WEAK MIXING AND A NOTE ON A STRUCTURE THEOREM FOR MINIMAL TRANSFORMATION GROUPS

BY

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Let (X, T) be a minimal transformation group (sometimes called a minimal set) with compact Hausdorff phase space.

Veech in his paper on point-distal minimal transformation groups obtained a structure theorem for point-distal minimal transformation groups [13]. (A minimal transformation group is point distal if it has a point that is not proximal to any other point.) This structure theorem led to the idea of a PD flow. A minimal transformation group (X, T) is a *PD* flow if there exists an ordinal Λ, transformation groups X_γ for $\gamma \leq \Lambda$ and homomorphisms $\phi_\gamma^\lambda: X_\lambda \to X_\gamma$ for $\gamma < \lambda$ such that $X_\Lambda = X$, X_0 is a singleton, $\phi_\gamma^{\gamma+1}: X_{\gamma+1} \to X_\gamma$ is a proximal or distal homomorphism, and $X_\gamma = \text{inv lim } \{X_\lambda: \lambda < \gamma\}$ if γ is a limit ordinal. The notion of *PD* flows has proved to be very useful; and it seems to be the natural approach to a structure theorem for minimal sets. Veech's structure theorem shows that every point distal minimal set (X, T) with metric phase space X is a factor of a *PD* flow (X^*, T) such that $(X^*, T) \to (X, T)$ is a proximal homomorphism. In [6], the following structure theorem for minimal sets was proved.

THEOREM. *For every minimal transformation group (X, T) there exist minimal sets (X^*, T), (Y, T) and homomorphisms α, β such that Y is a PD flow, $\alpha: X^* \to X$ is a proximal homomorphism, and $\beta: X^* \to Y$ is open and satisfies the condition that the almost periodic points in $(R_n(\beta), T)$ are dense in $R_n(\beta)$ and that $S(\beta) = R(\beta)$, where*

$$R(\beta) = \{(x, x') \in X^* \times X^*: \beta(x) = \beta(x')\},$$

$$R_n(\beta) = \{(x_1, \ldots, x_n) \in X^* \times \cdots \times X^*: \beta(x_1) = \cdots = \beta(x_n)\},$$

and $S(\beta)$ is the relativized equicontinuous structure relation. If X is metric, X^* can be taken to be metric.

In this paper we are concerned with the structure of the homomorphism β. We show that if (X, T) and (Y, T) are minimal transformation groups with metric phase spaces and if $\phi: (X, T) \to (Y, T)$ is a homomorphism such that the almost periodic points in $(R(\phi), T)$ are dense in $R(\phi)$ and $S(\phi) = R(\phi)$, then there exists a point in $(R(\phi), T)$ with dense orbit. This is the relativized concept of weak mixing. When Y is a singleton this says that there exists a point in $X \times X$ whose orbit is dense in $X \times X$. We give an example that shows the

Received October 18, 1974.

condition that the almost periodic points in $(R(\phi), T)$ be dense in $R(\phi)$ cannot be dropped (see Example 1.8).

We also show that given (X, T) and (Y, T) metric minimal set and a homomorphism $\phi: X \to Y$, if $S(\phi) = R(\phi)$ and the almost periodic points in $(R_n(\phi), T)$ are dense in $R_n(\phi)$, then there is a point in $(R_n(\phi), T)$ with dense orbit. When T is abelian and Y is a singleton this implies the known result that if $(X \times X, T)$ has a point with dense orbit, then $(X \times X \times X \times X, T)$ has a point with dense orbit. We then show that if (X, T) is a minimal transformation group with metric phase space and proximal relation P such that $P(x)$ is residual in X for every x in X, then any open invariant subset of $(X \times X, T)$ that contains an almost periodic point is dense. These results might shed some light on the study of weakly mixing minimal transformation groups when T is non-abelian.

One of the important studies in topological dynamics is the characterization of the equicontinuous structure relation $S(X)$ of a minimal transformation group (X, T), that is, the least closed invariant equivalence relation $S(X)$ such that $(X/S(X), T)$ is almost periodic. It is known now that under some conditions $S(X)$ is the same as the regionally proximal relation $Q(X)$ of (X, T) [4]. A natural question is: does $S(X) = Q(X)$ for all (X, T). Here we present a simple example that shows the answer is no (see Example 1.8).

