C_L(v)$ does not stabilize $\{1, 2\}$, $L = X$. If $\bar{u}$ is not a transposition then $C_L(u)$ is transitive on $\Omega - \{7\}$ and $C_L(v)$ moves 7, so X^Ω is 2-transitive. $\bar{X}$ contains an element of order 12, so $\bar{X} \cong A_7$ and $X = L$.

So we may take $L \cong SL_2(q)$, q odd. If $u \in U^\#$ induces an outer automorphism on $\bar{L}$ then u induces a field automorphism or an automorphism in $PGL_2(q)$. If u_1 and u_2 induce automorphisms of the first and second types, respectively then u_1u_2 does not induce an involutory automorphism on $\bar{L}$, a contradiction. Hence some $v \in U^\#$ induces an inner automorphism on $\bar{L}$.

Then $C_L(v)$ is cyclic of order $q - \varepsilon$, $\varepsilon = \pm 1 \equiv q \bmod 4$. Further unless $q \leq 9$, $C_L(v)$ is contained in a unique maximal subgroup M of L, which is the preimage of a dihedral group of order $q - \varepsilon$ in $\bar{L}$. (Here, and later in the proof we use Dickson's list of the maximal subgroups of $L_2(q)$ (p. 285, [2]). Further M contains the centralizer of no other involutory automorphism, so (1) holds.

Suppose u induces an automorphism in $PGL_2(q)$ on $\bar{L}$. Then $C_L(u)$ and

$C_L(uv)$ are cyclic of order $q + \varepsilon$. $X = \langle C_L(u), C_L(uv)\rangle$ is a subgroup of L and $\langle u\rangle X$ contains an abelian subgroup of order $2(q + \varepsilon)$, so we conclude $L = X$.

Suppose u induces a field automorphism on $\bar{L}$. We may represent $\langle u\rangle L$ on $\Omega = \{1 \leq i \leq 6\}$ as S and take $\bar{u} = (1, 2)$. Then $C_L(u) \cong C_L(uv) \cong SL_2(3)$ and $X = \langle C_L(u), C_L(v)\rangle$ is transitive on Ω. As no maximal subgroup of L containing $C_L(u)$ is transitive on Ω we conclude $L = X$. This completes the proof of (2).

Essentially the same proof shows:

LEMMA 3.6. *Lemma* 3.5 *holds if L is assumed to have dihedral Sylow* 2*-groups, and in* (1), $\hat{A}_7$ *and* $SL_2(q)$ *are replaced by* A_7 *and* $L_2(q)$.

LEMMA 3.7. *Let t be an involution acting on the semisimple group A. Assume a Sylow* 2*-group of* $C_A(t)$ *is cyclic of order* 4. *Then* $m(A) = 1$.

Proof. Let $R = \langle x\rangle \in \mathrm{Syl}_2\,(C_A(t))$. As R is cyclic, $t \notin A$. Let $R \leq S \in \mathrm{Syl}_2\,(A\langle t\rangle)$. Again as R is cyclic, $tr \in t^A$, where $r = x^2$. So $C_S(t)$ is the central product of $\langle t, tr\rangle = t^A \cap C_S(t)$ with R. Let $C_S(t) \leq X \leq S$ be maximal subject to being the central product of a dihedral group $D = \langle t, d\rangle = \langle t^A \cap X)$ with R. Here we choose $d \in t^A$.

Then $X \cap A = \langle td, x\rangle$ is abelian of exponent greater than 2, and thus is not Sylow in the semisimple group A. So we may choose $u \in N_S(X) - X$ with $u^2 \in X$. As $C_S(t) \leq X$, $t^u \notin t^X$, so we may take $t^u = d$. Set $X_1 = \langle u\rangle X$.

Suppose $u \in t^A$. Then $\langle u, t\rangle = D_1$ is dihedral. Let y be an element of order 4 in $\langle ut\rangle$. If $|D| > 4$ then $R = Z(X)$. If $|D| = 4$, then $R = X \cap A$. So in any event u normalizes R. Thus u either inverts or centralizes x. If u inverts x then u centralizes the 4-group $\langle r, yx\rangle \leq A$, impossible as $u \in t^A$. So X_1 is the central product of $D_1 = \langle t^A \cap X_1\rangle$ with R, contradicting the maximality of X.

So $u \notin t^A$. Thus $D = \langle X_1 \cap t^A\rangle \trianglelefteq N_S(X_1)$, so $N_S(X_1) = X_1 C_S(t) = X_1$ and then $S = X_1$. If $\Omega_1(S \cap A) \leq X \cap A$ then either $S \cap A$ is quaternion of order 8 or $|\Omega_1(S \cap A)| = 4$. In the latter case $S \cap A$ is not isomorphic to a Sylow 2-group of a semisimple group with that property. If $\Omega_1(S \cap A) \not\leq X \cap A$ we may pick u to be an involution, so $S \cap A$ is dihedral or the central product of the dihedral group $\langle u, td\rangle$ with either R or $\langle yr\rangle$, where y is an element of order 4 in $\langle td\rangle$. As A is semisimple, $S \cap A$ is dihedral and then $A \cong A_7$ or $L_2(q)$. But none of these groups admit an automorphism t of order 2 such that a Sylow 2-group of $C_A(t)$ is cyclic of order 4.

LEMMA 3.8. *Let U be a* 4*-group acting on a semisimple group A such that* $m(\Gamma_{1,U}(A)) = 1$. *Then* $m(A) = 1$.

Proof. Let S be a U-invariant Sylow 2-subgroup of A. If W is a U-invariant 4-subgroup of S, then $C_U(W) \neq 1$, contrary to hypothesis. So every U-invariant abelian subgroup of S is cyclic. Hence S is the central product of subgroups E and R where either $E = 1$ or E is extra special, and R is cyclic, quaternion, dihedral, or semidihedral (e.g. 5.4.9, [6]).

We may assume $m(A) > 1$. Suppose S is dihedral or semidihedral. By 3.6, S is semidihedral. Then A and its automorphism group are known. Indeed the outer automorphism group of A is cyclic, so some element $u \in U^{\#}$ induces an inner automorphism on A. But then $m(C_A(u)) > 1$.

Therefore S is not dihedral or semidihedral. Let $Z = \Omega_1(Z(S))$ and $V/Z = \Omega_1(Z(S/Z))$.

Claim S is extraspecial. Assume not. Then either R or R' is cyclic with a subgroup T of order 4 characteristic in S. Thus replacing S by TV if necessary we may assume R is cyclic of order 4. Now S is not cyclic, dihedral, or semidihedral so $S \neq R$ and hence $E \neq 1$. So there exists a U-invariant subgroup Y of S of order 8 containing R. $R \leq Z(S)$, so $\Omega_1(Y)$ is a U-invariant 4-group.

So S is extraspecial of order at least 2^5. Now for each involution $x \in S$, $Z = C_S(X)'$, so the involution $z \in Z$ is isolated. Thus $Z \leq Z(A)$ and then $A/Z(A)$ has abelian Sylow 2-groups. As A is semisimple its components have 2-rank 1. By 2.8 and 3.5, $\Gamma_{1,U}(AU)$ contains a Sylow 2-group of A, so $m(A) = 1$.

LEMMA 3.9. *Let $U = \langle a, b\rangle$ be a 4-group acting on a simple group L with dihedral Sylow 2-groups. Assume $C_L(a)$ and $C_L(b)$ have cyclic Sylow 2-groups and*

$$L \nleq \langle a, b, C_L(a), C_L(b)\rangle.$$

Then $[ab, L] = 1$.

Proof. $L \cong L_2(q)$, q odd, or A_7. As $m(C_L(a)) = 1$, $L \cong L_2(q)$ and $\langle a\rangle L \cong PGL_2(q)$, and inspecting Dickson's list of maximal subgroups (p. 285, [2]), $\langle a\rangle C_L(a)$ is a maximal subgroup of $\langle a\rangle L$ with $\langle a\rangle = Z(\langle a\rangle C_L(a))$. Hence the result follows.

4. The proof of Theorem 3

In this section we assume T is a nontrivial 2-group acting on a group G with $T \cap G = 1$. Let S be a Sylow 2-group of GT containing a Sylow 2-group of $N_{GT}(T)$. The proof of the following lemma is straightforward.

LEMMA 4.1. $N_S(T) = T \times (N_S(T) \cap G)$.

LEMMA 4.2. *Assume T is weakly closed in S with respect to G, and $O(G) = 1$. Then $[G, T] = 1$.*

Proof. Let GT be a counter example of minimal order. As T is weakly closed in S, $S \leq N(T)$. So by 4.1, $S = T \times (S \cap G)$.

Assume first $[E(G), T] = 1$. Then minimality of G implies $E(G) = 1$. So as $O(G) = 1$, $F^*(G) = O_2(G)$. But $T \leq C(O_2(G))$, and as $O_2(G) = F^*(G)$, $C_G(O_2(G)) = Z(O_2(G))$. Thus $TZ(O_2(G)) = C(O_2(G)) \trianglelefteq GT$. As T is weakly closed in S, $T \trianglelefteq GT$, so $[G, T] = 1$.

So $[E(G), T] \neq 1$ and then minimality of G implies $G = E(G)$, and $Z(G) = 1$. Let $Z = Z(T)$ and $1 \neq Y \leq (S \cap G)Z$. Then $T \leq N(Y) < G$, so minimality of G implies $N(Y) = O(N(Y))(N(Y) \cap N(T))$. Thus $N(Z)$ is weakly embedded in ZG, so a result of Goldschmidt [4] implies $Z \trianglelefteq GZ$. Now considering GT/Z, minimality of GT yields a contradiction.

For the remainder of this section we operate under the following hypothesis:

HYPOTHESIS 4.3. *$Q \trianglelefteq H \leq GT$, with $T \in \mathrm{Syl}_2(Q)$ and $T \cap T^g = 1$ for $g \in G - H$.*

LEMMA 4.4.

