ISOMETRIES INDUCED BY COMPOSITION OPERATORS AND INVARIANT SUBSPACES[1]

BY

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1. In this note, we consider some relations between some subspaces of $H^p(D)$ invariant under multiplication by z and some classes of isometries induced by linear fractional transformations mapping D onto D (l.f.t.). Here $D = \{|z| < 1\}$ and $H^p(D)$, $\infty > p \geq 1$, denotes the standard Hardy class of holomorphic functions. Given a l.f.t. ϕ, let C_ϕ and V_ϕ be defined on H^p by $C_\phi f = f \circ \phi$ and $V_\phi f = (\phi')^{1/p} C_\phi f$. (Note that the definition of V_ϕ depends on its domain H^p.) C_ϕ is a standard composition operator and is well known to be a bounded linear map of H^p onto H^p (see [5] for a discussion of composition operators). V_ϕ is clearly an isometry of H^p onto H^p, and further, F. Forelli has shown that for $p \neq 2$, every isometry of H^p onto H^p has the form bV_ϕ for some l.f.t. ϕ, where $b \in \mathbf{C}$, $|b| = 1$ [4]. We consider here the case where ϕ has a fixed point on $T = \{|z| = 1\}$, and for simplicity, we will assume $\phi(1) = 1$. Our main results are for H^2; in Theorem 1 we show that V_ϕ is a bilateral shift, and in Theorem 2 we show that a subcollection of $\{V_\phi\}$ generates a reflexive algebra which is related to a reflexive-type property of some other algebras.

2. For $c > 0$, $t \in \mathbf{R}$, let

$$\alpha_{c,t} = [t + i(c - 1)][t + i(c + 1)]^{-1},$$

and let $\phi_{c,t}(z) = (1 - \bar{\alpha})(1 - \alpha)^{-1}(z - \alpha)(1 - \bar{\alpha}z)^{-1}$ be the unique l.f.t. such that $\phi_{c,t}(\alpha_{c,t}) = 0$, $\phi_{c,t}(1) = 1$. Let $C_{c,t}$ and $V_{c,t}$ denote the corresponding maps induced by $\phi_{c,t}$, and for $r > 0$, let

$$\Delta_r(z) = \exp\,[-r(1 + z)(1 - z)^{-1}].$$

We note that by Beurling's theorem, $\{\Delta_r(z)H^p\}$ forms a decreasing family of invariant subspaces of H^p with $\bigcap_r \Delta_r H^p = \{0\}$.

LEMMA 1. *For $\alpha \in D$, there exists a unique $c > 0$, $t \in \mathbf{R}$ such that $\alpha = \alpha_{c,t}$. For $r > 0$, $V_{c,t}(\Delta_r H^p) = (\Delta_{rc^{-1}} H^p)$.*

Proof. Consider $\psi: \prod^+ \to D$ by $\psi(w) = (w - 1)(w + 1)^{-1}$, where $\prod^+ = \{\text{Re } w > 0\}$. Then $\text{Re } w = c$ iff $\psi(w) = \alpha_{c,t}$, and, in fact,

$$\alpha_{c,t} \in \{z \mid |z - c(c + 1)^{-1}| = (c + 1)^{-1}\},$$

the circle in D of radius $(c + 1)^{-1}$ tangent to T at 1. A direct computation

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shows that $\Delta_r(\phi_{c,t}(z)) = \Delta_{rc^{-1}}(z) \cdot \exp(itc^{-1})$; since $V_\phi(H^p) = H^p$, the lemma follows. This can also be seen by observing that

$$\phi'_{c,t}(1) = (1 - |\alpha|^2)[(1 - \alpha)(1 - \bar{\alpha})]^{-1} = c.$$

THEOREM 1. *$V_{c,t}\colon H^2 \to H^2$ is unitarily equivalent to a bilateral shift of infinite multiplicity.*

