HOMOTOPY TREES: ESSENTIAL HEIGHT AND ROOTS

BY

MICHEAL N. DYER

In this note we study some general properties of homotopy trees $HT(\pi, m)$. We show that for π a finite group, the trees are a single stalk from some point on (Theorem 2) and if $m \geq 3$, that roots can occur only at the lowest two levels of the tree (Corollary 1).

A (π, m)-complex X is a finite, connected m-dimensional CW-complex such that $\pi_1(X) \cong \pi$ and $\pi_i(X) = 0$ for $i = 2, \ldots, m-1$. The homotopy tree $HT(\pi, m)$ is the directed tree whose vertices are homotopy classes of (π, m)-complexes. If X and Y are (π, m)-complexes, then the vertex $[X]$ is connected by an edge to the vertex $[Y]$ iff Y has the homotopy type of the one-point union $X \vee S^m$ of X with the m-sphere S^m. $HT(\pi, m)$ is connected by Theorem 14 of [23, page 49] and has no circuits. The tree $HT(\pi, m)$ is measured by the *directed Euler characteristic* $\vec{\chi} = (-1)^m \chi$: vertices $(HT) \to Z$. Let

$$\vec{\chi}_{\min} = \min \{\chi[X] \mid X \text{ is a } (\pi, m)\text{-complex}\}.$$

Thus χ divides the tree into *levels* $\vec{\chi}^{-1}(j)$ $(j \geq \vec{\chi}_{\min})$. We call $\vec{\chi}^{-1}(i + \vec{\chi}_{\min})$ the ith *level of the tree*. For each $j \geq \chi_{\min}$, the *successor function* $s_j\colon \vec{\chi}^{-1}(j) \to \vec{\chi}^{-1}(j+1)$ is given by $s_j([X]) = [X \vee S^m]$. A vertex $x \in HT$ is a *root* if x has no predecessor; a *minimal root* if $x \in \vec{\chi}^{-1}(\vec{\chi}_{\min})$. The stalk $\langle x \rangle$ generated by the vertex x is the subtree whose vertices consist of

$$\{x, s(x), s^2(x), \ldots, s^n(x), \ldots\}.$$

For the purpose of classifying the homotopy type of (π, m)-complexes, we will identify the fundamental group of each (π, m)-complex with π. This can be done by simply choosing (and fixing) an isomorphism $\alpha_X\colon \pi \to \pi_1(X)$ for each X and using α_X to convert each $\pi_1(X)$-module into a π-module. Then any argument we make over π can be easily translated to $\pi_1(X)$. If $m \geq 3$, we may use a lemma of C. T. C. Wall [22, Lemma 1.2, page 59] to find a (π, m)-complex $Y \in [X]$ such that the two-skeleton $Y^{(2)}$ is the one-point union of a given $(\pi, 2)$-complex and a finite bouquet of 2-spheres. In this case, we may trivially identify the fundamental groups.

The homotopy type of a (π, m)-complex X is completely determined by (the isomorphism class of) its algebraic m-type $\mathbf{T}(X)$. This consists of the triple $\mathbf{T}(X) = (\pi, \pi_m(X), k(X))$ where $\pi_m(X)$ is a π-module and $k(X) \in H^{m+1}(\pi; \pi_m(X))$ is the first k-invariant of X (see [17, page 41], [7, Section 2]).

Received April 1, 1975.

Let us briefly define the k-invariant. Let

$$\begin{array}{c} 0 \longrightarrow \pi_m(X) \longrightarrow C_m(\tilde{X}) \xrightarrow{\partial_m} C_{m-1}(\tilde{X}) \xrightarrow{\partial_{m-1}} \cdots \\ \xrightarrow{\partial_1} C_0(\tilde{X}) \xrightarrow{\varepsilon} Z \longrightarrow 0 \end{array} \tag{0.1}$$

be the cellular chain complex of the universal cover $\tilde{X}$ of X. We will denote this by $0 \to \pi_m(X) \to C_*(\tilde{X}) \to Z \to 0$. This is an exact sequence of π-modules forming a portion of length m of a free, finitely generated resolution (each $C_i(\tilde{X})$ is a free, finitely generated π-module, $i = 0, 1, \ldots, m$) of the trivial π-module Z. Let

$$\mathscr{P}: 0 \to \pi_m(X) \to P_* \to Z \to 0$$

denote an exact sequence of length m of π-modules, where each P_i ($i = 0, 1, \ldots, m$) is finitely generated. Each such exact sequence determines an element k of $H^{m+1}(\pi; \pi_m(X))$ as follows. Cover the identity map $Z = Z$ by a chain map $\mathscr{F}: C_*(\tilde{X}) \to P_*$ as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_m(X) & \longrightarrow & C_*(\tilde{X}) & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & \pi_m(X) & \longrightarrow & P_* & \longrightarrow & Z & \longrightarrow & 0. \end{array} \tag{0.2}$$

