THE DYER-LASHOF ALGEBRA AND THE Λ-ALGEBRA

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Introduction

The Dyer-Lashof algebra R is an algebra of operations which act on the homology of infinite loop spaces. The algebra Λ may be considered as an algebra of operations which act on the homotopy of simplical restricted Lie algebras. The purpose of this paper is to describe the relation between R and Λ. As an application, we use this relation, together with the Adams spectral sequence, to obtain information about possible spherical classes in $H_*(\Omega^n S^n)$.

For each integer $i \geq 0$, there is a Kudo-Araki operation Q^i which acts on the mod-2 homology of each infinite loop space. The Dyer-Lashof algebra R is the free associative algebra over Z_2 generated by the Q^i, modulo the ideal of relations which hold in every infinite loop space. There are two types of relations: (1) Q^i is 0 when applied to a homology class of dimension greater than i, and (2) Adem-type relations which hold among iterates of the Q's. The structure of R is known from the work of Araki-Kudo, Browder, Dyer-Lashof, Madsen, May, and Nishida. The properties of R that we use are summarized in Section 1. In particular, certain iterates of the Q's (those which are called allowable of non-negative excess) form a basis for the vector space R. Let $\Omega^\infty S^\infty$ be the component containing the constant map of the space $\lim_n \Omega^n S^n$. The mod-2 homology of $\Omega^\infty S^\infty$ is a polynomial algebra with generators in 1-1 correspondence with the allowable basis elements of positive excess of R.

The algebra Λ is obtained (in [6]) as the homotopy of the free simplical restricted Lie algebra on one generator. Λ is shown to be the free associative algebra generated by certain elements λ_i, as $i = 0, 1, 2, \ldots$, modulo an ideal which turns out to be the same as the ideal of Adem relations for R. Not only is the algebraic structure of R similar to that of Λ, but, as we shall show, the action of the Steerod algebra and higher operations on Λ is related to the differential ∂ on Λ.

For each space X, the (unstable) Adams spectral sequence $\{E_r(X)\}$, $r = 1, 2, \ldots$, is a sequence of differential groups, which, roughly speaking, goes from the homology of X to the homotopy of X. Here we use the methods of Bousfield and the author [5], (modifications of those of Massey-Peterson [12]), to obtain the Adams spectral sequence for $\Omega^\infty S^\infty$. The term $E_1(\Omega^\infty S^\infty)$, defined by means of $H_*(\Omega^\infty S^\infty)$ and Λ, is shown to be itself isomorphic to Λ. This isomorphism is *not* filtration preserving, *nor* differential respecting. The precise formulation of this isomorphism (Lemma (5.1)) is the basis of our calculations. We then show (in Sections 6 and 7) that, except for elements related

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either to the Hopf invariant, or to the Kervaire invariant, all of the elements of $E_1^{0,*}(\Omega^\infty S^\infty)$ have nonzero differentials, and so cannot persist to $E_\infty^{0,*}(\Omega^\infty S^\infty)$. Thus the group of spherical classes in $H_*(\Omega^\infty S^\infty)$ can consist at most of the Hopf classes in dimensions 1, 3, 7, and (possibly) the Kervaire classes in dimensions $2(2^n - 1)$ This includes the result of Browder [9] that dimensions other than $2(2^n - 1)$ cannot contain a framed manifold, for such a manifold would, by the Pontrjagin-Thom construction and [16], give rise to a spherical class in $H_*(\Omega^\infty S^\infty)$.

All the vector spaces, modules, algebras, etc , are to be taken over the field Z_2. For a topological space X, $H_*(X)$ will denote the homology groups of X with Z_2 Coefficients. Each space X is to have a base point, and ΩX stands for the space of loops in X starting and ending at the base point ΩX is an H-space, and $u \cdot v$ denotes the Pontrjagin product of u and v in $H_*(\Omega X)$.

The symbol $C(m, n)$ is the binomial coefficient $m!/n!\,(m - n)!$ reduced modulo 2, with the usual conventions: $C(m, 0) = 1$, and $C(m, n) = 0$ if $m < n$ or $n < 0$.

1. Homology operations

This section summarizes some of the results of [2], [8], [10], [11], [13], [14], [15], and establishes notation A space X is called an infinite loop space if there is a sequence of spaces $\{X_k\}$, $k \geq 0$, with $X = X_0$ and $X_k = \Omega X_{k+1}$ for each $k \geq 0$ If X is an infinite loop space, the Kudo-Araki operations

$$Q^i\colon H_q(X) \to H_{q+i}(X)$$

are defined for each integer $i \geq 0$. We let the operations act on the *right* in homology, and the index i refers to the dimension increase. These operations have the following properties.

(1.1)
$$(u)Q^i = 0 \quad \text{if } \dim(u) > i$$
$$(u)Q^i = u^2 \quad \text{if } \dim(u) = i$$

(1.2) (Suspension). Let $\sigma\colon H_*(\Omega X) \to H_{*+1}(X)$ be the homology suspension homomorphism Then $\sigma((u)Q^i) = (\sigma(u))Q^i$.

(1.3) (Co-product). Let $\psi\colon H_*(X) \to H_*(X) \otimes H_*(X)$ be the coproduct (induced from the diagonal map $\Delta\colon X \to X \times X$), with

$$\psi(u) = \sum_j u_j' \otimes u_j''.$$

Then

$$\psi((u)Q^i) = \sum_{j,\, 0 \leq k \leq i} (u_j')Q^k \otimes (u_j'')Q^{i-k}.$$

(1.4) (Adem relations). If $j < 2i$, then

$$(u)Q^iQ^j = \sum_{m \geq 0} C(m - i - 1, 2m - j)(u)Q^mQ^{i+1-m}.$$

(1.5) (Nishida relations). Let the Steenrod algebra act on the right in homology, dual to its left action in cohomology; then

$$(u)Q^i Sq^j = \sum_{m \geq 0} C(j - i, i - 2m)(u)Sq^m Q^{i-j+m}.$$

(1.6) (Cartan formula). $(u \cdot v)Q^i = \sum_{0 \leq k \leq i} ((u)Q^k) \cdot ((v)Q^{i-k})$.

The homology operations also satisfy formulas arising from the composition action of the space G (the set of homotopy equivalence of the sphere with itself) on infinite loop spaces, but these will not be needed in this paper.