Given $\phi: (X, T) \to (Y, T)$. The above shows the value of the assumption that the almost periodic points in $(R(\phi), T)$ are dense in $R(\phi)$ in the study of relativized problems and in the generalization of results that assume T is abelian. Another assumption that seems to be useful in such problems is that for some y in Y and for some idempotent u in the enveloping semigroup of (X, T) the set $\phi^{-1}(y)u = \{xu: x \in \phi^{-1}(y)\}$ is dense in $\phi^{-1}(y)$. In Section 2 we provide some examples to aid in the study of these concepts (see Example 2.1).

The author would like to express his appreciation to Professor Ta-Sun Wu for his helpful comments.

Definitions and notation. Suppose $\phi: (X, T) \to (Y, T)$ is a homomorphism. In general we will assume that ϕ is onto. Let $R_n(\phi)$ denote the set

$$\{(x_1, \ldots, x_n) \in X \times \cdots \times X: \phi(x_1) = \cdots = \phi(x_n)\}.$$

Let $D_n(\phi)$ denote the set of almost periodic points in the transformation group $(R_n(\phi), T)$. Also let $R(\phi)$ denote $R_2(\phi)$ and $D(\phi)$ denote $D_2(\phi)$. If Y is a singleton let D_n and D denote $D_n(\phi)$ and $D(\phi)$ respectively.

A transformation group is called point-transitive if it has a point with dense orbit. A homomorphism ϕ of a transformation group (X, T) onto (Y, T) is called weakly mixing if $(R(\phi), T)$ is point-transitive.

Suppose $\phi: (X, T) \to (Y, T)$. Then $P(\phi) = \{(x, x') \in R(\phi)$: there is a net t_n in T with $\lim xt_n = \lim x't_n\}$ is the relativized proximal relation; $Q(\phi) = \{(x, x') \in R(\phi)$: there exist nets t_n in T and (x_n, x_n') in $R(\phi)$ such that $(x_n, x_n') \to$

(x, x') and $\lim xt_n = \lim x't_n\}$ is the relativized regionally proximal relation. $S(\phi)$ will denote the relativized equicontinuous structure relation and is the smallest closed invariant equivalence relation containing $Q(\phi)$. ϕ is an almost periodic homomorphism if and only if $Q(\phi)$ equals the diagonal of $X \times X$. When Y is a singleton, we will use the notation P, $P(X)$, or P_X, Q, $Q(X)$, or Q_X, S, $S(X)$, or S_X.

Given a transformation group (X, T), we will denote the enveloping semigroup by $E(X, T)$. Let I denote one of the minimal right ideals in $E(X, T)$ and J denote the set of idempotents in I. The properties of $E(X, T)$, I, and J are developed in [2] and [3].

Section 1

(1.1) LEMMA. *Suppose (X, T) is a minimal set, X is metric, $D(\phi)$ is dense in $R(\phi)$, and $S(\phi) = R(\phi)$. Fix z in Z and let $X_0 = \phi^{-1}(z)$. If A is a countable subset of X_0, then there exists a point x_0 in X_0 that is proximal to each point of A.*

Proof. By 2.11 of [6], there is a closed nonempty subset $\hat{X}$ of X_0 such that for x in X_0, $P(\phi)(x) \cap \hat{X}$ is a residual subset of $\hat{X}$. Then

$$\bigcap \{P(\phi)(x): x \in A\}$$

is nonempty; take x_0 in this intersection.

The assumption $S(\phi) = R(\phi)$ is made in Theorems 1.2 and 1.3 only so Lemma 1.1 may be applied.

(1.2) THEOREM. *Suppose (X, T) is a minimal set, X is metric, ϕ is a homomorphism of (X, T) onto (Z, T) and $D(\phi)$ is dense in $R(\phi)$. If $S(\phi) = R(\phi)$, then ϕ is weakly mixing.*

Proof. Fix a minimal right ideal I in the enveloping semigroup $E(X, T)$ and let J be the set of idempotents in I. Fix an idempotent u on J and an element z_0 in Z with $z_0u = z_0$. Let $X_0 = \phi^{-1}(z_0)$, let A be a countable dense subset of X_0u and take x_0 as in 1.1. Let $y_0 \in X_0u$, we now show that (x_0, y_0) has dense orbit in $R(\phi)$.