(1) *Let X be a 2-group containing T with $|X: T| \leq |T|$. Then $T \trianglelefteq X$.*

(2) *If $T \neq T^g \leq S \cap H$ then $TT^g = T \times T^g$ is abelian.*

Proof. Choose X as in (1). If T is weakly closed in $N_X(T) = X \cap H$ with respect to X, then $T \trianglelefteq X$. So we may take $T \neq T^g \leq X \cap H$. $T \in \mathrm{Syl}_2(Q)$, so $T = X \cap Q$. Thus $g \notin H$, so $T \cap T^g = 1$. Thus

$$|X| \geq |X \cap H| \geq |TT^g| = |T|^2 \geq |T|\,|X: T| = |X|,$$

so $TT^g = X = X \cap H \leq N(T)$. By symmetry, $T^g \trianglelefteq X$, so $X = T \times T^g$. By 4.1,

$$X = T \times (X \cap G) = T^g \times (X \cap G),$$

so $X \cap G$ is the center of $TT^g = X$. Then $T \cong X/T^g \cong X \cap G$ is abelian, so X is abelian.

LEMMA 4.5. *Assume $W = T \times T^g$ is the weak closure of T in $H \cap S$ with respect to G. Then:*

(1) *$H \cap S = C_S(t)$ for each $t \in T^\#$.*

(2) *$H \cap S = C_S(T^g)$.*

(3) *If $m(N_S(H \cap S)/(H \cap S)) > 1$ then T is elementary.*

(4) *$\Omega_1(W \cap G) \leq Z(\Omega_1(N_S(W)))$.*

(5) *Let $m(T) > 1$. If V is a conjugate of $\Omega_1(T)$ in $N_S(W)$, $V \not\leq W$, then $V \cap H = 1$ and $T = \Omega_1(T)$. In particular if W is the weak closure of T in S then $\Omega_1(W)$ is the weak closure of $\Omega_1(T)$ in S.*

Proof. By 4.4, T is abelian. Let $t \in T^\#$. By 4.3,

$$C_S(t) \leq H \cap S = T \times (H \cap S \cap G),$$

so as T is abelian, $H \cap S = C_S(t)$. Set $Z = \Omega_1(W \cap G)$. Then

$$Z \cap Z(H \cap S) = Z_0 \neq 1 \quad \text{so} \quad 1 \neq TZ_0 \cap T^g \leq Z(H \cap S).$$

Hence (1) implies (2). Also $Z \leq TT^g \leq Z(H \cap S)$.

Let $x(H \cap S)$ be an involution in $N_S(H \cap S)/(H \cap S)$. If $T^x = T$ then

$$x \in H \cap S \leq C(Z).$$

If $T^x \neq T$ then x inverts $[x, W] = W \cap G$ and hence centralizes Z.

Suppose $U/(H \cap S)$ is a 4-group in $N_S(H \cap S)/(H \cap S)$. We have shown each $u \in U - H$ inverts $W \cap G$, so $W \cap G$ is elementary. Thus T is elementary.

Finally assume V is a conjugate of $\Omega_1(T)$ contained in S but not in W. If $V \cap H \neq 1$ then by (1) and (2), $W \leq C(V \cap H)$ and then $V \leq W$. So $V \cap H = 1$. Assume $m(T) > 1$. Then (3) implies $T = \Omega_1(T)$.

LEMMA 4.6. *Assume $O(G) = 1$ and $W = TT^g$ is the weak closure of T in S with respect to G. Then one of the following holds:*

(1) $m(T) = 1$.

(2) $W \trianglelefteq GT$.

(3) $\langle T^G \rangle = O_2(\langle T^G \rangle) \times [T, G]$ *with* $O_2(\langle T^G \rangle) \cong E_n$ *and* $[T, G] \cong L_2(2^n)$, $Sz(2^n)$ *or* $U_3(2^n)$.

Proof. Let GT be a minimal counter example. By 4.4, T is abelian. We may take $m(T) > 1$.

Assume $G \neq E(G)$. Then minimality of G implies either $TE(G)$ is as in (3) or $[T, E(G)] = 1$. In the former case

$$[O_2(G), T] \leq O_2(G) \cap (W \cap G) \leq W \cap O_2(G) \cap E(G) = 1.$$

So $O_2(TE(G)) \leq C(F^*(GT) \leq O_2(GT)$ and then $TE(G) \leq F^*(GT)$. So $TE(G) = \langle T^G \rangle$. In the latter case minimality of G implies $E(G) = 1$, so $F^*(G) = O_2(G)$. If $[T, O_2(G)] = 1$ then arguing as in 4.2, $W \trianglelefteq GT$. So $W \cap G = [T, O_2(G)] \leq O_2(G) = P$. Thus T/P is weakly closed in S/P with respect to G/P, so by 4.2, $TO_{2,2'}(G) \trianglelefteq G$. Thus we may take $G = O_{2,2'}(G)$.

As $m(T) > 2$, 2.6.2 implies $O(Q) \leq C(P) \leq P$, so $O(Q) = 1$. Thus $W \trianglelefteq N(Q)$. So by 4.5 and induction on $|T|$ we may assume $T = \Omega_1(T)$.

Also $G = \langle N_G(\langle t \rangle P) \colon t \in T^\# \rangle$ and if $G \neq N_G(\langle t \rangle P)$, then by induction on the order of G, $W \trianglelefteq N_G(\langle t \rangle P)$. Therefore we may assume $G = N_G(\langle t \rangle P)$ for some $t \in T^\#$. By 4.5, $W \cap G \leq C(t^g) \leq C(T^g)$ for each $g \in G$. We may take $G = \langle T^G \rangle$, so $W \cap G \leq Z(G)$. Then $T/(W \cap G)$ is weakly closed in $S/(W \cap G)$ with respect to $G/(W \cap G)$, so by 4.2, $W/(W \cap G)O(G/(W \cap G)) \trianglelefteq G/(W \cap G)$, and we may assume $S = W$. So $W \cap G = F^*(G)$ and then $C_{GT}(W \cap G) = TC_G(W \cap G) = T(W \cap G) = W$. So $W \trianglelefteq G$.

Therefore $G = E(G)$. Let $U = \Omega_1(T)$ and $Z = \Omega_1(W \cap G)$. By 4.5, $Z \leq Z(\Omega_1(S))$ and $\Omega_1(W)$ is the weak closure of U in S with respect to G. Let $z \in Z^\#$ and assume $z^g \in S - Z$. Then as z^g centralizes Z, we may assume $Z^g \leq S$. By Alperin we may choose $ZZ^g \leq X \leq S$ and $g \in N(X)$. As $Z \leq X$, $U \leq N(X)$. Thus by minimality of G, $UO(N(X)) \trianglelefteq N(X)$ and we may choose $g \in N(U)$, impossible as $W \cap G$ is Sylow in $N_G(U)$. So Z is strongly closed in S with respect to G. Therefore by a theorem of Goldschmidt [5], either $\langle Z^G \rangle$ is a 2-group or $E(\langle Z^G \rangle)$ is a product of Goldschmidt groups.

In the first case UZ/Z is weakly closed in S/Z with respect to G/Z, so by 4.2, $UZ \trianglelefteq GT$, so $W \trianglelefteq GT$. In the second case minimality of G yields $G = \langle Z^G \rangle$. As $|U| = |Z|$ and $G \neq H$, G is a Bender group and $TG = O_2(GT) \times G$. As $T \cong W \cap G \leq Z(S)$, $T \cong E_n$ and $G \cong L_2(2^n)$, $Sz(2^n)$, or $U_3(2^n)$.

LEMMA 4.7. *Assume* $|T| \geq |S \cap H: T|$ *and let* W *be the weak closure of* T *in* S *with respect to* G. *Then one of the following holds:*

(1) $W = T$.

(2) *T is cyclic and S is wreathed, dihedral, or semidihedral.*

(3) $W = T \times T^g$ *is abelian.*

(4) *T is elementary,* $|W| = |T^3|$, $Z(W) = W' = \Phi(W)$ *and* $|S: W| = 2$.

Proof. Assume $W \neq T$. Then by 4.4, T is abelian and $T \times T^g \leq S \cap H$ for some $T^g \neq T$. As $|T| \geq |S \cap H: T|$, $S \cap H = T \times T^g$. If T is cyclic and $W = S \cap H$, then $|S: W| = 2$ so S is wreathed. So take $S \cap H < W$. Then $S \cap H$ is not the weak closure of T in $N_S(S \cap H)$, so if $m(T) = 1$ then $|S \cap H| = 4$. Then by a result of Suzuki, S is dihedral or semidihedral. So take $m(T) > 1$.

By 4.5, T is elementary and if V is a conjugate of T in $N_S(S \cap H)$ not contained in H, then $V \cap H = 1$. Hence $|V(H \cap S)| = |T|^3$. Let $Z = H \cap S \cap G$, $A = V(H \cap S)$. By 4.5, $Z = Z(A)$ and $ZV = N_S(V)$. As all elements in A but not in ZV or TZ are of order 4, $(TZ)^G \cap A = \{TZ, VZ\}$. Thus

$$|N_S(A): N_S(TZ)| \leq 2.$$

But $N_S(TZ) = AN_S(T) = ATZ = A$. Thus as $|T| > 2$, if $T^x \leq N_S(A)$ then $T^x \cap ZT$ or $T^x \cap ZV$ are nontrivial and hence $T^x \leq ZT$ or $T^x \leq ZV$. So $W = A$. As $Z = [T, V]$, $Z = W' = \Phi(W)$.

THEOREM 4.8. *Assume* $|T| \geq |S \cap H: T|$, $O(G) = 1$ *and* $m(T) > 1$. *Let* W *be the weak closure of* T *in* S *with respect to* G. *Then one of the following holds:*

(1) $W \trianglelefteq G$.

(2) $O^{2'}(GT) = O_2(GT) \times [T, G] \cong E_n \times L_2(2^n)$.