For each sequence of nonnegative integers, $I = (i_1, i_2, \ldots, i_s)$, let Q^I be the iterated operation $(\cdot)Q^I = (\cdot)Q^{i_1} \cdots Q^{i_s}$. Then define

$$l(I) = \text{length of } I = s,$$

$$\deg(I) = \text{degree of } I = i_1 + \cdots + i_s,$$

$$e(I) = \text{excess of } I = i_s - (i_1 + \cdots + i_{s-1}).$$

We note that $e(I) = 2i_s - \deg(I)$. A sequence I is called *allowable* if $2i_j \geq i_{j+1}$ for each $j = 1, 2, \ldots, s - 1$.

The Dyer-Lashof algebra R is defined to be the quotient A/J where A is the free associative, not commutative, algebra over Z_2 generated by $\{Q^i, i \geq 0\}$, and J is the homogeneous ideal generated by the relations

$$Q^I \qquad \text{if excess } (I) < 0$$

$$Q^i Q^j - \sum_{m \geq 0} C(m - i - 1, 2m - j)Q^m Q^{i+j-m} \qquad \text{if } j > 2i$$

That is, J is the ideal of relations satisfied by iterated operations applied to a homology class in any infinite loop space. The calculation in [10] of $H_*(\Omega^\infty S^\infty)$ shows that no further relations hold in general. The relations imply that R has a vector space basis $\{Q^I\}$, where I varies over all allowable sequences of excess ≥ 0.

2. The spaces $\Omega^n S^{n+k}$

Let $G_*(n)$ be the space of all continuous maps (of any degree, not necessarily basepoint preserving) of S^{n-1} to itself, with the compact-open topology. The evaluation map $\phi \colon G_*(n) \to S^{n-1}$ is defined by $\phi(f) = f(p)$, where p is a fixed basepoint in S^{n-1}. Then ϕ is a fibration, and the fibre is $(\Omega^n S^n)_*$, the set of all basepoint preserving maps of S^{n-1} to itself (of any degree). There are inclusions $G_*(n) \subset G_*(n + 1)$ for all n, and G_* is defined to be $\lim_n G_*(n)$. Similarly, $(\Omega^\infty S^\infty)_* = \lim_n (\Omega^n S^n)_*$. The inclusion $(\Omega^\infty S^\infty)_* \to G_*$ is a homotopy equivalence. Each map f of a sphere to itself has a degree, and for each integer j, let G_j (or $(\Omega^\infty S^\infty)_j$) be the subspace of G_* (respectively of $(\Omega^\infty S^\infty)_*$) of maps of degree j. Then each G_j and each $(\Omega^\infty S^\infty)_j$ is a component of G_* or of $(\Omega^\infty S^\infty)_*$

and they all have the same homotopy type. We denote the component G_1 by SG, and the component $(\Omega^\infty S^\infty)_0$ by $\Omega^\infty S^\infty$.

The space $(\Omega^\infty S^\infty)_*$ is an infinite loop space, taking as the kth space $X_k = \lim_n \Omega^n S^{n+k}$. The homology class of a point in the component $(\Omega^\infty S^\infty)_j$ will be called $[j]$. The class $[0]$ is the unit for the Pontrjagin algebra $H_*((\Omega^\infty S^\infty)_*)$. In general, multiplication by $[j]$ sends $H_*((\Omega^\infty S^\infty)_k)$ isomorphically onto $H_*((\Omega^\infty S^\infty)_{k+j})$. For $i > 0$, the operation Q^i yields 0 when applied to $[0]$, but not when applied to the other $[j]$. The result of Dyer-Lashof ([10]) is that

$$H_*(\Omega^\infty S^\infty) \cong P([1]Q^{i_1} \cdot [-2])Q^{i_2} \cdots Q^{i}q)$$

where $I = (i_1, \ldots, i_q)$ varies over all allowable sequences of excess ≥ 1, and $P(\cdots)$ stands for the polynomial algebra on the stated generators.

The space SG has the same homotopy type as $\Omega^\infty S^\infty$, hence

$$H_*(SG) \cong H_*(\Omega^\infty S^\infty),$$

even as coalgebras over the Steenrod algebra. With composition as multiplication, $H_*(SG)$ has a different ring structure and a different action by the Dyer-Lashof algebra ([14], [11]), but we do not need these here.

3. Unstable A-coalgebras

As in [5], let MA be the category of right A-modules, and CA the category of right homology A-coalgebras, where A is the mod-2 Steenrod algebra. That is, M in MA is to be a non-negatively graded vector space with a right A action: for $x \in M_n$, $(x)Sq^i \in M_{n-1}$ with $(x)Sq^i = 0$ if $2i > n$. C in CA is to be simultaneously an unstable right A-module and a connected, cocommutative coalgebra, where the structures are compatible as follows. The comultiplication of C satisfies a Cartan formula and the square root map $\sqrt{\cdot}$ of C (the dual of the squaring map for algebras) satisfies

$$\sqrt{\cdot} = (\cdot)Sq^n \colon C_{2n} \to C_n.$$

For example, if X is any connected space, $H_*(X)$ is in CA, and depends only on the homotopy type of X.

For each M in MA with $M_0 = 0$, let $U_*(M)$ in CA be the free unstable right A-coalgebra generated by M; $U_*(M)$ may be defined by a universal mapping property. If M is of finite type, then $U_*(M)$ is dual to $U(M^*)$, the free unstable (left) A-algebra generated by M^* (see [17; p. 29]).

Let $M(\Omega^\infty S^\infty)$ be the vector space with basis the symbols $\{x_0(I)\}$, as $I = (i_1, \ldots, i_q)$ varies over all allowable sequences of excess ≥ 0 and degree > 0; put $\dim x_0(I) = \text{degree}\,(I)$. The Dyer-Lashof algebra and the Steenrod algebra are to act on $M(\Omega^\infty S^\infty)$ by the formulas (1.4) and (1.5). Specifically,

$$x_0(i_1, \ldots, i_q)Q^i = \begin{cases} x_0(i_1, \ldots, i_q, i), & i \geq \deg(I) \\ 0, & i < \deg(I) \end{cases}$$

with the convention that the Adem relations are to hold. That is, if $2i_q < i$, then

$$x_0(i_1, \ldots, i_q, i) = \sum_m C(m - i_q - 1, 2m - i_q)(x_0(i_1, \ldots, i_{q-1}, m, i + i_q - m)).$$

Also,

$$x_0(i_1, \ldots, i_q)Sq^i = \sum_m C(i - i_q, i_q - 2m)x_0(i_1, \ldots, i_{q-1})Sq^m Q^{i-i_q+m}.$$

This defines the action of the Sq^i on $M(\Omega^\infty S^\infty)$ inductively by length; thereby $M(\Omega^\infty S^\infty)$ is in MA, and $U_*(M(\Omega^\infty S^\infty))$ is in CA.