This induces a homomorphism $f: \pi_m(X) \to \pi_m(X)$, which, in turn, determines an element

$$k = \{f\} \in H^{m+1}(\pi; \pi_m(X)) = \mathrm{End}_\pi(\pi_m(X))/B^m$$

where $B^m = \{\alpha \in \mathrm{End}_\pi(\pi_m(X)) \mid \alpha \text{ extends to } \alpha': C_m(\tilde{X}) \to \pi_m(X)\}$ (see [16, Theorem 3.6, page 74, and Section 6, page 84] as a general reference).

(0.3) *Note.* $k(X)$ is the class of $1: \pi_m(X) \to \pi_m(X)$.

DEFINITION. $k \in H^{m+1}(\pi; \pi_m(X))$ is said to be *projective* if one (and hence, all [8, Corollary 6.4]) realizing partial resolution(s) for k may be chosen with each P_i projective ($i = 0, 1, \ldots, m$).

Let $[X]$ be a vertex of $HT(\pi, m)$ and consider the π-module $\pi_m = \pi_m(X)$. Such a π-module is called *realizable*. Let $\tilde{K}_0 Z\pi$ denote the reduced projective class group of the integral group ring $Z\pi$ of π. The following theorem is proved in [8, Theorem 1].

THEOREM 1. *Let π be a group such that $H^{m+1}(\pi, Z\pi) = 0$. For each finitely generated topologically realizable π-module π_m, the group $H^{m+1}(\pi, \pi_m)$ supports the structure of a ring with identity such that the units U of $H^{m+1}(\pi, \pi_m)$ are the projective k-invariants. Furthermore, there exists a homomorphism $\mathscr{K}: U \to \tilde{K}_0 Z\pi$ such that* ker $\mathscr{K} = SF(\pi, m)$ *is the set of k-invariants arising from (π, m)-complexes, provided $m \geq 3$.*

Note that the hypothesis $H^{m+1}(\pi; Z\pi) = 0$ implies that $H^{m+1}(\pi; \pi_m(X)) \cong H^{m+1}(\pi; \pi_m(Y))$ for any two (π, m)-complexes X, Y. This follows because the theorem of J. H. C. Whitehead mentioned in paragraph two implies that there are integers s, t such that

$$X \vee sS^m \simeq Y \vee tS^m \tag{1.1}$$

where $iS^m = S^m \vee \cdots \vee S^m$ (i times). Hence, there is a π-module isomorphism

$$\pi_m(X) \oplus (Z\pi)^s \cong \pi_m(Y) \oplus (Z\pi)^t \tag{1.2}$$

for any two (π, m)-complexes X, Y [4, Appendix, Theorem, page 198].

For example, if π is a finite group of order n, then $H^i(\pi; Z\pi) = 0$ for all $i > 0$ [3, Proposition 8.2a, page 198]. It follows that $H^{m+1}(\pi; \pi_m) \cong Z_n$, the integers modulo n, (as a ring) for any realizable π_m [7, Section 2] and $\mathscr{K}: Z_n^* \to \tilde{K}_0 Z\pi$ is given by sending $p + nZ$ (p prime to n) to $-[(p, N)]$, the negative of the class represented by the projective ideal (p, N) generated by the integer p and $N = \sum_{x \in \pi} x$ [19, Section 6, page 278, and 7, Theorem 2.2]. This homomorphism has been extensively studied in [14] for π periodic and in [21] for more general finite π.

As another example, let π be a one-relator group with presentation

$$\{x_1, \ldots, x_n; Q^q\}$$

where Q is not a proper power and $q \geq 1$. It is known that for $i \geq 3$, $H^i(\pi; Z\pi) = 0$ [15, Corollary 11.3, page 663]. Recently, S. Jajodia [13] has shown that the ring $H^{i+1}(\pi; \pi_i) \cong Z_q$ for all $i \geq 2$ and realizable π_i.