Proof. Our proof will also give a representation for $V_{c,t}\colon H^p \to H^p$. Consider

$$T_{c,t}\colon H^p(\textstyle\prod^+) \to H^p(\textstyle\prod^+)$$

by $(T_{c,t}f)(w) = c^{-1/p}f((w - it)c^{-1})$, and

$$U\colon H^p(\textstyle\prod^+) \to H^p(D)$$

by $(Uf)(z) = 2^{1/p}(1 - z)^{-2/p}f((1 + z)(1 - z)^{-1})$. Then $T_{c,t}$ and U are unitary, and

$$(U^*f)(w) = 2^{1/p}(w + 1)^{-2/p}f((w - 1)(w + 1)^{-1}).$$

We compute

$$\begin{aligned}
&(UT_{c,t}U^*f)(z)\\
&\quad = U(2^{1/p}c^{1/p}(w - it + c)^{-2/p}f((w - it - c)(w - it + c)^{-1}))\\
&\quad = 2^{2/p}c^{1/p}(z(1 - c + it) + 1 + c - it)^{-2/p}\\
&\qquad \times f([z(1 + c + it) + 1 - c - it][z(1 - c + it) + 1 + c - it])\\
&\quad = (4c(c + 1 - it)^{-2}(1 - \bar{\alpha}_{c,t}z)^{-2})^{1/p}f(\phi_{c,t}(z))\\
&\quad = [\phi'_{c,t}(z)]^{1/p}f(\phi_{c,t}(z)) = V_{c,t}(z).
\end{aligned}$$

When $p = 2$, we take Fourier transforms and we get $V_{c,t}$ unitarily equivalent to $S_{c,t}$ on $L^2(0, \infty)$, where

$$(S_{c,t}f)(x) = c^{1/2}e^{-itx}f(cx).$$

Clearly, $L^2(1, c) \subset L^2(0, \infty)$ is a complete wandering subspace for $S_{c,t}$, so $V_{c,t}$ is a bilateral shift of infinite multiplicity (see [2] for basic facts about shifts).

COROLLARY 1. *The spectrum of $V_{c,t}$ is the whole unit circle T.*

For any algebra of operators $\mathscr{A}$, Lat $(\mathscr{A})$ denotes the lattice of closed invariant subspaces of $\mathscr{A}$, and for a lattice of invariant subspaces $\mathscr{L}$, Alg $(\mathscr{L})$ denotes the algebra of all operators leaving invariant all elements of $\mathscr{L}$. An algebra $\mathscr{A}$ is said to be reflexive if $\mathscr{A} = \text{Alg}(\text{Lat}(\mathscr{A}))$. Let $\mathscr{A}$ be the weakly closed algebra generated by $\{V_{1,t}\}_{t\in\mathbf{R}}$.

THEOREM 2. *Fix $c > 0$ and let Φ be a bounded linear map on H^2. If*

$$\Phi((\Delta_r H^2)^\perp \ominus (\Delta_s H^2)^\perp) \subset ((\Delta_{rc^{-1}} H^2)^\perp \ominus (\Delta_{sc^{-1}} H^2)^\perp)$$

for all $0 < s < r$, then for any t, $\Phi \in V_{c,t} \circ \mathscr{A} = \{V_{c,t} \circ A \mid A \in \mathscr{A}\}$.

Proof. By Lemma 1 and the representation obtained in the proof of Theorem 1, we have the spectral representation $V_{1,t} = \int_0^\infty e^{it\lambda}\, dP_\lambda$, where P_λ is the projection of H^2 onto $(\Delta_\lambda H^2)^\perp$. This yields a unitary $\mathscr{F}\colon L^2(0, \infty) \to H^2$ such that $\mathscr{F}^{-1} V_{1,t} \mathscr{F}$ is multiplication by $e^{it\lambda}$ and