PROPOSITION (3.1). *As members of CA*, $H_*(\Omega^\infty S^\infty) \cong U_*(M(\Omega^\infty S^\infty))$.

Proof. As asserted in Section 2, $H_*(\Omega^\infty S^\infty)$ is a polynomial algebras with generators

$$\{([1]Q^{i_1} \cdot [-2])Q^{i_2} \cdots Q^{i_q}\}$$

where $I = (i_1, \ldots, i_q)$ varies over all allowable sequences of excess ≥ 1. Thus $H_*(\Omega^\infty S^\infty)$ has a simple system of generators of the same form except that now the $I = (i_1, \ldots, i_q)$ vary over all allowable sequences of excess ≥ 0. Let

$$\alpha\colon H_*(\Omega^\infty S^\infty) \to M(\Omega^\infty S^\infty).$$

be the homomorphism defined by

$$\alpha(([1]Q^{i_1} \cdot [-2])Q^{i_2} \cdots Q^{i_q}) = x_0(i_1, \ldots, i_q)$$

for the simple generators, and $\alpha(y) = 0$ when y is a product of two or more distinct simple generators. From the universality of $U_*(\cdot)$, we obtain a homomorphism

$$\tilde{\alpha}\colon H_*(\Omega^\infty S^\infty) \to U_*(M(\Omega^\infty S^\infty)).$$

It follows from Madsen's calculations that $\tilde{\alpha}$ is an isomorphism ([11, Proposition 4.13], see also [13]).

We also need to consider the homology of the various spaces $\Omega^n S^{n+k}$. For each $n \geq 1$, $k \geq 0$, let $M(\Omega^n S^{n+k})$ be the vector space with basis $\{x_k(I)\}$, as $I = (i_1, \ldots, i_q)$ varies over all allowable sequences of excess $\geq k$, and with $i_1 < n + k$. For $k = 0$, we are considering the component $(\Omega^n S^n)_0$, so the empty sequence is to be excluded from $M(\Omega^n S^n)$. The Dyer-Lashof algebra and the Steenrod algebra act on $M(\Omega^n S^{n+k})$ by the formulas (1.4) and (1.5), taking into account that dim $(x_k) = k$, and that $\Omega^n S^{n+k}$ is only an H^{n-1}-space. In these cases, $H^*(\Omega^n S^{n+k})$ is not a polynomial algebra: for $k \geq 1$, it is an exterior algebra, while for $k = 0$, it is a truncated polynomial algebra. A result of Araki-Kudo asserts that for $n \geq 1$, $k \geq 1$, $H_*(\Omega^n S^{n+k}) \cong P(x_k Q^I)$ as $I = (i_1, \ldots, i_q)$ over all allowable sequences of excess $\geq k$ and with $i_1 < n + k$. Again Madsen's calculations show the following ([11], [13]).

PROPOSITION (3.2). *As members of CA*, $H_*(\Omega^n S^{n+k}) \cong U_*(M(\Omega^n S^{n+k}))$.

We next consider the James map ([18; p. 21]), $h\colon \Omega S^{n+k+1} \to \Omega S^{2n+2k+1}$. The fiber of h (localized at the prime 2) is S^{n+k}. Thus, after looping n times, there is a fibration (at the prime 2):

$$\Omega^n S^{n+k} \longrightarrow \Omega^{n+1} S^{n+k+1} \xrightarrow{\Omega^n h} \Omega^{n+1} S^{2n+2k+1}.$$

This corresponds to a short exact sequence in MA,

$$(3.3) \qquad 0 \to M(\Omega^n S^{n+k}) \xrightarrow{\iota} M(\Omega^{n+1} S^{n+k+1}) \xrightarrow{\eta} M(\Omega^{n+1} S^{2n+2k+1}) \to 0$$

where ι is the natural inclusion, and

$$\eta(x_k(I)) = \begin{cases} x_{2k+n}(i_2, \ldots, i_q) & \text{if } I = (k+n, i_2, \ldots, i_q) \\ 0 & \text{otherwise.} \end{cases}$$

To verify that $\Omega^n h$ induces η in homology, observe that h is not a loop map, and that $h_*\colon H_*(\Omega S^{n+k+1}) \to H_*(\Omega S^{2n+2k+1})$ does not commute the homology operations. Instead (as in [18]),

$$h_*(x_{n+k}) = 0,$$
$$h_*(x_{n+k} Q^{n+k} Q^{2(n+k)} \cdots Q^{2^q(n+k)}) = x_{2(n+k)} Q^{2(n+k)} \cdots Q^{2^q(n+k)}.$$

Then $\Omega^n h$ is an H^{n-1}-map, and $(\Omega^n h)_* = \eta$ on $M(\Omega^{n+1} S^{n+k+1})$.

4. The Unstable Adams spectral sequence

In [6], (see also [5]), there is constructed for each space X, a spectral sequence $\{E_r^{s,t}(X)\}$, $r = 1, 2, \ldots$, with the following properties.

(1) For a connected nilpotent space X, the $E_r(X)$ converge to $\pi_*(X)$ modulo the subgroup of elements of odd order. This convergence is valid when X is an H-space, in particular, for the space $\Omega^\infty S^\infty$.

(2) $E_2^{s,t}(X) \cong \mathrm{Ext}_{CA}^{s,t}(Z_2, H_*(X))$.

(3) The Hurewicz homomorphism (reduced mod 2) factors as the composite

$$\pi_t(X) \to E_\infty^{0,t}(X) \subset \cdots \subset E_2^{0,t}(X) \subset H_t(X).$$

We shall be dealing with spaces X for which $H_*(X) \cong U_*(M)$. In this situation, a theorem of Massey and Peterson ([12]) asserts that

$$E_2^{s,t}(X) \cong \mathrm{Ext}_{MA}^{s,t}(Z_2, M).$$

We retain the notation of [5], where it is further shown that $\mathrm{Ext}_{MA}^{s,t}(Z_2, M)$ may be calculated as the homology of a complex which we shall describe shortly.

First the algebra Λ is defined to be the free associative, not commutative, algebra with unit, which has

(i) for each integer $i \geq 0$, a generator λ_i of degree i;

(ii) for each pair of integers $i \geq 0$, $m \geq 0$, a relation

$$\lambda_i \lambda_{2i+1+m} = \sum_{j \geq 0} C(m-1-j, j) \lambda_{i+m-j} \lambda_{2i+1+j};$$

(iii) a differential ∂, $\partial(\lambda_i) = \sum_{j \geq 1} C(i-j, j) \lambda_{i-j} \lambda_{j-1}$.