For a third example, let A be a finitely generated abelian group of rank $r > 0$. It follows from [2, Proposition 3.1, page 112] that $H^i(A; ZA) = 0$ for all $i \neq r$. Among the k-invariant rings $H^{i+1}(A; A_i)$ there are *noncyclic* examples, for any realizable A_i and $i \geq r$.

The ring $H^{m+1}(\pi, \pi_m) = R(\pi, m)$ is called the *classifying ring* of the tree $HT(\pi, m)$ and the homomorphism $\mathscr{K}: U(\pi, m) \to \tilde{K}_0 Z\pi$, the *classifying homomorphism*.

Briefly, let us define isomorphisms between algebraic m-types [17, page 41]. Let π be a group, π_m a π-module, and $k \in H^{m+1}(\pi, \pi_m)$. An algebraic m-type is a triple $\mathbf{T} = (\pi, \pi_m, k)$. We say that $\mathbf{T}$ is isomorphic to $\mathbf{T}' = (\pi, \pi_m', k')$ iff there exists an automorphism $\theta: \pi \to \pi$, a θ-automorphism

$$\beta: \pi_m \to \pi_m' \quad (\beta(x \cdot y) = \theta(x)\beta(y),\ x \in \pi,\ y \in \pi_m)$$

such that $k = \beta_*^{-1} \cdot \theta^*(k')$ in the diagram

$$H^{m+1}(\pi; \pi_m) \xrightarrow{\beta^*} H^{m+1}(\pi; (\pi_m)_\theta) \xrightarrow{\theta^*} H^{m+1}(\pi; \pi_m').$$

Here $(\pi_m)_\theta$ is the π-module with action $\alpha * y = \theta(\alpha) \cdot y$ ($\alpha \in \pi$, $y \in \pi_m$). It is shown in [17, Theorem 1, page 42] that $X \simeq Y$ iff $\mathbf{T}(X) \cong \mathbf{T}(Y)$.

DEFINITION. Let $HT(\pi, m)^N = \vec{\chi}^{-1}([N + \vec{\chi}_{\min}, \infty))$ denote the subtree whose vertices are at level greater than or equal to N. We say that HT^N is an *evergreen* iff the successor function

$$s_i: \vec{\chi}^{-1}(i) \to \vec{\chi}^{-1}(i + 1)$$

is *surjective* for all $i \geq N + \vec{\chi}_{\min}$. HT has *essential height* $\leq l$ if HT^l is a single stalk.

THEOREM 2. *Let π be a finite group of order n and m be an integer ≥ 2. The tree $HT(\pi, m)$ always has finite essential height. For $m \geq 3$, the subtree $HT(\pi, m)^1$ is an evergreen; for m even and ≥ 4, the whole tree $HT(\pi, m)$ is evergreen.*

Proof. If π_m is a realizable π-module and $\alpha: \pi_m \to \pi_m$ is an automorphism, then we say that $\alpha_*: H^{m+1}(\pi, \pi_m) \to H^{m+1}(\pi, \pi_m) \cong Z_n$ has *degree* k if $\alpha_*(1) = k$. Let X be a minimal root and let v be the number of m cells in X. Then, for each $p \in SF = SF(\pi, m) \subset Z_n^*$, there exists an automorphism

$$\alpha_p: \pi_m(X) \oplus (Z\pi)^S \to \pi_m(X) \oplus (Z\pi)^S \tag{2.1}$$

of degree p, where $S = \max(v, 2)$. To see this we argue as follows.

Consider the boundary homomorphism $\partial_m: C_m(\tilde{X}) \to C_{m-1}(\tilde{X})$ in the cellular chain complex of the universal cover $\tilde{X}$ of X. Let π_{m-1} denote the image of ∂_m. For $m \geq 3$, $\pi_{m-1} = \pi_{m-1}(X^{(m-1)})$; if $m = 2$, π_1 is a so-called relation module of π. The sequence

$$0 \longrightarrow \pi_m(X) \xrightarrow{i} C_m(\tilde{X}) \longrightarrow \pi_{m-1} \longrightarrow 0$$

is an exact sequence of π-modules. Represent $p \in SF$ by a homomorphism $p': \pi_m(X) \to \pi_m(X)$ (multiplication by any integer $p' \in p$ will do) and consider the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_m(X) & \xrightarrow{i} & C_m(\tilde{X}) & \longrightarrow & \pi_{m-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p'} & & \downarrow & & \Vert & & \\ 0 & \longrightarrow & \pi_m(X) & \longrightarrow & p'C_m(\tilde{X}) & \longrightarrow & \pi_{m-1} & \longrightarrow & 0 \end{array} \tag{2.2}$$

