THE COMPONENT OF THE ORIGIN IN THE NEVANLINNA CLASS

BY

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1. Introduction

The Nevanlinna class N is the algebra of functions f analytic in the open unit disc U whose characteristic function

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{it})| \, dt$$

is bounded for $0 \leq r < 1$ where $\log^+ x = \max\{\log x, 0\}$. In [6], J. H. Shapiro and A. L. Shields define a metric d on N by

$$d(f, g) = \lim_{r \to 1} \tfrac{1}{2}\pi \int_0^{2\pi} \log(1 + |f(re^{it}) - g(re^{it})|) \, dt. \tag{1.1}$$

Although the metric d is both complete and translation invariant, they also show that N is not connected and scalar multiplication is not continuous in the scalar variable. Now if $f \in N$, then

$$\lim_{r \to 1} f(re^{it}) = f(e^{it}) \tag{1.2}$$

where the limit holds for almost every e^{it} in the unit circle T and $\log |f(e^{it})|$ is integrable on T [1, p. 17]. N^+ is the class of functions $f \in N$ such that

$$\lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{it})| \, dt = \int_0^{2\pi} \log^+ |f(e^{it})| \, dt$$

(see [1, Section 2.5]). N^+ may alternately be defined as the set of $f \in N$ such that

$$\begin{aligned} d(f, 0) &= \lim_{r \to 1} \tfrac{1}{2}\pi \int_0^{2\pi} \log(1 + |f(re^{it})|) \, dt \\ &= \tfrac{1}{2}\pi \int_0^{2\pi} \log(1 + |f(e^{it})|) \, dt \end{aligned} \tag{1.3}$$

(see [6, Proposition 1.2]). In [6], J. H. Shapiro and A. L. Shields pose the problem of characterizing the component of the origin in N (and more generally in $N(U^n)$). They show in Corollary 2 of Theorem 3.1 that every finite dimensional subspace of N/N^+ has the discrete topology. This fact suggests that quite possibly the space N/N^+ is totally disconnected and equivalently N^+ is the component of the origin in N. We shall prove that this is false and, in particular,

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we shall characterize the component of the origin in N. It will further be shown that for a metric ρ (equivalent to the metric d) the open ρ-balls in the component of the origin are connected. Thus the component of the origin is locally connected.

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2. Preliminaries

From our point of view the most important feature of N is the canonical factorization property. A function $f \in N$ can be factored uniquely as follows [1, p. 25]:

$$f(z) = B(z) \frac{S_{\mu_1}(z)}{S_{\mu_2}(z)} F(z) \tag{2.1}$$

where $B(z)$ is the Blaschke product with respect to the zeroes of f, $F(z)$ is an outer function, and $S_{\mu_1}(z)$ and $S_{\mu_1}(z)$ are singular inner functions with respect to the nonnegative singular measures μ_1 and μ_2 which are mutually singular and

$$S_{\mu_j}(z) = \exp\left[\int \frac{z + e^{it}}{z - e^{it}}\, d\mu_j(t)\right], \quad j = 1, 2. \tag{2.2}$$

A function $f \in N$ is in N^+ if and only if $S_{\mu_2}(z) = 1$, i.e., $\mu_2 = 0$ [1, p. 26]. Thus every function $g \in N$ can be written in the form $g = f/S_\mu$ where $f \in N^+$. In particular if $f = BS_\nu F$ and ν is mutually singular with respect to μ we shall say that f/S_μ is in reduced form. If f/S_μ is in reduced form, then we define

$$\left\| \frac{f}{S_\mu} \right\| = d(f, 0) + \mu(T). \tag{2.3}$$

In particular, if $f \in N^+$, $\|f\| = d(f, 0)$ so that $\|\cdot\|$ is an extension of $d(\cdot, 0)$ to N. We now define

$$\rho(f, g) = \|f - g\| \quad \text{for } f, g \in N. \tag{2.4}$$

PROPOSITION 2.1. (1) *If $f \in N^+$ and S_μ is a singular function, then* $\|S_\mu f\| = \|f\|$.