Proof. Suppose $\langle T^G \rangle = O_2(\langle T^G \rangle) \times [T, G]$ with $[T, G]$ a Bender group. If $s \in S - W$, then s centralizes an element u of $O_2(\langle T^G \rangle)^\#$ and an element v of $S \cap [T, G]$. Further we may take $uv \in T$, so by 4.5, $s \in H \cap S = W \leq \langle T^G \rangle$. Hence $O^{2'}(GT) = \langle T^G \rangle$. Therefore by 4.2, 4.6, and 4.7, we may take W as in 4.7.4. Let GT be a minimal counterexample. Define $Z = Z(W)$ and let V be a conjugate of T in $S - H$. Then ZT and ZV contain all involutions in W, so $Z = \Omega_1(R)$ where $R = W \cap G$. Minimality of G implies R is not Sylow in a normal subgroup of G. As usual either $G = E(G)$ or $F^*(G)$ is a 2-group. Let $u \in S \cap G - R$.

Assume $Z \leq O_2(G)$. Claim $Z \trianglelefteq GT$. This is clear if $Z = \Omega_1(O_2(G))$, so we may assume u is an involution in $O_2(G) - Z$. Then $R = [u, T] \leq O_2(G)$ and therefore $S \cap G = O_2(G)$. Now $G = \langle N_G(\langle t \rangle(S \cap G)) : t \in T^\# \rangle$ so by induction on the order of G we may take $G = N_G(P)$, where $P = \langle t \rangle(S \cap G)$, for some $t \in T^\#$. But

$$|P: C_P(Z)| \leq 2 > |P: C_P(u)|,$$

so Z is characteristic in P and hence normal in GT.

Suppose $O_2(G) = F^*(G)$. Then by 2.6.2, $O(Q) \leq C(O_2(G))$ and by 2.2, $O(Q) = 1$. Therefore $H \cap S \trianglelefteq H$ and then $(H \cap S)/Z$ serves in the role of T in G/Z for Lemma 4.6. Hence 4.6 implies R is Sylow in a normal subgroup of G, a contradiction.

So $G = E(G)$. Now G/Z has abelian Sylow 2-groups, so as $Z \leq Z(G)$, G is the central product of copies of $SL_2(5)$. As T centralizes a hyperplane of $(S \cap G)/Z$, T fixes each component of G, so G is quasisimple. Then $|Z| = 2$, a contradiction.

Arguing as in the last paragraph, G is quasisimple. By Corollary 4 in [5] there exists an element of $Z^{\#}$ fused to $u \in S - W$, and $|[Z, u]| \leq 2$. Hence $A = C_Z(u) = Z(S)$ is of index at most 2 in Z. Notice as R' and $[R, u]$ are contained in $Z(R)$, $(S \cap G)' \leq \langle R', [R, u]\rangle \leq A$. Thus if $A \leq Z(G)$, then $(S \cap G)/A$ is abelian of order at least 8, so G/A has a trivial multiplier, a contradiction.

So $A \not\leq Z(G)$. Now $W \leq N(A)$, so minimality of G implies $O(N(A))W \trianglelefteq N(A)$ and hence $Z \leq O_{2',2}(N(A))$. Therefore by Corollary 2 in [5], there exists $A^g \leq S$, $A^g \not\leq W$.

By Alperin's fusion theorem we may take $AA^g \leq X \leq S$ and $g \in N(X)$. $A \leq Z(S)$, so $A^g \leq Z(X)$. Hence $A = Z \cap X$ and $AA^g = \Omega_1(Z(X))$, so we may take $X = AA^g$. $(S \cap G)' \leq A \leq X$, so $S \cap G \in \mathrm{Syl}_2\,(N(X))$. Then we may take

$$g \in Y = \langle S \cap G, S^g \cap G\rangle.$$

Notice Y centralizes $B = A \cap A^g$ and then acts on X/B. As $|X/B| = 4$,

$$|O^2(Y): O^2(Y) \cap C(X)| \leq 3,$$

so $S \cap G$ has a subgroup D of index 2 with $DC(X) \trianglelefteq Y$. We may take $g \in N(D)$ by a Frattini argument. But $A = Z(D)$, or $D \cap R$ is quaternion and $A = D'$, a contradiction.

Lemma 4.5, Lemma 4.7, and Theorem 4.8 immediately yield Theorem 3.

LEMMA 4.9. *Let Q be a solvable, G semisimple, and A a component of G. Assume $m(Q) > 1$, $A \not\leq H$, and $m(T) \geq m(C_{Z(A)}(T))$. Then $C_A(T) - Z(A)$ contains a 2-element.*

Proof. Assume GQ is a counterexample. By 2.8 we may take $G = A$. Let $R = S \cap G$ and $U = C_R(T)$. $U \leq Z(G)$, so $m(T) \geq m(U)$. Let $r \in N_R(TU) - U$ with $r^2 \in U$. Then r inverts $TT^r \cap G = [T, r] = Z \cong T$. As $Z \leq Z(G)$ it follows that $\Phi(Z) = 1$, so $\Phi(T) = 1$. Therefore as $m(T) \geq m(U)$, $TZ = \Omega_1(TU)$ is the weak closure of T in $S \cap H$ with respect to G.

Set $\theta(T) = O(\langle C_G(t): t \in T^{\#}\rangle)$. As $\Phi(T) = 1$ and Q is solvable we may take $Q = \theta(T)T$. Let $t \in T^{\#}$ and $t^x \in TU$. Then $t^x = tz$, some $z \in Z \leq Z(GT)$, so $C_{GT}(t) = C_{GT}(t^x)$. Thus $[T, T^x] = 1$, so as $TU \in \mathrm{Syl}_2\,(H)$, $TU = T^xU$. Also $\theta(T) = \theta(T^x)$. Thus $(QU)^x = \theta(T^x)T^xU = QU$. Hence $\bar{Q}$ is tightly embedded in $\bar{G}\bar{T} = GT/U$. Further $C_{\bar{R}}(\bar{T}) = N_T(\bar{T}\bar{Z})$ and $|N_R(TZ): U| \leq |T|$, so $|C_{\bar{R}}(\bar{T})| \leq |\bar{T}|$. Hence. by 4.8, either $[\bar{T}, \bar{G}] = 1$ or $\bar{G}\bar{T} \cong E_n \times L_2(2^n)$.

In the first case $[T, G] = 1$ by 2.1. In the second case $\bar{G}$ has a cyclic multiplier. So in any event we have a contradiction.

5. The proof of Theorem 2

Assume the hypothesis of Theorem 2. If $T = 1$ for each $g \in G - H$, then Theorem 3.3 implies that $\langle Q^G \rangle \cap H$ is strongly embedded in $\langle Q^G \rangle$ or $Q \trianglelefteq G$.

Therefore assume $T \neq 1$. By symmetry between Q and Q^g we may assume $|T| \geq |N_Q(T)|_2$. Now, with suitable change of notation, Hypothesis 4.3 is satisfied with respect to the action of T on Q, so by 4.4, either $T \trianglelefteq S$ or $N_S(T) = T \times (N_S(T) \cap Q) \cong T \times T$. In the former case as $|T| \geq |N_Q(T)|_2 = |Q|_2 = |T|$, $S = T \times (S \cap Q) \cong T \times T$.

6. The proof of Theorem 5

In this section we assume the following hypothesis.

HYPOTHESIS 6.1. *G is a group, $D \leq G$, $\bar{D} = D/O(D)$. $\bar{D}$ is the central product of semisimple subgroups $\bar{A}_i$, $1 \leq i \leq r$, permuted by $H = N_G(D)$ under conjugation. Further:*

(1) *If t and t^g are 2-elements centralizing $\bar{A}_i$ and $\bar{A}_j$ respectively, and $t \notin Z(G)$, then $g \in H$.*

(2) *If $X \leq H^g$ with $XO(D) \geq A_i$ then $g \in H$.*

We wish to prove the following theorem:

THEOREM 5. *Assume Hypothesis* 6.1 *with $H \neq G$. Then either*

(1) *$D = A_1$, or*

(2) *$D = A_1A_2$, $m(A_i) = 1$, and if a_i, $i = 1, 2$, are commuting involutions with $\bar{a}_i \in Z(\bar{A}_i)$, then either $a_1 = a_2$ or $a_1a_2 \in Z(G)$.*

Throughout this section and until the proof of Theorem 5 is complete, we assume G is a counter example to Theorem 5. Set $A = A_1$.

LEMMA 6.2. $Z(G) = 1$.

Proof. Let $g \in N(DZ(G))$. Then $D^g \leq DZ(G) \leq H$, so by 6.1.2, $g \in H$. That is, $N(DZ(G)) = N(D)$. Similarly if t is a 2-element centralizing $A_iZ(G)/Z(G)O(D)$ then $A_i^t \leq A_iZ(G) \leq H$, so $t \in H$ and hence t centralizes $\bar{A}_i$. Therefore $DZ(G)/Z(G)$ satisfies Hypothesis 6.1 in $G/Z(G)$. Assume $Z(G) \neq 1$. Then minimality of G implies (1) or (2) of Theorem 5 holds in $G/Z(G)$, and as G does not satisfy (1), it must be (2). Choose a_i as in (2) and let $a = a_1a_2$. Then $aZ(G)$ is in the center of $G/Z(G)$ and we may assume $a \notin Z(G)$. Now for $g \in G$, $a^g \in aZ(G)$ centralizes $\bar{A}_i$ as above, so by 6.1.1, $G \leq H$, a contradiction.

LEMMA 6.3. *H acts transitively on the groups A_i.*

Proof. Assume not and let $\bar{B}$ be the product of all those $\bar{A}_i$ conjugate to $\bar{A}$ in H. By Theorem 2 there exists $g \in G - H$ such that a Sylow 2-group T of

$B^g \cap H$ is nontrivial. By 6.1.2, $[T, \bar{A}] \neq 1$, so by 2.1, $\bar{T} \not\leq O_2(\bar{B}\bar{T})$. Hence Theorems 2 and 3 imply either T is cyclic and B has dihedral or quaternion Sylow 2-groups, or $\bar{B}\bar{T} \cong E_n \times L_2(2^n)$. Hence $\bar{A} = \bar{B}$ is quasisimple. Similarly setting $\bar{L} = E(C_{\bar{D}}(\bar{A}))$, L has cyclic or dihedral Sylow 2-groups, or $\bar{L} \cong L_2(2^m)$. Choose $m(A) \geq m(L)$.