For each M in MA, let $(M \hat{\otimes} \Lambda, \delta)$ be the chain complex as follows. $M \hat{\otimes} \Lambda$ is the subspace of $M \otimes \Lambda$ spanned by $x_k \otimes \lambda_I$ where $x_k \in M_k$ and $I = (i_1, \ldots, i_s)$ is allowable with $i_1 < k$ (or I is empty). The differential δ on $M \hat{\otimes} \Lambda$ is defined by

$$\delta(x \otimes \lambda_I) = x \otimes (\partial \lambda_I) + \sum_{j \geq 0} (x) Sq^j \otimes \lambda_{j-1} \lambda_I.$$

As the relations in Λ are homogeneous,

$$\Lambda = \oplus_{s \geq 0} \Lambda^s, \qquad M \hat{\otimes} \Lambda = \oplus_{s \geq 0} M \hat{\otimes} \Lambda^s.$$

The term $x_k \otimes \lambda_I$ is given bi-degree (s, t), where $s = \text{length}(I)$, and $t = s + k + \text{degree}(I)$. Theorem (3.3) of [5] asserts that if $H_*(X) \cong U_*(M)$ then,

$$E_2^{s,t}(X) \cong H^{s,t}(M \hat{\otimes} \Lambda).$$

In particular,

$$E_2^{s,t}(\Omega^n S^{n+k}) \cong H^{s,t}(M(\Omega^n S^{n+k}) \hat{\otimes} \Lambda).$$

Passing to the homology of the sequence (3.3), we obtain for each $n \geq 0, k \geq 1$, a long exact sequence

$$\text{(4.2)} \quad \begin{aligned} \cdots \longrightarrow E_2^{s,t}(\Omega^n S^{n+k}) &\xrightarrow{\iota_*} E_2^{s,t}(\Omega^{n+1} S^{n+k+1}) \\ \xrightarrow{\eta_*} E_2^{s,t}(\Omega^{n+1} S^{2n+2k+1}) &\xrightarrow{\partial} E_2^{s+1,t}(\Omega^n S^{n+k}) \longrightarrow \cdots \end{aligned}$$

which is a form of the EHP sequence at the E_2-level.

5. The complex $M \hat{\otimes} \Lambda$

Let $M(\Omega^\infty S^\infty)$ be the vector space as defined in Section 3, and Λ the algebra of Section 4, with $\bar{\Lambda}$ the ideal of positive dimensional members of Λ. There is an isomorphism θ: $M(\Omega^\infty S^\infty) \hat{\otimes} \Lambda \cong \bar{\Lambda}$ defined by

$$\theta((x_0)I' \otimes \lambda_{I''}) = \lambda_{I'} \lambda_{I''}.$$

To see that θ is an isomorphism, observe that for each allowable sequence $I = (i_1, \ldots, i_s)$ there is a unique index q for which the sequences $I' = (i_1, \ldots, i_q)$ and $I'' = (i_{q+1}, \ldots, i_s)$ satisfy excess $(I') \geq 0$ and $i_{q+1} <$ degree (I'). We write $I = (I' \mid I'')$ to indicate this decomposition.

The isomorphism θ is *not* filtration preserving, as the filtration degree of $x_0(I') \oplus \lambda_{I''}$ is length (I''), while the filtration degree of $\lambda_{I'} \lambda_{I''}$ is length (I', I''). *Nor* does θ commute with the differentials. However, the differentials δ of $M(\Omega^\infty S^\infty) \hat{\otimes} \Lambda$ and ∂ of Λ are related as follows.

LEMMA (5.1). *Let $I = (i_1, \ldots, i_s)$ be allowable with $I = (I' \mid I'')$. Suppose that*

$$\partial\lambda_I = \sum \alpha_J\lambda_J, \quad \alpha_J \in Z_2$$

where the J vary over allowable sequences of length $s + 1$. Then

$$\delta((x_0)I' \otimes \lambda_{I''}) = \sum{}^* \alpha_J x_0(J') \otimes \lambda_{J''}$$

where the sum $\sum^*$ is taken over those allowable sequences $J = (J' \mid J'')$ for which length (J'') = length $(I'') + 1$.

Proof. Consider first the special case where I is allowable of excess ≥ 0; that is, $I = I'$ and I'' is empty. Suppose $\partial\lambda_I = \sum \alpha_J\lambda_J$. We shall show by induction on length (I) that

$$\delta(x_0(I)) = \sum{}^* \alpha_J x_0(J') \otimes \lambda_{J''} \tag{5.2}$$

where the sum $\sum^*$ is taken for those $J = (J' \mid J'')$, with length $(J'') = 1$. Observe that this is equivalent to the assertion that for each positive integer j, $x_0(I)Sq^j = \sum \alpha_J x_0(J')$ the sum taken for those $J = (J' \mid J'')$ for which $J'' = (j - 1)$.

For length $(I) = 1$, the sequence I is merely (i). Then

$$\begin{aligned}\delta(x_0(i)) &= \sum_{j\geq 1} (x_0(i))Sq^j \otimes \lambda_{j-1} \\ &= \sum_{j\geq 1} C(i - j, j)x_0(i - j) \otimes \lambda_{j-1}\end{aligned}$$

by the Nishida relations which define the action of the Sq^j on $M(\Omega^\infty S^\infty)$. As the expression for ∂ is given by (Section 4, (iii)), the formula (5.2) is valid for length 1.