(2) *If $f \in N^+$ and S_μ is a singular inner function, then $\|f/S_\mu\| \leq \|f\| + \mu(T)$.*
(3) *ρ is a translation invariant metric on N.*
(4) *$\rho \geq d \geq \frac{1}{2}\rho$, i.e., the metrics ρ and d are equivalent.*

Proof. (1) follows directly from the definition and the fact that $|S_\mu(e^{it})| = 1$ a.e. on T.

Suppose that $f/S_\mu = g/S_\nu$ where g/S_ν is in reduced form. We have $f = BS_\gamma F$

the canonical factorization of f. Let δ be the infimum of μ and γ. Then $\mu - \delta$ is mutually singular with respect to $\gamma - \delta$ and

$$\frac{f}{S_\mu} = \frac{BS_{\gamma-\delta}S_\delta F}{S_{\mu-\delta}S_\delta} = \frac{BS_{\gamma-\delta}F}{S_{\mu-\delta}}.$$

But then the expression on the right is in canonical form, so $\nu = \mu - \delta$ and $g = BS_{\gamma-\delta}F$. Thus

$$\left\|\frac{f}{S_\mu}\right\| = \|g\| + \nu(T) \leq \|S_\delta g\| + (\nu + \delta)(T) = \|f\| + \mu(T).$$

(3) will follow easily once we show that $\|\cdot\|$ is subadditive. Suppose f/S_μ, $g/S_\nu \in N$ are in reduced form. Then

$$\begin{aligned}\left\|\frac{f}{S_\mu} + \frac{g}{S_\mu}\right\| &= \left\|\frac{fS_\nu + gS_\mu}{S_{\mu+\nu}}\right\| \\ &\leq \|fS_\nu + gS_\mu\| + \mu(T) + \nu(T) \\ &\leq \|fS_\nu\| + \|gS_\mu\| + \mu(T) + \nu(T) \\ &= \left\|\frac{f}{S_\mu}\right\| + \left\|\frac{g}{S_\nu}\right\|.\end{aligned}$$

To prove (4) we show that $\|\cdot\| \geq d(\cdot, 0) \geq \frac{1}{2}\|\cdot\|$. Now suppose that $f/S_\mu \in N$ is in reduced form. Then

$$\begin{aligned}d\left(\frac{f}{S_\mu}, 0\right) &= \lim_{r\to 1} \tfrac{1}{2}\pi \int_0^{2\pi} \log\left(1 + \left|\frac{f}{S_\mu(re^{it})}\right|\right) dt \\ &= \lim_{r\to 1} \tfrac{1}{2}\pi \int_0^{2\pi} \log\left(|S_\mu(re^{it})| + |f(re^{it})|\right) dt \\ &\quad -\tfrac{1}{2}\pi \int_0^{2\pi} \log |S_\mu(re^{it})|\, dt \\ &= \lim_{r\to 1} \tfrac{1}{2}\pi \int_0^{2\pi} \log\left(|S_\mu(re^{it})| + |f(re^{it})|\right) dt \qquad (2.5) \\ &\quad -\log |S_\mu(0)| \\ &\leq \lim_{r\to 1} \tfrac{1}{2}\pi \int_0^{2\pi} \log\left(1 + |f(re^{it})|\right) dt + \mu(T) \\ &= \left\|\frac{f}{S_\mu}\right\|.\end{aligned}$$