Assume $\bar{T}\bar{A} \cong E_n \times L_2(2^n)$. Let $T \leq V \in \mathrm{Syl}_2(TA)$, and let Y be a cyclic subgroup of order $2^n - 1$ in $N_A(V)$. Let $U = C_V(A)$. Then Y is transitive on $[V, Y]^{\#} = (V \cap A)^{\#}$. Let Y_1 be a similar subgroup in A^g normalizing V and set $X = \langle Y, Y_1 \rangle$. Now for $x \in X$, $U \cap U^x$ centralizes A and A^x, so by 6.1.1, either $x \in H \cap X$ or $U \cap U^x = 1$. Therefore the action of X on V satisfies the Hypothesis of 3.1, and that lemma implies that either $|V| = 8$ or X normalizes $[Y, V]$. In any case as $T = [Y_1, V] \neq V \cap A = [Y, V]$ and $T[Y, V] = V$, we have a contradiction.

So T is cyclic and A has quaternion or dihedral Sylow 2-groups. If $m(L) = 1$ and $Z^*(L) \not\leq A$ then we can apply Theorems 2 and 3 with $Z(L)A$ in the role of Q and obtain a contradiction. Hence we may assume A and L have dihedral Sylow 2-groups. Let t be the involution in T and let a and b be involutions in A and L, respectively, centralized by t. Let $K = A^g$. $\langle C(a), C(b) \rangle \leq H$, so by 6.1.2, $\bar{K} \not\leq \langle C_{\bar{K}}(a), C_{\bar{K}}(b) \rangle$. Further $C_K(a)$ and $C_K(b)$ have cyclic Sylow 2-groups, so by 3.9, ab centralizes $\bar{K}$. Hence by 6.1.1, $C(ab) \leq H^g$. But ab centralizes a 4-group in A whereas $m(A \cap H^g) = 1$, a contradiction.

Given a conjugate H^g of H define $U(H^g) = U_H(H^g)$ to be the set of 4-groups U contained in H such that $C_G(u) \leq H^g$ for each $u \in U^{\#}$. Define $\Delta(H)$ to be the set of conjugates H^g distinct from H such that $U(H^g)$ is nonempty.

THEOREM 6.4. *$\Delta(H)$ is empty.*

For the remainder of this section assume Theorem 6.4 to be false and let $H^g \in \Delta(H)$. Let $L_i = A_i^g$, and $L = L_1$. Let $T \in \mathrm{Syl}_2(N_L(A))$ and S a T-invariant Sylow 2-group of A with $S \cap H^g \in \mathrm{Syl}_2(A \cap H^g)$. In the next two lemmas let $U \in U(H^g)$.

LEMMA 6.5.
(1) *$U \leq N(A)$.*
(2) *If $m(A) = 1$ then $A \cap H^g - Z^*(A)$ contains a 2-element.*

Proof. $A \not\leq \Gamma_{1,u}(H) \leq H^g$, so by 2.8, $U \leq N(A)$. If the components of $\bar{A}$ satisfy Hypothesis I, then (2) is obvious. Hence by 2.9 we may assume $m(Z^*(A)) > 1$. In this case we will apply 4.9. Now $m(\Gamma_{1,U}(Z^*(A)) > 1$, so $Z^*(A)$ contains a member V of $U_{H^g}(H)$. Considering the action of V on $Z^*(D^g)$, $Q = Z^*(D^g) \cap H$ has 2-rank at least 2. Notice $Z^*(D)$ is tightly embedded in G. Hence by Theorem 2,

$$m(Q) \geq m(Z^*(D) \cap H^g).$$

Further if $X \in \mathrm{Syl}_2(Q)$ then by (1), $\Omega_1(X) \leq N(A)$. Hence

$$m(N_X(A)) \geq m(Z^*(A) \cap C(N_X(A)).$$

Now (2) follows from 4.9.

LEMMA 6.6. $m(A) > 1$.

Proof. Assume $m(A) = 1$. $A \leq \Gamma_{1,U}(N(A))$, so by 3.5, $U \leq \bar{A}\bar{C}(\bar{A})$, $\mathrm{Aut}_G(\bar{A})$ contains no 2-subgroup W such that W contains a quaternion group and $m(Z(W)) = 2$, and if V is a 2-group in $N(A)$ with $m(V) > 2$, then $A \leq \Gamma_{1,V}(N(A))$.

As $U \leq \bar{A}\bar{C}(\bar{A})$, $\Gamma_{1,U}(N(A)) \leq H^g$ contains a Sylow 2-group Q of A. Similarly $A_i \cap H^g$ contains a Sylow 2-group Q_i of A_i for each i. Now $X = Q_1 \cdots Q_{r-1}$ is a 2-group in H^g with $\Gamma_{1,X}(G) \leq H$. Suppose $r > 2$. Then $m(X) \geq 2$ and as $m(X) \leq 2$, $r = 3$, and $Q_1 \cap Q_2 = 1$. But as $r = 3$ is odd, we may assume X acts faithfully on L, contradicting 3.5.3.

So $r = 2$. We may assume $Z^*(A_1) \neq Z^*(A_2)$, so $Q_1 \cap Q_2 = 1$. Now $Y = Q_1Q_2$ has a subgroup W of index at most 2 fixing L. As $C(z) \leq H$ for each $z \in Z(Y) = \Omega_1(Y)$, $C_W(\bar{L}) = 1$, so W is isomorphic to a subgroup of $\mathrm{Aut}_G(\bar{L})$. But as $Q_1 \cap Q_2 = 1$, W contains a quaternion subgroup, and $m(Z(W)) = 2$, a contradiction.

LEMMA 6.7. *$m(L \cap H) \geq 2 \leq m(A \cap H^g)$, so $U(H^g) \cap L$ is nonempty.*

Proof. This follows from 6.5, 6.6, and 3.8.

LEMMA 6.8. *If $|T| \geq |N_A(L)|_2$ then there exists $x \in A - N(L)$ such that $T \cap T^x$ is of even order, and thus $Z^*(L) \cap Z^*(L^x) \neq 1$.*

Proof. If not, apply Theorem 3 to the action of T on A. By 6.7, $m(T) > 1$. If $\bar{T}\bar{A} \cong E_n \times L_2(2^n)$ the argument in the second paragraph of the proof of 6.3 can be repeated verbatim.

LEMMA 6.9. $D = A_1A_2$.

Proof. Assume $r > 2$. Let P be a 2-group in $L \cap H$ containing T and maximal with respect to $\langle A^P \rangle \neq D$. By 6.5 and 6.7, $P \neq 1$. Let Q be a P-invariant Sylow 2-group of $\langle A^P \rangle \cap H^g$ and $R = N_Q(P)$. Claim $R = Q$. Assume not and let $x \in N_Q(RP) - R$. Then $P < PP^x \leq PR$. As $x \in H^g$, $LL^x \cap \langle A^P \rangle$ is of odd order. But this is impossible as $P < PP^x \leq \langle A, P \rangle$. So $R = Q$. $[P, Q] = 1$ by 4.1. By 6.5, $\bar{Q} \cap \bar{A} \nleq Z(\bar{A})$, so $P \leq N(A)$. Hence $P = T \in \mathrm{Syl}_2(L \cap H)$.

Next let $T_2 \in \mathrm{Syl}_2(L_2 \cap H)$ and $V = TT_2$. We have shown $R = S \cap H^g = N_S(V)$. Claim $R = S$. Assume not and let $x \in N_S(RV) - R$, $x^2 \in R$. If $V \cap V^x \neq 1$ then

$$x \in N_S(V \cap V^x) \leq S \cap H^g = R,$$

a contradiction. So $V \cap V^x = 1$, and thus $|R| \geq |V|$. But by symmetry between H and H^g we may choose $|T| \geq |R|$, so $V = T \leq L$. But by 6.5, $\bar{T}_2 \nleq Z(\bar{L}_2)$, so $V \nleq L$. Thus $R = S$ and then by 4.4.1, $T \in \mathrm{Syl}_2(L)$.

Let $T_i \in \mathrm{Syl}_2(L_i)$, $T_i \leq H$, $X = T_1 T_2 \cdots T_{r-1}$ and $Y = XT_r$. By 4.2 there exists $a \in A$ such that $Y \neq Y^a \leq YS$. Then

$$|X|\,|S| \geq |XS| \geq |XX^a| \geq \frac{|X|^2}{|X \cap X^a|}.$$

Thus $|X \cap X^a| \geq |X|/|S| = |X: T|$. Also $a \notin H^g$, so by 6.1, no element of $(X^a)^{\#}$ centralizes $\bar{L}$. Hence $|X \cap X^a| = 1$. It follows that $T = X$, a contradiction.

LEMMA 6.10. *If $A \cap H^g \nleq N(L)$ and $T \nleq Z^*(L)$ there exists $a \in S \cap H^g$ with $L_1^a = L_2$. Further $R = TT^a \cap S \neq 1$, $R \leq Z(TT^a)$, $RT = TT^a$, and TT^a is abelian.*

Proof. By hypothesis there exists $a \in A$ with $L_1^a = L_2$. By 6.9 we may pick a to be a 2-element and hence pick $a \in S \cap H^g$. By hypothesis $T \nleq Z^*(L)$, so $T \neq T^a$. Thus $TT^a \cap S = R \neq 1$. R is centralized by T and T^a, so $R \leq Z(TT^a)$. $T \cap T^a \leq Z^*(L)$, so R projects on $\bar{T}/Z(\bar{L})$. Thus as $R \leq Z(TT^a)$, T and then TT^a is abelian.