Assume inductively that (5.2) is valid for lengths $<s$, and let $I = (i_1, \ldots, i_s)$ be allowable of excess ≥ 0, and of length s. Then

$$\begin{aligned}\delta(x_0(I)) &= \sum_{j\geq 1} x_0(I)Sq^j \otimes \lambda_{j-1} \\ &= \sum_{j\geq 1}\sum_{m\geq 0} C(i_s - j, j - 2m)x_0(i_1, \ldots, i_{s-1})Sq^m Q^{i_s-j+m} \otimes \lambda_{j-1} \\ &= \sum_{j\geq 1} C(i_s - j, j)x_0(i_1, \ldots, i_{s-1}, i_{s-j}) \otimes \lambda_{j-1} \\ &\quad + \sum_{j\geq 1}\sum_{m\geq 1} C(i_s - j, j - 2m)x_0(i_1, \ldots, i_{s-1})Sq^m Q^{i_s-j+m} \otimes \lambda_{j-1}.\end{aligned}$$

Suppose now that

$$\begin{aligned}\partial(\lambda_{i_1}\cdots\lambda_{i_{s-1}}) &= \sum \alpha_K\lambda_K \\ &= \sum_{K''=m-1} \alpha_K\lambda_{K'}\lambda_{m-1} + \sum_{K''\neq m-1} \alpha_K\lambda_{K'}\lambda_{K''}\end{aligned}$$

where the first sum is taken for those $K \neq (K' \mid K'')$ with $K'' = m - 1$, and the second sum is taken for $K'' \neq m - 1$. The inductive assumption asserts that

$$x_0(i_1, \ldots, i_{s-1})Sq^m = \sum_{K''=m-1} \alpha_K x_0(K').$$

The expression for $\delta(x_0(I))$ becomes

$$\delta(x_0(I)) = \sum_{j \geq 1} C(i_s - j, j)x_0(i_1, \ldots, i_{s-1}, i_{s-j}) \otimes \lambda_{j-1}$$
$$+ \sum_{j \geq 1} \sum_{m \geq 1} \sum_{K''=m-1} C(i_s - j, j - 2m)\alpha_K x_0(K', i_{s-j+m}) \otimes \lambda_{j-1}.$$

Also,

$$\begin{aligned}\partial(\lambda_I) &= \lambda_{i_1} \cdots \lambda_{i_{s-1}}(\partial\lambda_{i_s}) + \partial(\lambda_{i_1} \cdots \lambda_{i_{s-1}})\lambda_{i_s} \\ &= \sum_{j \geq 1} C(i_s - j, s_j)\lambda_{i_1} \cdots \lambda_{i_{s-1}}\lambda_{i_{s-j}}\lambda_{j-1} \\ &= \sum \alpha_K \lambda_{K'}\lambda_{K''}\lambda_{i_s}.\end{aligned}$$

We next show that to obtain the expression for $\delta(x_0(I))$, we must delete from this sum those K for which length $(K'') > 1$, and that the expression for $\delta(x_0(I))$ then becomes the sum $\sum^*$ of (5.2). For this, we make use of the following, which is easily established inductively by length.

SUBLEMMA (5.3). *Let $K = (k_1, \ldots, k_s)$ be allowable, and $i > 2k_s$. Suppose that $\lambda_K\lambda_i = \sum \lambda_J\lambda_J$, J allowable, $\lambda_J \in Z_2$. Let $K = (K' \mid K'')$, and each $J = (J' \mid J'')$. Then for those J with $l(J'') < l(K'')$, λ_J must be* 0.

By means of this sublemma, the expression for $\partial(\lambda_I)$ becomes

$$\begin{aligned}\partial(\lambda_I) &= \sum_{j \geq 1} C(i_s - j, s_j)\lambda_{i_1} \cdots \lambda_{i_{s-1}}\lambda_{i_{s-j}}\lambda_{j-1} \\ &\quad + \sum_{m \geq 1} \sum_{K''=m-1} \alpha_K \lambda_{K'}\lambda_{m-1}\lambda_{i_s} + \sum_{l(K'')>1} \alpha_K \lambda_{K'}\lambda_{K''}\lambda_{i_s} \\ &= \sum_{j \geq 1} C(i_s - j, j)\lambda_{i_1} \cdots \lambda_{i_{s-1}}\lambda_{i_{s-j}}\lambda_{j-1} \\ &\quad + \sum_{m \geq 1} \sum_{K''=m-1} \sum_{j \geq 0} C(i_s - j, j - 2m)\alpha_K \lambda_K \lambda_{i_s-j+m}\lambda_{j-1} + \sum_{l(K'')>1} \cdots .\end{aligned}$$

Thus the formula (5.2) has been established for the special case when $l(I'') = 0$. The general case follows easily by further use of the sublemma.

Continuation. Let $M(\Omega^n S^{n+k})$ be the vector space as described in Section 3. Let $\Lambda(n + k)$ be the subspace of Λ spanned by allowable $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_s}$ with $i_1 < n + k$. (Thus $\Lambda(n + k) \cong E_1(S^{n+k})$ as in [4; (5.4)].) There is an isomorphism

$$\theta\colon M(\Omega^n S^{n+k}) \hat{\otimes} \Lambda \cong \Lambda(n + k)$$

where $\theta(x_k(I') \otimes \lambda_{I''}) = \lambda_{I'}\lambda_{I''}$. As before, to see that θ is an isomorphism, observe that for each allowable sequence $I = (i_1, \ldots, i_s)$, there is a unique index q for which the sequences

$$I' = (i_1, \ldots, i_q) \quad \text{and} \quad I'' = (i_{q+1}, \ldots, i_s)$$

satisfy excess $(I') \geq k$ and $i_{q+1} < k + \deg(I')$. We write $I = (I' \mid_k I'')$ to stand for this decomposition.

LEMMA (5.4). *Let I be allowable with $I = (I' \mid_k I'')$. Suppose that $\partial\lambda_I = \sum \alpha_J\lambda_J$, J allowable, $\alpha_J \in Z_2$. Then in $M(\Omega^n S^{n+k}) \hat{\otimes} \Lambda$,*

$$\delta(x_k(I') \otimes \lambda_{I''}) = {\textstyle\sum}^* \alpha_J x_k(J') \otimes \lambda_{J''}$$

where the sum $\sum^*$ is taken for those allowable sequences $J = (J' \mid_k J'')$ for which length (J'') = length $(I'') + 1$.

The proof is similar to the proof of (5.1).

6. Calculations in $E_2(\Omega^n S^{n+k})$

The results of Section 5 show that $E_2(\Omega^n S^{n+k})$ may be calculated as the homology of the complex $M(\Omega^n S^{n+k}) \hat{\otimes} \Lambda$. We shall use (5.1), (5.4), and the EHP sequence (4.2) inductively to calculate $E_2^{0,*}(\Omega^n S^{n+k})$, and also to make some partial calculations of $E_2^{1,*}(\Omega^n S^{n+k})$ and $E_2^{2,*}(\Omega^n S^{n+k})$.