where $p'C_m(\tilde{X})$ is the push out of i and p'. $p \in SF$ implies that $p'C_m(\tilde{X})$ is stably free [8, Corollary 6.4]. If v $(= \operatorname{rank}_\pi C_m(\tilde{X})) \geq 2$, then, by a theorem of H. Bass [1, Corollary 10.3, page 29], $p'C_m(\tilde{X}) \cong C_m(\tilde{X}) \cong (Z\pi)^v$; if $v < 2$, then $p'C_m(\tilde{X}) \oplus Z\pi$ is free. The isomorphism α_p then follows from Schanuel's lemma [19, Corollary 1.1, page 270].

We will show that $HT(\pi, m)$ has essential height $\leq S$. Let Y be a (π, m)-complex at level higher than $S - 1$; i.e.,

$$\vec{\chi}(Y) = |\chi(Y)| \geq S + \vec{\chi}_{\min}.$$

By (1.2), $\pi_m(X) \oplus (Z\pi)^u \cong \pi_m(Y) \oplus (Z\pi)^t$ for certain nonnegative integers u and t. A simple Euler characteristic argument shows that

$$u - t = \vec{\chi}(Y) - \vec{\chi}_{\min} \geq S.$$

Because S is greater than one, the cancellation theorem of H. Bass mentioned in the last paragraph implies that $\pi_m(Y) \cong \pi_m(X) \oplus (Z\pi)^k$ $(k \geq S)$. Thus

$\mathbf{T}(Y) \cong (\pi, \pi_m(X) \oplus (Z\pi)^k, p)$ for some $p \in SF$. We may assume (0.3) that $\mathbf{T}(X \vee kS^m) = (\pi, \pi_m(X) \oplus (Z\pi)^k, 1)$. Then the isomorphism

$$(\mathrm{id}, \alpha_p)\colon (\pi, \pi_m(X) \oplus (Z\pi)^k, 1) = \mathbf{T}(X \vee kS^m) \to (\pi, \pi_m(X) \oplus (Z\pi)^k, p)$$

given by (2.1) shows that $Y \simeq X \vee kS^m$.

We say that a π-module M has the *cancellation property* if any isomorphism $M' \oplus (Z\pi)^i \cong M \oplus (Z\pi)^j$ $(j \geq i)$ implies that $M' \cong M \oplus (Z\pi)^{j-i}$. The evergreen property for $HT(\pi, m)^1$ follows because, as in the preceding paragraph, $\pi_m(X) \oplus (Z\pi)^2$ has the cancellation property; the evergreen property for $HT(\pi, m)$ (m even) follows because $\pi_m(X) \oplus Z\pi$ has the cancellation property [7, Proposition 5.1].

For example, let us prove the final statement. Let $m > 3$ be even and Y be a (π, m)-complex such that $\vec{\chi}(Y) > \vec{\chi}_{\min}$. Then $\pi_m(Y) \cong \pi_m(X) \oplus (Z\pi)^i$ and $\mathbf{T}(Y) \cong (\pi, \pi_m(X) \oplus (Z\pi)^i, p)$ for some $p \in SF(\pi, m) \subset Z_n^*$. Let $\mathbf{T}_p = (\pi, \pi_m(X), p)$. Because $m \geq 3$, $\mathbf{T}_p \cong \mathbf{T}(W)$ for some (π, m)-complex W [19, Theorem 3.1, page 272]. Thus $W \vee iS^m \simeq Y$, which implies that $HT(\pi, m)$ is an evergreen and that the only roots of the tree are minimal ones. □

In fact, the proof shows even more. Among the minimal roots for $HT(\pi, m)$, let X be the one with the smallest number of m-cells. Denote that number by $\nu(\pi, m)$. Then

$$\text{essential height of } HT(\pi, m) \leq \begin{cases} \nu(\pi, m) & \text{if } m \text{ is even} \\ \max\{\nu(\pi, m), 2\} & \text{if } m \text{ is odd.} \end{cases}$$

For example, if π is the finite abelian group $Z_{\tau_1} \times \cdots \times Z_{\tau_S}$, where $\tau_i \mid \tau_{i+1}$ $(i = 1, \ldots, S-1)$, then the essential height of $HT(\pi, 2)$ is $\leq S + C(S, 2)$. See Theorem 3 for a better estimate.