LEMMA 6.11. *There exists $H^g \in \Delta(H)$ such that $W = N_S(L)$ is of maximal order and $W \nleq Z^*(A)$.*

Proof. Choose $H^g \in \Delta(H)$ so that W has maximal order. Let P be a WT-invariant Sylow 2-group of L containing a Sylow 2-group of $L \cap H$. By hypothesis $|W| \geq |T|$, so by 6.8 there exists $y \in P \cap H$ with $A \neq A^y$.

Assume $W \leq Z^*(A)$. $[W, \bar{L}] \neq 1$, by 6.7 we have $m(W) > 1$, and $Z(\bar{L}) \neq 1$, so by 4.8 applied to the action of W on L, $|W| < |N_P(W)| \leq 2|T| \leq 2|W|$. Therefore $|T| = |W|$ and there exists $a \in S$ with $L_2 = L^a$. Thus $T \leq Z^*(L)$, or else we could replace H^g by $H^{g^{-1}}$. Suppose $W^x \cap WTy$ is nonempty for some $W^x \leq WT\langle y \rangle$ and $x \in L$. Then $W^x \leq N(W) \leq H$ and W^x moves A, so substituting H^x for H^g we are done, since $T = Z^*(L) \cap T\langle y \rangle$ fixes A. Thus we may assume WT is the weak closure of W in $W(P \cap H)$.

Again as $[W, \bar{L}] \neq 1$, $m(W) > 1$, and $Z(\bar{L}) \neq 1$, 4.6 applied to the action of W on L implies WT is not the weak closure of W in WP, so by 4.5.5, $W = \Omega_1(W)$ and there is a conjugate W_1 of W under L acting regularly on $\Delta - \{T\}$, where $\Delta = W^G \cap WT$. Similarly there is a conjugate T_1 of T under A acting regularly on $\Delta - \{W\}$. Set $X = \langle TW, T_1, W_1 \rangle$. Then $(W_1TW)^\Delta = N_P(\Delta)^\Delta$ is regular on $\Delta - \{T\}$ and Sylow in $C_X(T)^\Delta \trianglelefteq N_X(T)^\Delta$, so by 3.3, $X^\Delta \cong L_2(q)$ in its doubly transitive representation on $q + 1$ letters, where $q = |W|$. In particular the stabilizer of W and T in X is transitive on $T^{\#}$. As $\langle y \rangle T \in \mathrm{Syl}_2(C_L(TW))$, a Frattini argument implies this fusion takes place in $N(\langle y \rangle T)$.

Now $y^2 \in T$ and as y moves A, 6.5.1 implies $y^2 \in T^{\#}$. Thus $\langle y^2 \rangle = \Phi(T\langle y \rangle)$, impossible as $N(\langle y \rangle T)$ is transitive on $T^{\#}$.

LEMMA 6.12. *Let* $U = \Omega_1(T)$. *Then:*

(1) $m(T) = m(N_S(L))$.

(2) *Either* $\{U, U^a\} = U^A \cap SU$, *some* $a \in A \cap H^g$, *and if* $L \cap H \nleq N(A)$ *then* $\Omega_1(S)$ *is abelian, or there exists* $b \in A - H^g$ *with* $UU^b = U \times U^b$.

Proof. Assume $\{U, U^a\} = U^A \cap SU$, some $a \in A \cap H^g$. By 4.2, $U \neq U^a$. $UU^a \trianglelefteq SU$, so $|S: N_S(U)| = 2$. By 6.5, $\Omega_1(S) \leq N_S(L) = N_S(U)$, so $m(T) \leq m(N_S(L))$. Also $N_S(L) \nleq Z^*(A)$, so if $L \cap H \nleq N(A)$, then by 6.10, $N_S(L) \geq \Omega_1(S)$ is abelian.

So we may assume there exists $b \in A - H^g$ with $U^b \leq SU \cap H^g$. As $b \notin H^g$, $U \cap U^b = 1$. By 6.5, $U^b = \Omega_1(U^b) \leq N(L)$, so by 4.1, $UU^b = U \times U^b$. Thus $m(T) \leq m(N_S(L))$. So in any event $m(T) \leq m(N_S(L))$. By symmetry, $m(T) = m(N_S(L))$.

LEMMA 6.13. *Choose* $H^g \in \Delta(H)$ *such that* $W = N_S(L) \nleq Z^*(A)$ *and* W *is of maximal order. Set* $U = \Omega_1(T)$ *and* $V = \Omega_1(W)$. *Then*

(1) $\Omega_1(S) = V$ *is abelian, and*

(2) UV *contains the weak closure of* U *in* SU *with respect to* A.

Proof. Maximality of (W) and 6.8 implies there exists $y \in L \cap H$ with $A \neq A^Y$. By 6.12 we may assume there exists $b \in A - H^g$ with $UU^b = U \times U^b$. Also by 6.12, $m(U) = m(V)$, so $UU^b = UV$.

If UV is the weak closure of U in US then it remains to show only that $V = \Omega_1(S)$, so we may choose b to be an involution. If UV is not the weak closure of U in US then let $U^x \leq N_{SU}(UV)$, $U^x \nleq UV$. If $U^x \cap UV \neq 1$ then $U \leq C(U^x \cap UV) \leq H^{gx}$ and then $U = \Omega_1(U) \leq N(L^x)$, so that U centralizes U^x. Hence $U^x = \Omega_1(U^x) \leq N_{SU}(U)$, so $U^x \leq \Omega_1(N_{SU}(L)) = UV$, a contradiction. Thus $U^x \cap UV = 1$, so there exists an involution $i \in U^x - (S \cap H^g)U$. $i = du$, $d \in S$, $u \in U$. Then $i^2 = 1$ and $d = iu^{-1} \in N(UV)$, so $1 = (du)^2 = d^2u^du \in d^2UV$. Thus $d^2 \in UV$. In this case set $b = d$, so that in any event, $b^2 \in UV$.

Then b centralizes $V = [UV, b]$. By 6.10, $WW^y = W^yR$, $R = WW^y \cap L \neq 1$. Hence $w \in W$ is of the form $w_1^y r$. As $[b, A^y] = 1$ and $C_U(b) = 1$, $[w, b] = 1$ if and only if $w = w_1^y$ and $w \in Z^*(A)$. So $V \leq Z^*(A)$ and $[V, y] = 1$.

Let P be a $T\langle y\rangle W$-invariant Sylow 2-group of L. By 4.2 there is $x \in L$ with $VV^x = VxV^x = K \leq PV$. $V \leq Z^*(A)$ and $Z^*(A)$ is tightly embedded in G. Also by 6.12, $m(W) = m(V)$, so K is the weak closure of V in $VP \cap H$. Thus we can apply 4.6 to the action of V on L and conclude there is a conjugate V_1 of V in PV acting on K with $V_1 \nleq K$. If $V_1 \cap K \neq 1$ then $V \leq C(V_1 \cap K) \leq C(V_1)$, so $V_1 \cap K = 1$. V_1 acts on $WT\langle y\rangle = C(V^d)$ for each $V^d \leq K$. Further $C_W(y) = W \cap A^y = W \cap Z^*(A)$, so unless $|W: W \cap Z^*(A)| = 2$, WT is the unique abelian subgroup of index 2 in $WT\langle y\rangle$, and then V_1 acts on WT. As $|W|$ is maximal, $|W| \geq |T|$, so as $W \cap W^d = 1$ for $d \in V_1^{\#}$, $WT = W \times W^d$. Thus $[d, WT] = T$ is inverted by d, so as $m(V_1) > 1$, $T \cong W$ is elementary abelian. Therefore $W = V \leq Z^*(A)$, contrary to hypothesis.

So $|W: W \cap Z^*(A)| = 2$ and $|WT: C_{WT}(y)| = 2$. Indeed the same argument shows $|N_{Y_1}(WT)| \leq 2$. Now $Z = C_{WT}(y) = Z(TW\langle y\rangle$ is of index 4 in $TW\langle y\rangle$, so $|V_1: N_{V_1}(WT)| \leq 2$. Therefore $|V_1| = 4$ and $TW = W \times W^d$ where $\langle d\rangle = N_{V_1}(WT)$.

Let $X = V_1 WT\langle y\rangle$. $V_1 \cap \Phi(X) = 1 \neq V \cap \Phi(X)$, so V_1 is not conjugate to V in $N(X)$. Thus $N_{PW}(X) = XN_{PW}(UV) = X$. Therefore $PW = X$. Then $P/U \cong X/UV$ is the split extension of a 4-group $VT\langle Y\rangle/UV$ by the 4-group VV_1U/UV and thus is abelian or order 16 or isomorphic to $Z_2 \times D_8$. P/U is Sylow in the semisimple group $A/Z^*(A)$, so $A/Z^*(A) \cong L_2(16)$ or $L_2(q_1) \times L_2(q_2)$, $q_i \equiv \pm 3 \bmod 8$. In the first case $A/Z^*(A)$ has a trivial multiplier, a contradiction. In the second case we must have $\bar{A} \cong SL_2(q_1) \times SL_2(q_2)$. But then $\bar{A}$ does not admit the action of a 4-group U tightly embedded in $U\bar{A}$.

We are now in a position to derive a contradiction and establish Theorem 6.4. Choose $H^g \in \Delta(H)$ with $W = N_S(L)$ of maximal order and $W \not\leq Z^*(A)$. This is possible by 6.11. By 6.13, $\Omega_1(S) = V$ is abelian and setting $U = \Omega_1(T)$, UV contains the weak closure of U in US with respect to A. Now as V is abelian and $V \cap Z^*(A) \neq 1$, Goldschmidt's fusion theorem [5] implies either $A/Z^*(A)$ has a component X isomorphic to $Sz(8)$, or $V \leq Z^*(A)$. In the first case V centralizes each involution in a Sylow 2-group of X^g and the other automorphism group of X is of odd order. So $U \leq \bar{X}C(\bar{X})$ and $|U| \geq |V|$ so $C_U(\bar{X}) \neq 1$. Let $\bar{B}$ be the product of all such components. Then BUV/BV is weakly closed in BUS/BV with respect to A/BV, so by 4.2, $[U, A/BV] = 1$. Thus $A \leq O(D)(A \cap H^g)$, a contradiction.