To simplify notation, the term $x_k(Q^{I'}) \otimes \lambda_{I''}$ will sometimes be written as $x_k(I') \otimes (I'')$, or $x_k(I)$, where $I = (I' \mid_k I'')$. For such a term, let the *width* be the length of I', and the *filtration degree* be the length of I''. As the differential preserves the width, the homology of the complex becomes tri-graded; we consider elements homogeneous in width, filtration, and dimension. The basis $\{x_k(I') \otimes I''\}$ is ordered according to the sequences $I = (I', I'')$, lexicographically from the left. If

$$x = \sum \alpha_I x_k(I' \mid_k I'')$$

where the $I = (I' \mid_k I'')$ vary over allowable sequences (of a fixed length), and the $\alpha_I \in Z_2$, then the greatest term for which $\alpha_I \neq 0$ is called the leading term of x, and the other terms are called lower terms.

Facts about Λ. (6.1). Let i and j be nonnegative integers, with dyadic expansions $i = \sum_\nu a_\nu 2^\nu$, $j = \sum b_\nu 2^\nu$ respectively. The binomial coefficient $C(i, j) = i!/j!\,(i-j)!$ satisfies

$$C(i, j) \equiv \prod_{\nu \geq 0} C(a_\nu, b_\nu) \mod 2$$

and is nonzero mod 2 if and only if $a_\nu \geq b_\nu$ for all ν.

For $i = \sum_{\nu \geq 0} a_\nu 2^\nu$, let $\rho(= \rho(i))$ be the least index ν for which $a_\nu = 0$. Then $i = 2^\rho - 1 + 2^{\rho+1}N$. For any j, the binomial coefficient $C(i - j, j)$ mod 2 can be nonzero only if $j \equiv -1 \mod 2^{\rho(i)}$. Thus for the element $\lambda(i) = \lambda_i$ in Λ, its differential, given by (Section 4, (iii)), is a sum of terms of the form

$(i - 2^\rho n, 2^\rho n - 1)$, and the leading term is $(i - 2^\rho, 2^\rho - 1)$, unless $i = 2^\rho - 1$, in which case $\partial(i) = 0$.

The relations of (4.1) imply that if $(i_1, \ldots, i_q)$ is a sequence of nonnegative integers with $i_j \equiv -1 \bmod 2^\theta$, for each $j = 1, 2, \ldots, q$, then the allowable expression for $\partial(i_1, \ldots, i_q)$ is a sum of terms $(m_1, \ldots, m_q)$ with also $m_j \equiv -1 \bmod 2^\theta$ for each $j = 1, 2, \ldots, q$. Similarly, the allowable expression for $\partial(i_1, \ldots, i_q)$ will be a sum of terms $(k_1, \ldots, k_{q+1})$, with $k_j \equiv -1 \bmod 2^\theta$, for each $j = 1, 2, \ldots, q + 1$.

We notice that if $I = (i_1, \ldots, i_q)$ is an allowable sequence of excess ≥ 0, then the allowable expression for $\partial(I)$ will be a sum of terms $(k_1, \ldots, k_{q+1})$, each of negative excess, and with excess $(k_1, \ldots, k_q) \leq$ excess $(i_1, \ldots, i_q)$.

LEMMA (6.2). *Let $I = (i_1, \ldots, i_q)$ be an allowable sequence of excess ≥ 0 which also satisfies $2i_j - i_{j+1} < 2^{\rho(i_j)}$ for each $j = 1, 2, \ldots, q - 1$. Then the allowable expression for $\partial(I)$ is a sum of sequences $(k_1, \ldots, k_q, k_{q+1})$, each of which satisfies*

$$\text{excess } (k_1, \ldots, k_q) \leq \text{excess } (i_1, \ldots, i_q) - 2^{\rho(i_q)}.$$

Proof. It is sufficient to show that $(\partial i_1, i_2, \ldots, i_q)$ is a sum of such sequences, and this will be done by induction on the length q. For $q = 1$, it is true because the leading term of $\partial(i)$ is $(i - 2^{\rho(i)}, 2^{\rho(i)} - 1)$. Assume inductively the above statement for lengths $\leq q - 1$, and let $(i_1, \ldots, i_q)$ satisfy the hypotheses. The hypotheses imply that $\rho(i_1) \geq \rho(i_2) \geq \cdots \geq \rho(i_q)$. The inductive assumption implies that the allowable expression for $(\partial i_1, i_2, \ldots, i_{q-1})$ is a sum of sequences $(m_1, \ldots, m_q)$, each of which satisfies excess $(m_1, \ldots, m_{q-1}) \leq$ excess $(i_1, \ldots, i_{q-1}) - 2^{\rho(i_{q-1})}$.

From this, it follows that

$$2m_{q-1} < 2i_{q-1} - 2^{\rho(i_{q-1})} - m_q$$

and hence, using the hypothesis that $2i_{q-1} - i_q < 2^{\rho(i_{q-1})}$, we find that

$$2m_{q-1} < i_q - m_q - 1.$$

Let (m_q, i_q) be expressed as a sum of allowable sequences of the form (n_q, n_{q+1}). Then each n_q must be of the form $i_q - m_q - 1 - 2^{\rho(i_q)}t$, with $t \geq 0$. Thus the allowable expression for $(m_1, \ldots, m_{q-1}, n_q, n_{q+1})$ is either 0, or is a sum of sequences $(k_1, \ldots, k_{q+1})$, with $k_q \leq i_q - m_q - 1 - 2^{\rho(i_q)}$. Hence excess $(k_1, \ldots, k_q) \leq$ excess $(i_1, \ldots, i_q) - 2^{\rho(i_q)}$, and the lemma is proven.

THEOREM (6.3). *A basis for $E_2^{0,*}(\Omega^n S^{n+k})$ consists of those $x_k(i_1, \ldots, i_q)$ which satisfy*

(1) $k \leq i_1 < n + k$,

(2) $0 \leq e(I) - k < 2^{\rho(i_q)}$,

(3) $0 \leq 2i_j - i_{j+1} < 2^{\rho(i_j)}$ *for* $j = 1, 2, \ldots, q - 1$.

Proof. By induction on the width q. For $q = 1$, we are dealing with elements of the form $x_k(i)$, with $k \leq i < n + k$. Then

$$\delta(x_k(i)) = x_k(i - 2^{\rho(i)}, 2^{\rho(i)} - 1) + \text{lower terms}.$$

Thus $x_k(i)$ is a cycle if and only if $i - 2^{\rho(i)} < k$; that is, if and only if $i - k < 2^{\rho(i)}$.