As another example, let π be a finite group of minimal *free* period k (see [7, Section 7] for a definition) and let g be the minimal number of generators of π. Then the essential height of $HT(\pi, ki + 1)$ $(i \geq 1)$ is $\leq g$.

COROLLARY 1. *If π is finite and $m > 2$, then roots of $HT(\pi, m)$ may only occur at level* 0 *for m even and level* 0 *or* 1 *for m odd.*

It is shown in [9, Corollary 3.7] that $HT(GQ(32), 3)$ has nonminimal roots, where $GQ(32)$ is the generalized quaternion group of order 32. Also, M. J. Dunwoody has shown that roots exist at level 1 in $HT(T, 2)$, where T is the group of the trefoil knot [5].

Finally, we will improve theorem A of [11, page 115].

THEOREM 3. *Let $\pi = Z_{\tau_1} \times Z_{\tau_2} \times \cdots \times Z_{\tau_s}$ be a finite abelian group with torsion coefficients $\{\tau_1, \tau_2, \ldots, \tau_s\}$, where τ_i divides τ_{i+1} for $i = 1, \ldots, s-1$. Then $HT(\pi, 2)$ has essential height $\leq C(S, 2)$.*

COROLLARY 2. *The essential height of $HT(Z_{\tau_1} \times Z_{\tau_2}, 2)$ is less than or equal to one.*

Proof. In order to simplify the notation, we will prove only the corollary. Let $\mathscr{P}$: $\{x, y\colon x^{\tau_1}, y^{\tau_2}, [x, y]\}$ be the standard presentation of $\pi = Z_{\tau_1} \times Z_{\tau_2}$, P

be the realization of $\mathscr{P}$ as a $(\pi, 2)$-complex, and $\pi_2 = \pi_2(P)$. Let $\bar{x}$, $\bar{y}$ denote the images of x, y in the group π. Consider the 2-types

$$\mathbf{T}_p^i = (\pi, \pi_2 \oplus (Z\pi)^i, p) \quad \text{for } p \in Z^*_{\tau_1\tau_2} \text{ and } i \geq 0.$$

As a point of reference, we may assume (0.3) that $\mathbf{T}(P \vee kS^2) = \mathbf{T}_1^k$ $(k \geq 0)$. We will show that *for* $p \in SF(\pi, 2)$, *each* $\mathbf{T}_p^1 \cong \mathbf{T}_1^1$. Assuming this, let Y be any $(\pi, 2)$ complex such that $\vec{\chi}(Y) > \vec{\chi}_{\min} = 2$. By (1.2) and because $\pi_2 \oplus Z\pi$ has the cancellation property [7, Proposition 5.1], $\pi_2(Y) \cong \pi_2 \oplus (Z(\pi))^k$ $(k \geq 1)$. Hence $\mathbf{T}(Y) \cong \mathbf{T}_p^k$ for some $p \in Z^*_{\tau_1\tau_2}$. But Theorem 1 shows that because $\mathbf{T}_p^k \cong \mathbf{T}(Y)$ is 2-realizable, $p \in SF(\pi, 2)$. Thus

$$\mathbf{T}(Y) \cong \mathbf{T}_p^k \cong \mathbf{T}_1^k \cong \mathbf{T}(P \vee kS^2).$$

To show that $\mathbf{T}_p^1 \cong \mathbf{T}_1^1$ for each $p \in SF(\pi, 2)$, we use a theorem of S. MacLane and J. H. C. Whitehead [17, Theorem 2, page 42] to realize $\mathbf{T}_p^0$ as the 2-type of a *finite*, connected 3-dimensional CW complex X. Consider the following alteration of the cellular chain complex $C_*(\tilde{X})$:

$$\mathscr{C}: 0 \longrightarrow \pi_2 \longrightarrow (C_2/B_2) \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} Z\pi \xrightarrow{\varepsilon} Z \longrightarrow 0$$

where $C_i = C_i(\tilde{X})$, $B_2 = \operatorname{im}\{\partial: C_3 \to C_2\}$. As in the proof of Theorem 4.1 of [6, page 236], we may assume that $C_1 = (Z\pi)^2$ and $\partial_1 = (\bar{x} - 1, \bar{y} - 1)$ with respect to a natural basis for $C_1(\tilde{X})$ defined by the lifts of the (two) 1-cells of X. $\mathscr{C}$ realizes $\mathbf{T}_p^0$. $p \in SF(\pi, 2)$ implies that C_2/B_2 is a stably free projective module [7, Theorem 2.5]. π is finite abelian implies that stably free projectives are free [20, page 178]; hence C_2/B_2 is a free π-module.