7. More Theorem 5

In this section G continues to be a counterexample to Theorem 5. In addition we assume the following hypothesis:

HYPOTHESIS 7.1. *t is an involution in H with $C(t) \leq H^g \neq H$. Set $L_i = A_i^g$, $L = L_1$.*

By 6.4, H^g contains no 4-group U with $C(u) \leq H$, all $u \in U^\#$ and of course the same holds with the roles of H and H^g reversed.

LEMMA 7.2. $m(C_D(\bar{A}\bar{A}^t) \cap C(t)) \leq 1 \leq m(C_{\bar{A}}(t))$.

Proof. This follows from our initial remark and 6.1.

LEMMA 7.3. *Assume $A \neq A^t$. Then $D = AA^t$ and $m(A) > 1$.*

Proof. Let $B = C_{[A,t]}(t)^\infty$. If $D \neq AA^t$, then by 7.2, $m(B) = 1$. Thus $m(A) = 1$ and $Z^*(A) \neq Z^*(A^t)$.

So we may assume $m(A) = 1$ and $O_2(\bar{A}) \neq O_2(\bar{A}^t)$, and it remains to exhibit a contradiction. Now $A \not\leq H^g$, so $\bar{B} = E(D \cap H^g)^-$. Let z be an involution with $z \in Z^*(B)$. Then by 2.7, either $\bar{B} = \bar{L}$, $[\bar{B}, \bar{L}] = 1$, or $\bar{B} = C_{[\bar{L}, z]}(z)'$. In any case $[\bar{L}, z] = 1$, a contradiction.

LEMMA 7.4. *Assume $A^t = A$. Then $D = A_1A_2$ and $\bar{A}\langle t\rangle \cong PGL_2(q)$, q odd.*

Proof. By 7.3, t fixes each A_i. If $m(A) = 1$ then as $m(C(t) \cap Z^*(D)) \leq 1$, $Z^*(A) = Z^*(A_i)$ for each i. Further t inverts elements $x_i \in A_i$, $i = 1, 2$, of order 4, so $U = \langle x_1x_2, x_1^2\rangle$ is a 4-group in $C(t)$. Hence by 7.2, $D = A_1A_2$.

So $m(A) > 1$. Let $R \in \mathrm{Syl}_2\,(C_A(t))$. Then $m(R) = 1$. Let r be the involution in R. Similarly let $R_2 \in \mathrm{Syl}_2\,(C_{A_2}(t))$, and suppose $D \neq A_1A_2$. Then $m(RR_2) = 1$, so r is the unique involution in RR_2. Hence $R = R_2 = \langle r\rangle$, so $A\langle t\rangle$ has dihedral or semidihedral Sylow 2-groups. This is impossible as $m(A) > 1$ and $r \in Z^*(A)$. Thus $D = A_1A_2$.

Suppose A has dihedral Sylow 2-groups. Then $\bar{A} \cong L_2(q)$ or A_7, so as $m(C_A(t)) = 1$, $\bar{A}\langle t\rangle \cong PGL_2(q)$. Hence we may assume this is not the case. So $|R| > 2$, and r is in the subgroup R_0, of index at most 2 in R, fixing L. Let $t \in T \in \mathrm{Syl}_2\,(C_L(r))$ and $T_0 = N_T(A)$. Let $RT_0 \leq S \in \mathrm{Syl}_2\,(T_0A)$.

Suppose t induces an inner automorphism on $\bar{A}$. Then $t = ab$, $a \in R$, $b \in C(\bar{A})$ and as $m(R) = 1$, $|a| > 2$, $a^2 \in Z^*(A)$, and R is cyclic Sylow 2-group of $C_A(a)$. Then by 5.4.8 in [6], either $S \cap A$ is dihedral or semidihedral or $R = \langle a\rangle$. The former is impossible as $\bar{A}$ is semisimple and $|Z^*(\bar{A})|_2 \neq 1$. In the latter case

$$|C_{(S \cap A)/\langle a^2\rangle}(a)| = 4,$$

so $(S \cap A)/\langle a^2\rangle$ is dihedral or semidihedral, yielding the same contradiction.

So t induces an outer automorphism on $\bar{A}$. By 4.2 there exists $T_0^a \neq T_0 \leq T_0R$. If $a \in H^g$ then we may choose $r \in T_0T_0^a \cap R$, and r induces an inner automorphism on L. So $T_0^a \cap T_0 = 1$, $|R| \geq |T_0|$, and by 4.4, T_0 is abelian. If $|R| = |T_0|$ then by Theorem 3, A has dihedral Sylow 2-groups. So $|T_0| < |R|$ and then by symmetry between H and H^g, $|T| = |R|$, and for $x \in T - T_0$, $R_0^x = R_0 \leq A \cap A^x \leq Z^*(A)$. Hence R is cyclic. Also R and T_0 normalize each other so $RT_0 = R \times T_0$.

Let RT_0 be of index 2 in $X \leq S$. If $|T_0| > 2$ then RT_0 is the unique abelian subgroup of index 2 in X, so $N_S(X) = XC_S(t) = X$ and $X = S$. But then $|S \cap A : R_0| = 4$, a contradiction as above. Hence $T_0 = \langle t\rangle$ and R is cyclic of order 4. Now 3.7 yields the desired conclusion.

LEMMA 7.5. $A^t \neq A$.

Proof. Assume $A^t = A$. By 7.4, A has dihedral Sylow 2-groups. t centralizes involutions a and b in A and A_2. Suppose neither a or b acts on L. Then by 2.8, ab centralizes $\bar{L}$, so $C(ab) \leq H^g$. But ab centralizes a 4-group in A, against 7.2. Thus we may assume a acts on L, and $t \in C_L(a)$. As $[b, t] = 1$, b acts on L. Now we repeat the arguments in the last paragraph of the proof of 6.3 to reach a contradiction.

8. Still more Theorem 5

In this section G continues to be a counterexample to Theorem 5.

LEMMA 8.1. *Let a be an involution with $C(a) \leq H$. Then either*
(1) *$a \in H^g$ if and only if $g \in H$; or*
(2) *$D = A_1A_2$, and if $t = a^g \in H$, $g \notin H$, then $A_1^t = A_2$ and $m(A) > 1$.*

Proof. This is a direct consequence of 7.3, 7.4, and 7.5.

LEMMA 8.2. *$D = A_1A_2$.*

Proof. Assume $D \neq A_1A_2$, and let a be an involution in A. By 8.1, $a \in H^g$ if and only if $g \in H$. Thus if $a \neq a^g \in C(a)$ then $g \in H$ so $a^g \in A^g = A_i$, some i. Further $b = aa^g$ centralizes some $\bar{A}_j$, so by 6.1 and 8.1, $b \in H^x$ if and only if $x \in H$. Thus 3.3 yields a contradiction.

LEMMA 8.3. *$m(A) > 1$.*

Proof. Assume $m(A) = 1$. Then we may assume $Z^*(A_1) \neq Z^*(A_2)$. Thus $Z^*(D) = UO(D)$ where U is a 4-group. Further by 8.1, for each $u \in U^\#$, $u \in H^g$ if and only if $g \in H$. Hence 3.3 again yields a contradiction as $U^\#$ is not fused in H.

LEMMA 8.4. *Let $T_i \in \mathrm{Syl}_2(A_i)$ and $T_1T_2 \leq S \in \mathrm{Syl}_2(G)$. Then $T_1 \cup T_2$ is strongly closed in $H \cap S$ with respect to H and in $N_S(T_i)$ with respect to G.*

Proof. See 8.1 and 8.2.

Now 8.3, 8.4, and 3.4 yield a contradiction. This completes the proof of Theorem 5.

9. Groups of component type

In this section we operate under the following hypothesis:

HYPOTHESIS 9.1. *G is a finite group. For each involution $t \in G$,*

$$O_{2',E}(C(t)) = O(C(t))E(C(t)).$$

Recall $\mathscr{L}$ is the set of components of the groups $E(C(t))$ as t ranges over all involutions in G, and $\mathscr{L}^*$ is the set of maximal elements of $\mathscr{L}$ under the partial order defined in Section 1.

LEMMA 9.2. *Let t be an involution in G, $A \trianglelefteq E(C(t))$, $A \in \mathscr{L}^*$, and $X \leq G$. Assume $t \in X$, $A \leq E(X)$ and u is an involution centralizing t and $[E(X), A] = B$. Then:*
(1) *$B \leq E(C(u))$.*
(2) *Either $A \trianglelefteq E(X)$ or $A = C_{[L,t]}(t)'$ for some component $L \neq L^t$ of $E(X)$ and of $E(C(u))$.*

Proof. By 2.7, $A \leq O_{2',E}(C(u))$, so by 9.1, $A \leq E(C(u))$. Then $B = [B, A] \leq E(C(u))$. If $A \trianglelefteq E(C(u))$ we are done, so assume $A \ntrianglelefteq E(C(u))$. Then by 2.7 there is a component L of $E(C(u))$ such that either $L = [L, t]$ and $A \leq L$ or $L \neq L^t$ and $A = C_{[L,t]}(t)'$. As $A \in \mathscr{L}^*$, the first case is impossible. Now $A \leq B = [B, A] \leq LL^t$ and by 2.5, either $A = B \trianglelefteq E(X)$ or $LL^t = B \leq E(X)$.