In [6, Theorem 3.1], it is shown that $\lim_{a\to 0} d(af/S_\mu, 0) = \mu(T)$ where f/S_μ is in reduced form. It follows that $d(f/S_\mu, 0) \geq \mu(T)$. Observe also that since $|S_\mu(z)| \leq 1$ for all $z \in U$,

$$\lim_{r\to 1} \tfrac{1}{2}\pi \int \log(|S_\mu(re^{it})| + |f(re^{it})|)\, dt$$

$$\geq \lim_{r\to 1} \tfrac{1}{2}\pi \int \log(|S_\mu(re^{it})|(1 + |f(re^{it})|)\, dt$$

$$= \|f\| + \tfrac{1}{2}\pi \int \log|S_\mu(re^{it})|\, dt.$$

Thus by (2.5) $d(f/S_\mu, 0) \geq \|f\|$. But then

$$d\left(\frac{f}{S_\mu}, 0\right) \geq \max\{\|f\|, \mu(T)\} \geq \frac{1}{2}\left\|\frac{f}{S_\mu}\right\|.$$

COROLLARY 2.2. (1) *If* $\langle f_n/S_{\mu_n}\rangle$ *is a sequence of functions in* N *written in reduced form, then* $\lim_{n\to\infty} f_n/S_{\mu_n} = 0$ *if and only if* $\lim_{n\to\infty} f_n = 0$ *and* $\lim_{n\to\infty} \mu_n(T) = 0$.

(2) ρ *is a complete metric.*

In [6, Theorem 2.1] it is shown that if $\omega \in T$, f/S_μ, $g/S_\nu \in N$ are in reduced form, and $\mu(\{\omega\}) > \nu(\{\omega\})$, then there exists a set $V \subset N$ which is both closed and open such that $f/S_\mu \in V$ and $g/S_\nu \in V^c$. If μ is a measure on T, then μ is said to be continuous (or nonatomic) if $\mu(\{\omega\}) = 0$ for every $\omega \in T$. We now let

$$(2.6)\quad K = \left\{\frac{f}{S_\mu} : f \in N^+,\ \mu \text{ is a continuous nonnegative singular measure}\right\}.$$

PROPOSITION 2.3. *K is a closed subgroup of N which contains the component of the origin.*

Proof. By the above remarks K is the intersection of subsets of N which are both closed and open. Thus K is closed and K contains the component of the origin. It is easily verified that K is a group.

In this paper we shall prove that K is the component of the origin in N and, in particular, that every open ball (with metric ρ) is connected. In [7], M. Stoll shows that $K = F^+ \cap N$ thus obtaining a different formulation of K. For a definition of the class of analytic functions F^+ see [8].

3. *K* is the component of the origin

If $C = \langle f_i\rangle$, $1 \leq i \leq n$, is a finite sequence in N with $f = f_1$, $g = f_n$, and for some $\varepsilon > 0$, $\rho(f_i, f_{i+1}) < \varepsilon$, $1 \leq i \leq n - 1$, then C is called an ε-chain from f to g. Throughout this section we will adopt the somewhat abusive convention of identifying a finite sequence with its range. If $E \subset N$ and $\varepsilon > 0$, then we say that E is ε-chainable if for every $f, g \in E$ there exists an ε-chain C from f to g such that $C \subset E$.

Our method of attack will be to show that every ball in K of ρ-radius r is ε-chainable for every $\varepsilon > 0$. We will then use this fact to prove that every such ball is connected. This will show that K is both connected and locally connected. Since we already know that K contains the component of the origin, it will follow that K is the component of the origin.

PROPOSITION 3.1. *For every $\varepsilon > 0$, every open ball in K is ε-chainable.*

Proof. Let $\varepsilon > 0$ and let B be the ball centered at the origin with ρ-radius r. Further let $f/S_\mu \in B$ with f/S_μ in reduced form. Since μ is a continuous measure on the unit circle T, there exists open intervals I_i and closed intervals J_i in T, $1 \leq i \leq n$, such that

(i) $J_i \subset I_t$ for $1 \leq i \leq n$,
(ii) $\bigcup_{i=1}^n J_i = T$, and
(iii) $\mu(I_i) < \varepsilon/2$.