$$\mathscr{F}(L^2(s, r)) = (\Delta_r H^2)^\perp \ominus (\Delta_s H^2)^\perp.$$

(We note that $\mathscr{F}$ is given by $(\mathscr{F}a)(z) = \sqrt{2} \int_0^\infty a(\lambda)\, \Delta_\lambda(z)(1 - z)^{-1}\, d\lambda$, which is the map used in [1, p. 195] and [6]. We can also obtain the above spectral representation directly from this by a simple computation.) Hence, $\mathscr{F}$ produces a unitary equivalence between $\mathscr{A}$ and $\mathscr{M}$, the algebra of bounded multiplication operators on $L^2(0, \infty)$. Clearly, Lat $(\mathscr{M}) = \{L^2(E)\}$, for $E \subset (0, \infty)$ measurable, where

$$L^2(E) = \{f \in L^2(0, \infty) \mid f(x) = 0 \text{ a.e. } x \notin E\},$$

and this lattice is generated by

$$\{L^2(s, r) \mid 0 < s < r\} = \{\mathscr{F}^{-1}((\Delta_r H^2)^\perp \ominus (\Delta_s H^2)^\perp) \mid 0 < s < r\}.$$

Thus, if for all $0 < s < r$, $(\Delta_r H^2)^\perp \ominus (\Delta_s H^2)^\perp$ is invariant for Φ, all $L^2(s, r)$ are invariant for $\mathscr{F}^{-1} \circ \Phi \circ \mathscr{F}$, which must therefore be a multiplication operator. Hence, $\Phi \in \mathscr{A}$ which proves the theorem for the case $c = 1$. If

$$\Phi((\Delta_r H^2)^\perp \ominus (\Delta_s H^2)^\perp) \subset ((\Delta_{rc^{-1}} H^2)^\perp \ominus (\Delta_{sc^{-1}} H^2)^\perp)$$

for some $c > 0$, $0 < s < r$, then choose $u \in \mathbf{R}$ and apply (using Lemma 1) the above case to the map $V_{c^{-1},u} \circ \Phi$. This gives $V_{c^{-1},u} \circ \Phi \in \mathscr{A}$, and since $V_{c,t} \circ V_{c^{-1},u} = I$ where $t = -cu$, we get $\Phi \in V_{c,t} \circ \mathscr{A}$ and the theorem is proved.

COROLLARY 2. (i) *$V_{c,t}$ induces a one-parameter group given by*

$$(V_{1,t})^s = \int e^{ist\lambda}\, dP_\lambda \quad \textit{if} \quad c = 1,$$

$$(V_{c,t})^s = V_{c^s,u} \quad \textit{where} \quad u = t(1 - c^s)(1 - c)^{-1} \quad \textit{if} \quad c \neq 1.$$

(ii) *$\mathscr{A}$ is a reflexive algebra.*

Proof. For $c = 1$, (i) was shown in the proof of the theorem and for $c \neq 1$, (i) follows from the group structure of the l.f.t.'s; (ii) is a weaker statement than the theorem.

3. If ϕ is a l.f.t. with $\phi(e^{i\theta}) = e^{i\theta}$, $e^{i\theta} \neq 1$, then analogous results hold using $\tilde{\Delta}_r(z) = \exp[-r(e^{i\theta} + z)(e^{i\theta} - z)^{-1}]$ in place of $\Delta_r(z)$. This case does

not exclude the case $\phi(1) = 1$, since there exist l.f.t.'s (hyperbolic) fixing two points on T; a l.f.t. with a unique fixed point on T is called parabolic. Since

$$\phi_{c,t}(z) = [(it - (c+1))z + (-it + c - 1)][-it - (c+1) + (it + c - 1)z]^{-1},$$

we see immediately (see [3] or [5]) that $\phi_{c,t}$ is parabolic iff $c = 1$. If $c \neq 1$, then $\phi_{c,t}(1) = 1$ and $\phi_{c,t}(\gamma) = \gamma$ where

$$\gamma = (t + i(c-1))(t - i(c-1))^{-1}.$$

A l.f.t. without a fixed point on T is called elliptic, but our results do not apply in this case.

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