Assume inductively the theorem for widths $q - 1$. Consider the EHP sequence (4.2), with n decreased by 1:

$$0 \longrightarrow E_2^{0,*}(\Omega^{n-1}S^{n+k-1}) \xrightarrow{\iota_*} E_2^{0,*}(\Omega^n S^{n+k})$$
$$\xrightarrow{\eta_*} E_2^{0,*}(\Omega^n S^{2n+2k-1}) \xrightarrow{\partial} E_2^{1,*}(\Omega^{n-1}S^{n+k-1}) \longrightarrow \cdots.$$

The set of elements $x_k(i_1, \ldots, i_q)$ which satisfy (1), (2), (3) for $i_1 < n + k - 1$ form a basis of $E_2^{0,*}(\Omega^{n-1}S^{n+k-1})$ inductively on $n - 1$, and are mapped monomorphically by ι_*. To this set we must adjoin a basis for $\eta_*^{-1}(\ker \partial)$. Let $x = x_{k+i}(i_2, \ldots, i_q)$ be a basis element of $E_2^{0,*}(\Omega^n S^{2i+1})$, where $i = i_1 = n + k - 1$. Then

$$\partial x = x_k(\partial i, i_2, \ldots, i_q).$$

If $2i - i_2 < 2^{\rho(i)}$, then Lemma (6.2) and the inductive assumption imply that each nonzero term in ∂x has filtration ≥ 2; hence $\partial x = 0$ in $E_2^{0,*}(\Omega^{n-1}S^{n+k-1})$. Take $\eta_*^{-1}(x)$ to be $x_k(i_1, \ldots, i_q)$, which satisfies (1), (2), (3) as desired. On the other hand, as $x = x_{k+i}(i_2, \ldots, i_q)$ varies over the basis of $E_2^{0,*}(\Omega^n S^{i+1})$, with $2i - i_2 \geq 2^{\rho(i)}$, the leading terms of ∂x, namely

$$x_k(i - 2^{\rho(i)}, i_2 - 2^{\rho(i)}, \ldots, i_q - 2^{\rho(i)+q-2}) \otimes (2^{\rho(i)+q-1} - 1)$$

are nonzero and distinct, even as n varies. Thus no sum of such x can be in $\ker \partial$, which shows that a basis of $E_2^{0,*}(\Omega^n S^{n+k})$ is as described.

Remark (6.4). If $x_0(i_1, \ldots, i_q)$ is a basis element of $E_2^{0,t}(\Omega^n S^n)$ of dimension $t = i_1 + \cdots + i_q$, there is a family of elements of the form

$$x_0(i_1, \ldots, i_q, t, 2t, \ldots, 2^m t)$$

each in $E_0^{0,2^{m+1}t}(\Omega^n S^n)$. Some typical generators of these families are $x_0(1)$ (the rest of the family is $x_0(1, 1)$, $x_0(1, 1, 2)$, $x_0(1, 1, 2, 4), \ldots$), $x_0(3)$, $x_0(7), \ldots$, $x_0(2^\theta - 1), \ldots$, $x_0(3, 5, 9)$, $x_0(7, 9, 17)$, $x_0(7, 11, 19)$, $x_0(5, 9, 17, 33, 65), \ldots$, $x_0(15, 27, 51, 99, 195), \ldots$.

Proposition (6.5). *Let $x = x_0(i_1, \ldots, i_q)$ be a basis element of $E_2^{0,*}(\Omega^n S^n)$ which has none of the following forms*:

(1) $x_0(2^\theta - 1)$,
(2) $x_0(2^\theta - 1, 2^\theta - 1)$,
(3) $x_0(i_1, \ldots, i_q)$, *where* excess $= 0$, *and* i_q *is even.*

Then there is a nonzero class $y \in E_2^{2,*}(\Omega^n S^n)$, with $d^2 x = y$.

Proof. For $x = x_0(i_1, \ldots, i_q)$, consider its ancestors in the EHP-sequence. Namely, for each $j = 1, 2, \ldots, q-1$, let $z^{(j)} = x_k(i_j, \ldots, i_q)$, where $k = i_1 + \cdots + i_{q-1}$. We shall show inductively on the width $(q - j + 1)$, that if x has not one of the excluded forms, then $d^2 z^{(j)} = y^{(j)}$ is a nonzero class in $E_2^{2,*}(\Omega^n S^n)$.

For width one, we are dealing with $z^{(q)} = x_k(i)$, where $i_q = i$ is of the form $2^\rho - 1 + 2^{\rho+1}N$, and $k \leq i < k + n$. The element $y = x_k \otimes \partial(i)$ in $E_1^{2,k+i+1}(\Omega^n S^{n+k})$ is a d^1-cycle which is not a d^1-boundary because $E^{1,k+i+1}(\Omega^n S^{n+k})$ contains no terms of width zero (the candidate $x_k \otimes \lambda_i$ is not present as $k \leq i$; indeed, $x_k Q^i$ appears as the homology class under consideration). We consider the EHP-sequence

$$\cdots \pi_{2i}(\Omega S^{2i+1}) \xrightarrow{P} \pi_{2i-1}(S^i) \xrightarrow{E} \pi_{2i-1}(\Omega S^{i+1}) \xrightarrow{H} \cdots.$$

The Whitehead product $\langle x_i, x_i \rangle$ in $\pi_{2i-1}(S^i)$ is represented by $x_i \otimes (\partial\lambda_i)$ in $E_2^{2,*}(S^i)$, as in [7, p. 198]. As the Whitehead product suspends to zero in $\pi_{2i-1}(\Omega S^{i+1})$, we must have

$$d^2(x_i Q^i) = x_i \otimes (\partial\lambda_i)$$

in $E_2^{*,*}(\Omega S^{i+1})$. After looping $i - k$ times, we must also have

$$d^2(x_k Q^i) = x_k \otimes (\partial\lambda_i) + \text{lower terms}$$

which is nonzero in $E_2^{*,*}(\Omega^{i-k+1} S^{i+1})$.