Now let μ_i denote the measure μ restricted to the interval I_i. Then the support of $\mu - \mu_i$ is contained in $T \cap I_i^c$. Thus by a well known theorem [3, p. 68], $S_{\mu-\mu_i}$ is continuous everywhere in the plane except at points in $T \cap I_i^c$. Since $S_{\mu-\mu_i}$ is nonzero in the closed disc off $T \cap I_i^c$, $S_{\mu-\mu_i}$ is bounded away from zero on the sector L_i determined by J_i, i.e., $L_i = \{re^{i\theta}: \theta \in J_i\}$. Thus $|S_{\mu-\mu_i}| \geq \delta_i > 0$ on L_i. Since $U \subset \bigcup_{i=1}^n L_i$, $\sum_{i=1}^n |S_{\mu-\mu_i}| \geq \delta$ where $\delta = \min\ \{\delta^i: 1 \leq i \leq n\}$. But then by the corona theorem [1, p. 202] there exist functions $s_i \in H^\infty$ such that $\sum_{i=1}^n s_i S_{\mu-\mu_i} = 1$. Letting $g_i = fs_i$ we have $g_i \in N^+$ and $\sum_{i=1}^n g_i S_{\mu-\mu_i} = f$. Now let L be the at most n dimensional subspace of N^+ generated by $\{g_i S_{\mu-\mu_i}: 1 \leq i \leq n\}$ and let B_0 denote the ball of radius $r - \mu(T)$ in L. Note that $f \in B_0$. Since B_0 is an open connected set in L, there exist functions $K_1, \ldots, K_m$ such that for each K_j there exists i such that $K_j = \varepsilon_j g_i S_{\mu-\mu_i}$ with ε_i complex such that

(a) $\|K_j\| < \varepsilon/2$,
(b) $\sum_{j=1}^p K_j \in B_0$ for $1 \leq p \leq m$, and
(c) $\sum_{j=1}^m K_j = f$.

The proof of this assertion is precisely the same as the argument used to show that any two points contained in an open connected set in n-dimensional space can be connected by polygonal arcs. We now let f_j and v_j be defined by $v_j = \mu_i$ and $f_j = \varepsilon_j g_i$ where $K_j = \varepsilon_j g_i S_{\mu-\mu_i}$. Thus we have $K_j = f_j S_{\mu-v_j}$ and $\|K_j\| = \|f_j\|$. Hence

(1) $\sum_{j=1}^m f_j/S_{v_j} = \sum_{j=1}^m f_j S_{\mu-v_j}/S_\mu = \sum_{j=1}^m K_j/S_\mu = f/S_\mu$.

(2) $\|\sum_{j=1}^p f_i/S_{v_j}\| = \|\sum_{j=1}^p K_j/S_\mu\| \leq \|\sum_{j=1}^p K_j\| + \mu(T) < r - \mu(T) + \mu(T) = r$.

(3) $\|f_j/S_{v_j}\| \leq \|f_j\| + v_j(T) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Thus the sequence $\langle \sum_{j=1}^{p} f_j/S_{\nu_j} \rangle$ with $1 \leq p \leq m$ is an ε-chain from the origin to f/S_μ. Since any two points in B have ε-chains to the origin, any two points in B have ε-chains connecting them.

LEMMA 3.2. *If B is an open ball of ρ-radius r in K, $\varepsilon > 0$ and $f, g \in B$, then there exists an ε-chain $C = \{f_1, \ldots, f_n\}$ such that $f = f_1$, $g = f_n$, and there exist balls B_i each centered at f_i for $2 \leq i \leq n - 1$ such that $f_{i-1}, f_{i+1} \in B_i$, each B_i has radius less than ε, and $\bigcup_{i=1}^{n} \text{cl}(B_i) \subset B$.*

Proof. Without loss of generality we may assume that B is centered at the origin. Let $\delta = r - \max\{\|f\|, \|g\|\} > 0$. Now let $\varepsilon_0 = \min\{\varepsilon, \delta/2\}$ and let B_0 be the ball centered at the origin of radius $r - \delta/2$. Then, $f, g \in B_0$, and there exists an ε_0-chain, $C = \{f_1, \ldots, f_n\}$, with $f = f_1$ and $g = f_n$. It is clear that if B_i is the ball of radius ε_0 about f_i for $2 \leq i \leq n - 1$, then $B_1, \ldots, B_n$ satisfy the conditions of the lemma.