LEMMA 9.3. *Let t be an involution in G and $A \trianglelefteq E(C(t))$ where $A \in \mathscr{L}^*$. Let a be an involution centralizing $\langle t\rangle A$ with $A \trianglelefteq E(C(a))$. Let*

$$t \in S \in \mathrm{Syl}_2\,(C(\langle a\rangle A)) \quad \text{and} \quad S \leq T \in \mathrm{Syl}_2\,(C(A)).$$

Then:

(1) *There is a component $K \neq K^t$ of $E(C(a))$ with $A = C_{[K,t]}(t)'$.*

(2) *S has a subgroup R of index 2 centralizing K and $[K, t] \leq E(C(r))$ for each $r \in R^{\#}$. $[K, t] \trianglelefteq E(C(r))$ if $r \in C_R(t)$.*

(3) *If $y \in S - R$ with $t \in C(y)$ and $|C_S(y)| \geq 8$ then $A \trianglelefteq E(C(y))$.*

(4) *Either there exists an involution b in the center of T with $[K, t] \trianglelefteq E(C(b))$ or T is dihedral or semidihedral.*

(5) *If $\langle a\rangle = \Omega_1(O_2([K, t]))$ and $K \in \mathscr{L}^*$ then either $a \in Z^*(C(A))$ or T is dihedral or semidihedral, and a^x moves K for each $a^x \neq a$ in $a^{C(A)} \cap C(a)$.*

(6) *Either $m(C_T(t)) = 2$ or there exists a 4-group $U \leq C(A)$ with $KK^t \trianglelefteq E(C(u))$ for each $u \in U^{\#}$.*

Proof. (1) follows from 9.2. Now S acts on $[E(C(a)), A] = [K, t]$ and then has a subgroup R of index 2 fixing K. If $R \neq C_S(K)$ let $x \in R - C(K)$ with $x^2 \in C(K)$, and let $X = \langle x\rangle O_2(K)$. Then by 2.5, $K \leq \langle N_K(X), A\rangle \leq N(X)$, so $[X, K] = 1$, a contradiction. Now by 9.2, $[K, t] \leq E(C(r))$, each $r \in R^{\#}$ and if $[t, r] = 1$ then even $[K, t] \trianglelefteq E(C(r))$.

Suppose $y \in S - R$ with $[t, y] = 1$ and $C_S(y) = Y$ has order at least 8. Assume $A \ntrianglelefteq E(C(y))$. By (1) there is a component $L \neq L^t$ of $E(C(y))$ with $A = C_{[L,t]}(t)'$. If $[L, t] = [K, t]$ then $t \in \langle yt, y\rangle$ centralizes K, a contradiction. By (2) there is a subgroup Y_0 of index 2 in Y with $[L, t] \leq C(Y_0)$. As $|Y_0| \geq 4$, t centralizes an element $1 \neq z \in R \cap Y_0$. By (2) $[L, t]$ and $[K, t]$ are normal in $E(C(z))$, a contradiction.

Assume $\langle a\rangle = \Omega_1(O_2([K, t]))$ and $K \in \mathscr{L}^*$, but $a \notin Z^*(C(A))$. Then there exists $a \neq a^x \in S$, $x \in C(A)$. Let $r \in R$. By (2), $[K, t] \leq E(C(r))$. As $K \in \mathscr{L}^*$ and $a \in Z(K)$, 9.2 implies $[K, t] \trianglelefteq E(C(r))$. Thus $[K, t] = \langle A^{E(C(r))}\rangle$. As $a \neq a^x$,

$$[K, t] \neq [K, t]^x = \langle A^{E(C(a^x))}\rangle,$$

so $C_R([K, t]^x) = 1$. Hence (2) implies $|C_S(a^x)| = 4$. Thus either $S = C_S(a^x)$ is of order 4 and T is dihedral or semidihedral, or S is nonabelian dihedral or semidihedral and $\langle a\rangle$ is characteristic in S, so that $T = S$.

Notice that if $b \in Z(T)$ with $A \trianglelefteq E(C(b))$ and $|R| > 4$, then (2) and (3) imply $[K, t] \trianglelefteq E(C(b))$. So assume (4) is false and choose a with S maximal subject to $A \trianglelefteq E(C(a))$. Let $b \in N_T(S) - S$ with $b^2 \in S$. Then $Z = \langle a, a^b\rangle \leq Z(S)$

and by maximality of S, $A \trianglelefteq E(C(aa^b))$, so $a^b \in S - R$. As T is not dihedral or semidihedral, $|S| > 4$, so (3) implies $A \trianglelefteq E(C(a^b))$. As $b \in C(A)$, this is impossible.

Finally assume $m(C_T(t)) \geq 3$. Then by (2), $KK^t \trianglelefteq E(C(u))$ for each $u \in C_R(t)$. By (4) $S = T$, so $C_R(t)$ contains a 4-group.

HYPOTHESIS 9.4. *t is an involution in G and $A \trianglelefteq E(C(t))$ where $A \in \mathscr{L}^*$. Further one of the following holds:*

(1) *$t \in Z(A)$.*

(2) *t is contained in a 4-group U with $A \trianglelefteq E(C(u))$, each $u \in U^{\#}$.*

(3) *s is an involution in A with $A = [A, s]$, and $A \trianglelefteq \langle A^{C(s)} \rangle$.*

LEMMA 9.5. *Assume hypothesis 9.4. Let a be an involution centralizing $A\langle t \rangle$ and in case 9.4.2 assume a centralizes U. Then $A \trianglelefteq E(C(a))$.*

Proof. Assume $A \ntrianglelefteq E(C(a))$. By 9.3 there is a component $K \neq K^t$ of $E(C(a))$ with $A = C_{[K,t]}(t)'$. Then $Z(A) \leq Z(KK^t)$, so in 9.4.1, t centralizes K, a contradiction. Further in 9.4.2, U acts on $KK^t = [A, E(C(a))]$, so some $u \in U^{\#}$ fixes K. But we could have chosen $t = u$. Finally in 9.4.3, $C(s)$ normalizes the semisimple group $\langle A^{C(s)} \rangle = X$, while by 2.5, $KK^t = \langle C_K(s), A \rangle \leq N(X)$, contradicting 2.1.

LEMMA 9.6. *Assume hypothesis 9.4. Let a be an involution centralizing A. Then either $A \trianglelefteq E(C(a))$, or we are in 9.4.2, $C(A)$ has dihedral Sylow 2-groups, and $a \notin O^2(C(A))$.*

Proof. Let $a \in T \in \mathrm{Syl}_2\,(C(A))$. We may choose $t \in T$, and in 9.4.2, choose $U \leq T$. By 9.5 we may take $t \in Z(T)$. Then another application of 9.5 implies we are in 9.4.2. As U intersects any subgroup of T of index 2 nontrivially, 9.3.2 and 9.3.4 imply T is dihedral. As a is not conjugate to t in $C(A)$, $a \notin O^2(C(A))$.

THEOREM 9.7. *Assume Hypothesis 9.4.*

Then one of the following holds:

(1) *$A = \Delta_A(N(A)) = \Delta_A(C(a))$ for each involution $a \in C(A)$, and $[A, A^g] \neq 1$ for each $g \in G$.*

(2) *$D = \Delta_A(N(A)) = AA^x, x \in C(O_2(A))$. $m(A) = 1$. $D \trianglerighteq \Delta_A(C(a))$, each involution $a \in C(A)$ and if $[A, A^g] = 1$ then $A^g = A^x$.*

(3) *$A = \Delta_A(N(A))$ and $[A, A^g] \neq 1$ for $g \in G$. $C(A)$ has dihedral Sylow 2-groups, $O_2(A) = 1$, and $A = \Delta_A(C(a))$ for each involution a in $O^2(C(A))$.*

(4) *$A \trianglelefteq E(G)$.*

Proof. Set $D = \Delta_A(N(A))$. Suppose $[A, A^g] = 1$. By 9.6, $A \trianglelefteq E(C(a))$ for each involution $a \in A^g$. By 9.2, $A \trianglelefteq E(N(D^g))$. By symmetry $A^x \trianglelefteq D$ for each $A^x \trianglelefteq D^g$. Hence $D^g = D$.

Therefore if $A \trianglelefteq E(C(a))$, $A^x \trianglelefteq E(C(a^g))$, and $A^x \trianglelefteq D$, then $x \in N(D)$ and $A^{xg^{-1}} \trianglelefteq E(C(a))$, so xg^{-1} and then g normalizes D.

Assume $D \neq A$. By 9.6, $A^g \trianglelefteq E(C(a))$, each involution $a \in A$, so that $A \trianglelefteq \langle A^{C(a)} \rangle \leq D$. Hence 9.4.1 or 9.4.3 holds, and by 9.6, $A \trianglelefteq E(C(b))$ for each involution b in $C(A)$.

Suppose A normalizes D^g. Either $m(A) = 1$ and we pick $t \in Z(A)$, or A contains a 4-group U with $C(u) \leq N(D)$, each $u \in U^{\#}$. Then by 2.8 and 3.8 there is a 4-group $V \leq \Gamma_{1,U}(A^g)$ and by symmetry a 4-group W in $\Gamma_{1,V}(C_D(A))$. Now by 2.7 either $A = A^g$ or $[A, A^g] = 1$. Hence $g \in N(D)$.

Therefore we can apply Theorem 5 to D and conclude 9.7.2 or 9.7.4 holds. On the other hand if $D = A$ then 9.7.1. or 9.7.3 holds by 9.6.

10. The proof of Theorem 1

Assume the hypothesis of Theorem 1 and assume further that none of the conclusions hold. Then $m(C(A)) \geq 2$ and by 9.7, there is an involution $a \in C(A)$ with $A \trianglelefteq E(C(a))$. By 9.3 there is a component $K \neq K^t$ of $E(C(a))$ with $A = C_{[K,t]}(t)'$. A is a homomorphic image of K, so by hypothesis $K \in \mathscr{L}^*$.