Assume inductively for widths $\leq q - j + 1$, that if $z^{(j)} = x_k(i_j, \ldots, i_q)$ is a basis element of $E_2^{0,*}(\Omega^n S^{n+k})$ not of the excluded forms, then

$$d^2 z^{(j)} = x_k(i_j, \ldots, i_{q-1}) \otimes (\partial i_q) + \text{lower terms}$$

is a nonzero class $y^{(j)}$ in $E_2^{2,*}(\Omega^n S^{n+k})$. Let

$$z^{(j-1)} = x_{k-i}(i, i_j, \ldots, i_q)$$

be a basis element of $E_2^{0,*}(\Omega^{2i+1-k} S^{i+1})$, with $\eta_*(z^{(j-1)}) = z^{(j)}$ in the homomorphism

$$\eta_*\colon E_2^{0,*}(\Omega^{2i+1-k} S^{i+1}) \to E_2^{0,*}(\Omega^{2i+1-k} S^{2i+1}).$$

As η_* commutes with the differentials,

$$d^2 z^{(j-1)} = x_k(i, i_j, \ldots, i_q) \otimes (\partial i_q) + \text{lower terms } \pi$$

which is some nonzero element $y^{(j-1)}$ in $E_2^{2,*}(\Omega^{2i+1-k} S^{i+1})$. It is straightforward to verify that $y^{(j-1)}$ does not suspend to zero in any of the

$$E_2^{2,*}(\Omega^n S^{n+k-i})$$

for $n \geq 2i + 1 - k$, and the proposition is proved.

7. Spherical classes in $H_*(\Omega^\infty S^\infty)$

From the discussion of Section 4, we see that $E_\infty^{0,t}(X)$ is isomorphic to the group of spherical classes in $H_*(X; Z_2)$. Recall our notation

$$x_0(i_1, \ldots, i_q) = ([1]Q^{i_1} \cdot [-2])Q^{i_2} \cdots Q^{i_q}$$

in $H_*(\Omega^\infty S^\infty)$.

THEOREM (7.1). *The only possibilities for spherical classes in $H_*(\Omega^\infty S^\infty; Z_2)$ are $x_0(1)$, $x_0(3)$, $x_0(2^n - 1)$, and $x_0(2^n - 1, 2^n - 1)$ for $n = 1, 2, 3, \ldots$.*

The proof will be completed at the end of this section.

Remark. This recovers Browder's result [9] that dimensions other than $2(2^n - 1)$ cannot contain a framed manifold of Kervaire invariant one, because such a manifold would, by the Pontrjagin-Thom construction, give rise to a spherical class in $H_*(SG) \cong H_*(\Omega^\infty S^\infty)$, which would be nonzero [16]. The classes $x_0(2^n - 1, 2^n - 1)$ plus decomposables are spherical if and only if there is a manifold of Kervaire invariant one in dimension $2(2^n - 1)$. This is the case in dimensions 2, 6, 14, 30, and 62 (Barrat-Mahowald). The remaining dimensions $2(2^n - 1)$, $n \geq 6$, are undecided.

Towers. An element α in $E_r^{s,t}(X)$ is said to generate a tower if the elements $\alpha\lambda_0^n$ are nonzero for all $n \geq 0$, and α is not of the form $\beta\lambda_0$. The set $\{\alpha\lambda_0^n, n \geq 0\}$ is called a tower.

PROPOSITION (7.2). *The only towers in $E_2^{s,t}(\Omega^\infty S^\infty)$ occur in dimensions congruent to -1 or to 0 modulo 4.*

Proof. We use the method of [3] to locate the towers in $\mathrm{Ext}_{MA}(Z_2, M)$, for $M = M(\Omega^\infty S^\infty)$. The tower detector is the complex

$$T^s(M) = \begin{cases} M \otimes \lambda_0^s, & s = 0, 1 \\ M \otimes \lambda_0^s \oplus M_{2k} \otimes \lambda_{2k-1}\lambda_0^{s-1}, & s \geq 2. \end{cases}$$

$$\delta(x \otimes \lambda_0^s) = \begin{cases} xSq^1 \otimes \lambda_0^{s+1} + xSq^{2k} \otimes \lambda_{2k-1}\lambda_0^s & \text{for } x \in M_{4k},\ s \geq 1 \\ xSq^1 \otimes \lambda_0^{s+1} & \text{otherwise.} \end{cases}$$

$$\delta(x \otimes \lambda_{2k-1}\lambda_0^s) = 0.$$

The allowable monomial basis of Λ gives a projection of complexes

$$\gamma\colon M \hat{\otimes} \Lambda \to T(M).$$

In [3], it is shown that $(\ker \gamma, \delta)$ is a chain complex whose homology has no towers, so the towers in $H^*(M \hat{\otimes} \Lambda)$ correspond to those in $H^*(T(M))$.

To find the towers in $E_2(\Omega^\infty S^\infty)$, we consider $T(M(\Omega^\infty S^\infty))$. If $(i_1, \ldots, i_q)$ is allowable of excess ≥ 0, and i_q is odd, then $(i_1, \ldots, i_{q+1})$ is also allowable and

$$\delta(x_0(i_1, \ldots, i_{q+1}) \otimes \lambda_0^s) = x_0(i_1, \ldots, i_q) \otimes \lambda_0^{s+1} + \text{possibly another term.}$$

Thus when i_q is odd, neither $x_0(i_1, \ldots, i_q)$ nor $x_0(i_1, \ldots, i_{q+1})$ generate towers. The remaining elements of filtration zero are the $x_0(i_1, \ldots, i_q)$ with excess 0, and i_q even, which must have dimension $\equiv 0 \bmod 4$. In filtration one, we have elements $y_{2k} \otimes \lambda_{2k-1}$ for $y_{2k} \in M(\Omega^\infty S^\infty)_{2k}$ which occur in dimensions $\equiv -1 \bmod 4$. In particular, the elements $x_0(i_1, \ldots, i_q)$ described in (6.3) generate towers if $i_{q-1} = i_1 + \cdots + i_{q-2}$ and $2i_{q-1} = i_q$, and not otherwise.

For each $n \geq 1$, the groups $E_\infty^{s,t}(\Omega^\infty S^\infty)$, with $s + t = n$, are finite, and only finitely many are nonzero; they are the quotients of a filtration of $\pi_n(S)$, the stable n-stem. As only a tower can kill another tower by a differential d^r, the towers of $E_2(\Omega^\infty S^\infty)$ must be paired by the differentials. Thus, each tower generator α of dimension $4k$ and filtration 0, must have a differential $d^r\alpha = \beta \neq 0$. In particular, the elements

$$x_0(2^n - 1, 2^n - 1, \ldots, 2^q(2^n - 1))$$

for $q \geq 1$, do not persist to $E_\infty(\Omega^\infty S^\infty)$. The elements $x_0(2^n - 1)$, $n \geq 4$ are shown not to be spherical by Adams [1]. After excluding the elements accounted for by (6.5), this leaves for possible spherical classes in $H_*(\Omega^\infty S^\infty)$ only the Hopf classes $x_0(1)$, $x_0(3)$, $x_0(7)$, and the classes $x_0(2^n - 1, 2^n - 1)$, for $n = 1, 2, \ldots$.

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