For each $y \in P(\phi)(x_0) \cap X_0u$, there is a minimal right ideal I' in $E(X, T)$ such that $x_0q = yq$ for all $q \in I'$. By 3.6 of [2] there is an idempotent u_y in I' such that $uu_y = u$, $u_yu = u_y$. So $x_0u_y = yu_y$ and $yu_y = yuu_y = yu = y$. Let $N = \text{Cls}((x_0, y_0)T)$. Then

$$(x_0, y_0)u_y = (y, y_0u_y) = (y, y_0uu_y) = (y, y_0u) = (y, y_0) \in N$$

for y in A, and so $X_0u \times \{y_0\} \subseteq N$ and $(X_0u \times \{y_0\})T \subseteq N$. Let $(x, y) \in D(\phi)$. There exists an idempotent v in J such that $xv = x$, $yv = y$, then there exists pv, $qv \in I$ with $y_0qv = xv = x$ and $y_0qv = yv = y$. Consider $x' = y_0(pv)(qv)^{-1}u$, where $(qv)^{-1}$ is the inverse of qv in the group Iv (3.5 of [2]). Then $x' \in X_0u$ since

$$\phi(y_0(pv)(qv)^{-1}u) = \phi(y_0pv)(qv)^{-1}u = \phi(y_0qv)(qv)^{-1}u = \phi(y_0vu)$$
$$= \phi(y_0u) = \phi(y_0) = z_0.$$

And since $x'qv = y_0(pv)(qv)^{-1}u(qv) = x(qv)^{-1}(qv) = xv = x$, if t_n is a net in T (considered as a subset of X^X, see [2], Chapter 3) converging to qv, then

$$y_0t_n \to y_0qv = y \quad \text{and} \quad x't_n \to x'qv = x.$$

Thus $D(\phi) \subseteq N$ and therefore $N = R(\phi)$.

(1.3) THEOREM. *Suppose (X, T) is a minimal set, X is metric, ϕ is a homomorphism of (X, T) onto (Z, T) and $D_n(\phi)$ dense in $R_n(\phi)$. If $S(\phi) = R(\phi)$, then $(R_n(\phi), T)$ is point transitive.*

Proof. We will show that every closed invariant set C in $R_n(\phi)$ with non-empty interior equals $R_n(\phi)$, which implies the existence of a dense G_δ set of transitive points in $R_n(\phi)$.

Fix a minimal right ideal I in the enveloping semigroup $E(X, T)$ and let J be the set of idempotents in I.

Let V_i, $i = 1, \ldots, n$ be open sets in X such that

$$V = V_1 \times \cdots \times V_n \cap R_n(\phi) \subseteq C.$$

We will show that V_n may be replaced by X, that is,

$$V_1 \times \cdots \times V_{n-1} \times X \cap R_n(\phi) \subseteq C.$$

By the argument it will be clear that each V_i, $i = 1, \ldots, n-1$, could in turn be replaced by X and thus that $R_n(\phi) = X \times \cdots \times X \cap R_n(\phi) \subseteq C$. Since $D_n(\phi)$ is dense in $R(\phi)$, all we need to show is that

$$V_1 \times \cdots \times V_{n-1} \times X \cap D_n(\phi) \subseteq C.$$

Suppose $(y_1, \ldots, y_{n-1}, y_n) \in V_1 \times \cdots \times V_{n-1} \times X \cap D_n(\phi)$. For each open neighborhood

$$W_1 \times \cdots \times W_{n-1} = W$$

of $(y_1, \ldots, y_{n-1})$ in $X \times \cdots \times X$ ($n - 1$ times), there exists

$$(x_1^W, \ldots, x_n^W) \in V \cap D_n(\phi)$$

with $(x_1^W, \ldots, x_{n-1}^W) \in W$. Now there is an idempotent u_W in J with

$$(x_1^W, \ldots, x_n^W)u_W = (x_1^W, \ldots, x_n^W).$$

In the next paragraph we will show that

$$\{x_1^W\} \times \cdots \times \{x_{n-1}^W\} \times Xu_W \cap R(\phi) \subseteq C.$$

For now, fix $x_0 \in X$ independent of W and take $p_W \in I(X, T)$ with $x_0p_W = x_1^W$, and $p_Wu_W = p_W$. The collection of neighborhoods W of $(y_1, \ldots, y_{n-1})$ is a directed set directed by containment. Consider the net p_W, take a convergent subnet p_{W_j}, and suppose p is its limit. Then note $x_0p = y_1$. Now for some u in J, $pu = p$. Then

$$y_np^{-1}p_{W_j} \in Xu_{W_j} \quad \text{and} \quad y_nu = y_np^{-1}p = \lim y_np^{-1}p_{W_j}.$$