Suppose K has 9.4. Then by 9.7, K satisfies 9.7.2. So we may choose $a \in Z(K)$ and A has dihedral Sylow 2-groups. By 9.3.5 either $a \in Z^*(C(A))$ or $C(A)$ has dihedral or semidihedral Sylow 2-groups. Assume $[A, A^g] = 1$. If $a \in Z^*(C(A))$ we may assume a centralizes a subgroup $B = B^\infty$ covering A^g modulo $O(C(A))$. By 2.5, $B \leq C(KK^t)$, so $C(b) \leq N(KK^t)$ for each involution b in B. We may take a Sylow 2-group T of B in A^g. Then $v = a^g \in C(T) \leq N(KK^t)$, so v acts on $KK^t \cap C(A^g) \geq A$. As $v \in Z^*(C(A^g))$, $[v, A] = 1$. Now $C(u) \leq N(K^gK^{tg})$ for each involution u in A, so $K = \langle A, C_K(u) \rangle \leq N(K^gK^{tg})$, a contradiction. So assume $C(A)$ has dihedral Sylow 2-groups. $A^g \leq C(A)$ and A^g has dihedral Sylow 2-groups, so $C(A)$ has 2 classes of involutions, one in $A^gO(C(A))$, and one outside. Moreover A^g contains a 4-group U fused in A^g while by 9.7, A is not normal in $E(C(u))$ for some $u \in U^{\#}$. So $t \notin A^gO(C(A))$. Hence $a \in A^gO(C(A))$ and conjugating in $C(A)$ we may assume $a \in A^g$. Let $S \in \mathrm{Syl}_2(C(A))$ with $t \in S$ and $a \in Z(S)$. Let $a \neq a^x \in S$, $x \in C(A)$. By 9.3.5, a^x moves K, so $b = ta^x$ fixes K and then centralizes KK^t by 2.5. As $a \in A^g$, $d = a^{g^{-1}} \in A \leq KK^t$. Then $d = kk_1^t$ where k and k_1 are elements of order 4 in K. Let $u = b_1k$ and $U = \langle u, a \rangle$, where b_1 is an element of order 4 in $\langle b \rangle$. Then U is a 4-group centralizing K^t and by 9.7, $\Gamma_{1,U}(G) \leq N(KK^t)$. Moreover u centralizes d and then acts on $D = K^{g^{-1}}K^{tg^{-1}}$. By 9.7, D does not act on KK^t, so by 2.8 and 3.5, $U \leq DC(D)$. As a Sylow 2-group of $C(D)$ is cyclic, even $U \leq D$. But $D - Z(D)$ has one class of involutions $(d^{g^{-1}})^D \subseteq a^G$, so $u \in a^G$. So by 9.7, u is fused to a in $N(KK^t)$, a contradiction. Therefore $[A, A^g] \neq 1$ and conclusion (3) of Theorem 1 holds.

Therefore K does not satisfy 9.4. Thus by 9.3.6,

$$m(C(A) \cap C(t)) = 2.$$

If $m(C(K) \cap C(a)) > 2$ then arguing on K in place of A as above, since $[K, K^g] = 1$ we conclude K satisfies (4) of Theorem 1. Then $A \leq [K, t] \leq E(G)$. By 2.7 there is a component L of $E(G)$ such that either $A = C_{[L,t]}(t)'$ or

$A \leq L$, and as $A \in \mathscr{L}^*$ it must be the former. So this is not the case and $m(C(K) \cap C(a)) \leq 2$. If $a \in Z([K, t])$ then 9.2 implies $K \trianglelefteq E(C(u))$ for each involution $u \in Z(K)$, impossible as K does not satisfy 9.4. Further if $O_2(K) \neq 1$ then by 9.3.2 we may pick $a \in Z([K, t])$. So $O_2(K) = 1$. But now $\langle a \rangle K^t \leq C(K)$ and $m(C(a) \cap K^t\langle a \rangle) > 2$, a contradiction.

11. The proof of Theorem 4

Assume Theorem 4 to be false and let G be a counterexample of minimal order. Let $1 \neq T$ be a Sylow 2-group of $Q^g \cap N(Q)$ acting on a Sylow 2-group R of Q. T exists by Theorem 2. If R is not cyclic, quaternion, or dihedral, then by Theorem 3, $m(T) > 1$ and $T \leq O_{2',2}(QT)$, so that by 2.1, $[K, t] \leq O(Q)$. As $m(T) > 1$, $K = C_K(T)O(Q) = \Gamma_{1,T}(K) \leq N(Q^g)$. In particular if $m(Q) > 1 = m(K)$, then by 2.7, $K \trianglelefteq O_{2',E}(KQ^g)$ and then by 2.1, K centralizes a Sylow 2-group of Q^g. Thus we are in (2). So we may assume $m(K) > 1$.

For $X \leq G$ define $X \sim K^h$ if $K^h \leq (K^h \cap C(X))O(K^h)$. Minimality of G implies:

(11.1) *If $X \sim Q$ and $C(X) \neq G$, then $C_K(X) \trianglelefteq O_{2',E}(C(X))$ or*

$$\frac{O^{2'}(C_Q(X))P}{O(C_Q(X))} \cong E_n \times L_2(2^n)$$

for suitable P.

Assume X is a 2-group in 11.1. Let U be a Sylow 2-group of $C_K(X)$ and V a Sylow 2-group of Q containing U. Then in the second case of 11.1, $X = C_{XU}(K/O(K)) \leq N_{XV}(XU)$, so if $V \neq U$ then $N_V(X) > U$. Hence if $N(X) < G$, then as $C_K(X) \trianglelefteq O_{2',E}(C(X))$, $U = N_V(X)$, so $U = V$, and (3) is satisfied, contrary to the choice of G. Therefore:

(11.2) *If X is a 2-group with $G \neq N(X)$ and $X \sim K$, then*

$$C_K(X) \trianglelefteq O_{2',E}(N(X)).$$

(11.3) *If $K \sim K^h$ then $K \trianglelefteq O_{2',E}(N(K^h))$.*
(11.4) *$\sim$ is an equivalence relation on K^G.*

Let $\mathscr{C}(K)$ be the equivalence class under $\sim$ containing K. If $\mathscr{C}(K) = K^G$ then $\langle K^G \rangle \trianglelefteq O_{2',E}(N(K))$, and then $K \trianglelefteq O_{2',E}(G)$. So $\mathscr{C}(K) \neq K^G$. Set $D = \langle \mathscr{C}(K) \rangle$. If t is a 2-element in $G - Z(G)$ with $t \sim K$, then by 11.2, $\{K^h : t \sim K^h\} \subseteq \mathscr{C}(K)$. So if $t^x \sim K^g \in \mathscr{C}(K)$ then $x \in N(D)$. Also if $K = N_K(D^y)O(K)$ then similar arguments show $D = D^y$. Therefore applying Theorem 5, with the members of $\mathscr{C}(K)$ in the role of the A_i's, we conclude $\mathscr{C}(K) = \{K\}$. Then remarks in the initial paragraph imply Q has a dihedral Sylow 2-group R and a Sylow 2-group T of $H \cap Q^g$ is cyclic for suitable $a \in G - H$ and $H = N(Q)$. We conclude $R \leq K$ and $\bar{K}\langle t \rangle = K\langle t \rangle/O(K) \cong PGL_2(q)$, where $T = \langle t \rangle$ is of order 2. We may take $K = Q$.

Set $P = R\langle t\rangle$ and let $P \leq S \in \mathrm{Syl}_2(G)$. Suppose s is a conjugate of t in $S - P$. Then $\langle \bar{s}\rangle\bar{Q} \cong PGL_2(q)$, so we may assume $st \in C(\bar{Q})$ and $[s, t] = 1$. Let $z \in C_R(t)$. Then tz is conjugate to t under Q, so $st \in t^G$, impossible as $\{Q\} = \mathscr{C}(Q)$. Thus $P = \langle t^G \cap S\rangle$. It follows that $S \in \mathrm{Syl}_2(G)$. Further $PQ \trianglelefteq H$, so P is strongly closed in S with respect to G.

As P is strongly closed in S with respect to G, $G/Z(G)$ satisfies the hypothesis of Theorem 4, so minimality of G implies $Z(G) = 1$. As $\{Q\} = \mathscr{C}(Q)$, $N(X) \leq H$ for each X with $C_Q(X)O(Q) = Q$. Suppose a is an involution centralizing $\bar{P}\bar{Q}$. Let $\langle z\rangle = Z(P)$. z induces an automorphism in $PGL_2(q)$ on $\bar{L} = \bar{Q}^g$ so $C_{\bar{L}}(z)$ is maximal in $\bar{L}$. But $\bar{L} \nleq \langle C_{\bar{L}}(z), C_{\bar{L}}(a)\rangle \leq H$, so az centralizes $\bar{L}$. Hence $P \leq C(az) \leq H^g$, a contradiction. Thus $C(\bar{Q})$ has odd order.

Therefore $S \leq \mathrm{Aut}\,(L_2(q))$, so S/P is cyclic. Minimality of G implies $G = \langle P^G\rangle$. Suppose a is an involution in $S - P$. As P is strongly closed in S and S/P is cyclic, $a^G \cap S \subseteq Pa$. But now considering the transfer of G to S/P, P is Sylow in a proper normal subgroup of G, a contradiction. Thus as $\bar{P}\bar{Q} \cong PGL_2(q)$ and $S \leq \mathrm{Aut}\,(\bar{P}\bar{Q})$, $S = P$ is dihedral. But this contradicts 3.6, since $\Gamma_{1,R}(G)$ is not solvable.

References

1. M. Aschbacher, *Finite groups with proper 2-generated cores*, Trans. Amer. Math. Soc., vol. 197 (1974), pp. 87–112.
2. L. Dickson, *Linear groups*, New York, 1958.
3. G. Glauberman, *On the automorphism group of a finite group having no nonidentity subgroups of odd order*, Math. Zeitschr., vol. 93 (1966), pp. 154–160.
4. D. Goldschmidt, *Weakly embedded 2-local subgroups of finite groups*, J. Algebra, vol. 21 (1972), pp. 341–351.
5. ———, *2-fusion in finite groups*, Ann. of Math., vol. 99 (1974), pp. 70–117.
6. D. Gorenstein, *Finite groups*, Harper and Row, New York, 1968.
7. ———, *Centralizers of involutions in finite simple groups*, Actes. Congr. inter. Math. 1970, vol. 1 (1971), pp. 355–360.
8. D. Gorenstein and J. Walter, *Centralizers of involutions in balanced groups*, J. Algebra, vol. 20 (1972), pp. 284–319.
9. J. Walter, *Centralizers of involutions and the classification problem*, Finite Groups, North Holland, Amsterdam, 1973, pp. 147–155.

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