Now the argument of theorem A of [11, pages 119–123] applied to $\mathscr{C}$ yields the result that $\mathbf{T}_p^1 \cong \mathbf{T}_1^1$. Briefly, here is a sketch of the argument: choose $c \in C_2/B_2$ such that $\partial_2 c = \alpha = (1 - \bar{y}, \bar{x} - 1) \in (Z\pi)^2$. Here α is the total Fox derivative of the commutator $[x, y]$ [11, Section 2]. Define a new chain complex

$$\begin{array}{l} \mathscr{C}': 0 \longrightarrow \pi_2 \oplus Z\pi \longrightarrow C_2/B_2 \oplus Z\pi \xrightarrow{(\partial_2, \alpha)} (Z\pi)^2 \xrightarrow{(\bar{x}-1, \bar{y}-1)} Z\pi \xrightarrow{\varepsilon} Z \longrightarrow 0 \\ \qquad\qquad \pi_2 \oplus Z\pi = (\pi_2, 0) \oplus Z\pi(c, -1) \end{array}$$

obtained by adding a copy of $Z\pi$ to C_2/B_2 and defining the boundary operator to be multiplication by α on that factor. $\mathscr{C}'$ realizes $\mathbf{T}_p^1$ as a free complex. We prove this by comparing $\mathscr{C} \oplus (Z\pi, 2)$ to $\mathscr{C}'$ as in (0.2):

$\mathscr{C} \oplus (Z\pi, 2)$:

$$\begin{tikzcd} 0 \arrow[r] & \pi_2 \oplus Z\pi \arrow[r] \arrow[d, "f_2 = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}"'] & C_2/B_2 \oplus Z\pi \arrow[r, "(\partial_2, 0)"] \arrow[d, "g = \begin{pmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{pmatrix}"'] & (Z\pi)^2 \arrow[r] \arrow[d, equal] & Z\pi \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, equal] & 0 \\ \mathscr{C}': 0 \arrow[r] & \pi_2 \oplus Z\pi \arrow[r] & C_2/B_2 \oplus Z\pi \arrow[r, "(\partial_2, \alpha)"] & (Z\pi)^2 \arrow[r] & Z\pi \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

The induced map f_2 shows that both $\mathscr{C}'$ and $\mathscr{C} \oplus (Z\pi, 2)$ have the same k-invariant. The argument of [11, page 120, last paragraph, to page 123, first paragraph] shows that under these conditions we may choose a basis for $C_2/B_2 \oplus Z\pi$ so that $\mathscr{C}'$ then realizes $\mathbf{T}_1^1$ and, in fact, $\mathscr{C}' = C_*(P \vee S^2)$, with that basis (see also [10, pages 38–39]). Thus $\mathbf{T}_p^1 \cong \mathbf{T}_1^1$ *with an isomorphism inducing the identity on* π [10, Proposition 4, page 36]. $\square$

The following corollary is an easy consequence of the last sentence of the proof of Corollary 2.

COROLLARY 3. *With* $\pi_2 = \pi_2(P)$ *as in the proof of corollary* 2, *there is an automorphism* $\alpha_p\colon \pi_2 \oplus Z\pi \to \pi_2 \oplus Z\pi$ *of degree p for each* $p \in SF(\pi, 2)$ (see [7, Section 3] for a related discussion).

Note. E. Vogt has brought to my attention recent work of Wolfgang Metzler. He has shown that for certain finite abelian groups

$$\pi(\tau_1, \ldots, \tau_S) = Z_{\tau_1} \times Z_{\tau_2} \times \cdots \times Z_{\tau_S} \quad \text{with } S \geq 3,$$

there exist distinct minimal roots K_1, K_2 of $HT(\pi, 2)$ for which $K_1 \vee S^2 \simeq K_2 \vee S^2$ [18, Satz 2]. Thus, for certain finite abelian groups π, the homotopy tree $HT(\pi, 2)$ is not a single stalk.