THEOREM 3.3. *K is the component of the origin in N and every ρ-ball in K is connected.*

Proof. By Proposition 3.1 we need only show that the open ρ-balls are connected. Let B be an open ρ-ball in K and let $f, g \in B$. Further let $\langle \varepsilon_n \rangle$ be a monotone decreasing sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. By Lemma 3.2 there exists an ε_1-chain $C_1 = \{f_1, \ldots, f_m\}$ from f to g in B and there exist balls B_i each centered at f_i, $2 \leq i \leq m - 1$ and each with ρ-radius less than ε_1 such that $f_{i-1}, f_{i+1} \in B_i$, and $\text{cl}(B_i) \subset B$. Now let $E_1 = \bigcup_{i=1}^{m} B_i$. Observe that $C_1 \subset E_1$, $\text{cl}(E_1)$ can be finitely covered by balls of ρ-radius ε_1, and $\text{cl}(E_1) \subset B$. Now each pair f_i, f_{i+1} is contained in one of the balls B_j for $j = i$ or $j = i + 1$. Thus by the same procedure we can obtain an ε_2-chain from f_i to f_{i+1} in B_j with corresponding balls of radius less than ε_2 and with closures inside B_j. If we let C_2 denote the chain obtained by unioning (juxtaposing) the $m - 1$ ε_2-chains and if we let E_2 be the union of the balls, then C_2 is an ε_2-chain, $C_1 \subset C_2$, $C_2 \subset E_2$, $\text{cl}(E_2)$ can be finitely covered by ε_2-balls, and $E_2 \subset E_1$. Continuing inductively we obtain ε_n-chains C_n and sets E_n such that $C_n \subset C_{n+1}$, $E_{n+1} \subset E_n$, $C_n \subset E_n$, and each $\text{cl}(E_n)$ can be finitely covered by ε_n-balls. If we let $E = \text{cl}(\bigcup_{n=1}^{\infty} C_n)$, then $E \subset \text{cl}(E_n)$ for each n. Thus E is totally bounded and $E \subset B$. Since (K, ρ) is a complete metric space, E is compact. Also $f, g \in E$. By its construction E is ε-chainable for every $\varepsilon > 0$. Since a compact metric space is connected if and only if it is ε-obtainable for every $\varepsilon > 0$, E is connected [4, Theorem 5.1, p. 81]. But then for $f \in B$, B can be written as a union of connected sets containing f. Hence B is connected. This completes the proof.

4. Remarks

We note that K is arcwise connected since it is a connected, locally connected, complete metric space [2, Theorem 3-17, p. 118].

The question of characterizing the component of the origin in $N(U^n)$ is still open in the case $n > 1$. The component of the origin in $N(U^n)$ is definitely not $N^+(U^n)$. This follows since if we define $\phi: U^n \to U$ by $\phi(z_1, \ldots, z_n) = z_1$, then the map $C_\phi: N \to N(U^n)$ defined by $C_\phi(f) = f \circ \phi$ isometrically embeds N in $N(U^n)$. In particular $C_\phi(K)$ is connected in $N(U^n)$ but is not contained in $N^+(U^n)$.

The spaces N/K and K/N^+ could be of interest for further study. N/K is totally disconnected but is not discrete. K/N^+ is connected and locally connected, but by [6] every finite dimensional subspace of K/N^+ has the discrete topology. Hence neither of these is trivial and their study might shed more light on the space N.

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