Let $\pi(\tau_1, \ldots, \tau_S)$ have presentation

$$\mathscr{P} = \{x_1, \ldots, x_S\colon x_1^{\tau_1}, \ldots, x_S^{\tau_S}, \{[x_i, x_j] \mid 1 \leq i < j \leq S\}\}$$

and let P denote the cellular model of $\mathscr{P}$. It can be shown that the Z-rank of $\pi_2(P)^\pi$ is precisely the number $C(S, 2)$. We ask two questions:

(1) Is the essential height of $HT(\pi(\tau_1, \ldots, \tau_S), 2)$ equal to $C(S, 2)$?

(2) If π is an arbitrary finite group, and X a minimal root of $HT(\pi, m)$, is the essential height of $HT(\pi, m) \leq Z$-rank of $\pi_2(X)^\pi$?

One method of proof for (2) might go as follows. Let X be a minimal root and $\pi_m = \pi_m(X)$. By Schanuel's lemma and [7, Theorem 2.2] it follows that there exists an automorphism

$$\alpha_p\colon \pi_m \oplus (Z\pi)^M \to \pi_m \oplus (Z\pi)^M$$

of degree p for each $p \in SF(\pi, m)$. Here $M \leq v(\pi, m)$. The problem is then to *cancel* (in the style of Bass-Jacobinski [20, Chapter 9], [1], [12]) *while preserving the degree.*

BIBLIOGRAPHY

1. H. BASS, *K-theory and stable algebra*, Inst. Hautes Études Sci. Publ. Math., vol. 22 (1964), pp. 5–60.

2. R. BIERI AND B. ECKMANN, *Groups with homological duality generalizing Poincare duality*, Inventiones Math., vol. 20 (1973), pp. 104–124.

3. H. CARTAN AND S. EILENBERG, *Homological algebra*, Princeton Univ. Press, Princeton, New Jersey, 1952.

4. W. H. COCKCROFT AND R. G. SWAN, *On the homotopy types of certain two-dimensional complexes*, Proc. London Math. Soc. (3), vol. 11 (1961), pp. 194–202.
5. M. J. DUNWOODY, *The homotopy type of a two-dimensional complex*, preprint.
6. M. N. DYER, *On the 2-realizability of 2-types*, Trans. Amer. Math. Soc., vol. 204 (1975), pp. 229–243.
7. ———, *On the homotopy classification of* (π,m)*-complexes*, J. Pure and Applied Algebra.
8. ———, *Projective k-invariants*, preprint.
9. ———, *Non-minimal roots in homotopy trees*, preprint.
10. M. N. DYER AND A. J. SIERADSKI, *Trees of homotopy types of two-dimensional CW-complexes I*, Comm. Math. Helv., vol. 48 (1973), 31–44.
11. ——— ———, *Trees of homotopy types of two-dimensional CW complexes II*, Trans. Amer. Math. Soc., vol. 205 (1975), pp. 115–125.
12. H. JACOBINSKI, *Genera and decompositions of lattices over orders*, Acta Math., vol. 121 (1968), pp. 1–29.
13. S. JAJODIA, private communication.
14. R. LEE AND C. THOMAS, *Free, finite group actions on* S^3, Bull. Amer. Math. Soc., vol. 79 (1973), pp. 211–215.
15. R. LYNDON, *Cohomology theory of groups with a single defining relation*, Ann. of Math., vol. 52 (1950), pp. 650–665.
16. S. MACLANE, *Homology*, Springer-Verlag, Berlin, 1963.
17. S. MACLANE AND J. H. C. WHITEHEAD, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. U.S.A., vol. 36 (1950), pp. 41–48.
18. W. METZLER, *Uber den Homotopietyp zweidimensionaler CW-Komplexe und Elementartransformationen bei Darstellungen von Gruppen durch Erzeugende und definierende Relationen*, preprint.
19. R. G. SWAN, *Periodic resolutions for finite groups*, Ann. of Math., vol. 72 (1960), pp. 267–291.
20. R. G. SWAN AND E. G. EVANS, *K-theory of finite groups and orders*, Lecture notes in mathematics, vol. 149, Springer-Verlag, Berlin, 1970.
21. S. ULLOM, *Non-trivial lower bounds for class groups of integral group rings*, preprint.
22. C. T. C. WALL, *Finiteness conditions for CW-complexes I*, Ann. of Math., vol. 81 (1965), pp. 354–363.
23. J. H. C. WHITEHEAD, *Simple homotopy types*, Amer. J. Math., vol. 72 (1950), pp. 1–57.

UNIVERSITY OF OREGON
EUGENE, OREGON