Also, $y_1 p^{-1} = x_0 u$. So

$$\phi(x_0 u) = \phi(y_1 p^{-1}) = \phi(y_1)p^{-1} = \phi(y_n)p^{-1} = \phi(y_n p^{-1})$$

and thus $\phi(y_n p^{-1} p_W) = \phi(x_0 u p_W) = \phi(x_1^W)$. Therefore

$$(x_1^W, \ldots, x_{n-1}^W, y_n p^{-1} p_W) \in \{x_1^W\} \times \cdots \times \{x_{n-1}^W\} \times Xu_W \cap R_n(\phi) \subseteq C.$$

And so its limit $(y_1, \ldots, y_{n-1}, y_n u) \in C$.

Then

$$(y_1, \ldots, y_{n-1}, y_n u)T \subseteq C$$

and since $(y_1, \ldots, y_{n-1}, y_n)v = (y_1, \ldots, y_{n-1}, y_n)$ for some v in J, we see that

$$(y_1, \ldots, y_{n-1}, y_n) = (y_1, \ldots, y_{n-1}, y_n u)v \in C.$$

We now show that as claimed, $\{x_1^W\} \times \cdots \times \{x_{n-1}^W\} \times Xu_W \cap R_n(\phi) \subseteq C$. As in 1.1 take x' with $\phi(x') = \phi(x_1^W)$ and such that x' is proximal to a dense subset A of $Xu_W \cap R(\phi)(x_1^W)$. For each a in A there is an idempotent u_a in $E(X, T)$ such that $u_W u_a = u_W$, $u_a u_W = u_a$, and $x'u_a = a$. Also then $xu_a = x$, for $x \in Xu_W$. Since A is dense there is some a in $V_n \cap A$. Then

$$(x_1^W, \ldots, x_{n-1}^W, a) \in V \quad \text{and} \quad (x_1^W, \ldots, x_{n-1}^W, x')u_a = (x_1^W, \ldots, x_{n-1}^W, a) \in V.$$

Thus for some t in T,

$$(x_1^W, \ldots, x_{n-1}^W, x')t \in V; \quad \text{so } (x_1^W, \ldots, x_{n-1}^W, x') \in VT \subseteq C.$$

Now for every $b \in A$, $(x_1^W, \ldots, x_{n-1}^W, x')u_b = (x_1^W, \ldots, x_{n-1}^W, b)$ and so

$$(x_1^W, \ldots, x_{n-1}^W, b) \in C.$$

And since A is dense

$$\{x_1^W\} \times \cdots \times \{x_{n-1}^W\} \times Xu_W \cap R_n(\phi) = \{x_1^W\} \times \cdots \times \{x_{n-1}^W\} \times Xu_W \cap R(\phi)(x_1^W) \subseteq C.$$

This completes the proof.

(1.4) COROLLARY. *If (X, T) is a metric minimal set with T abelian and if its only almost periodic factor is the singleton transformation group, then $(X \times \cdots \times X, T)$ has a point with dense orbit.*

(1.5) *Remark.* Corollary 1.4 is well-known (see [9], Proposition 2.3).

The referee suggested that probably under the conditions of Corollary 1.4, if $(x_1, \ldots, x_{n-1})$ has dense orbit in $X \times \cdots \times X$ ($n - 1$ times), then

$$\{x: (x_1, \ldots, x_{n-1}, x) \text{ has dense orbit}\}$$

is residual. The following theorem is perhaps a suitable substitute for his suggested theorem.

THEOREM. *Suppose (X, T) is a metric minimal set with T abelian and its only almost periodic factor is the singleton transformation group. If*

$$x' = (x_1, \ldots, x_n)$$

is a point with dense orbit in $X' = X \times \cdots \times X$ *(n times), then the set*

$$\{x^* \in X' : (x', x^*) \text{ has dense orbit in } X' \times X'\}$$

is residual.

Proof. By [10], there exists an invariant Borel probability measure on X with support X. Then the product measure in an invariant Borel probability measure on X' with support X'. By 1.1 of [6], there exists a dense G_δ subset of points x^* in X' such that (x', x^*) has dense orbit in $(X' \times X', T)$.

If ϕ is a homomorphism from a minimal set (X, T) onto a minimal set (Z, T) that is open and if there is a point z in Z and an idempotent u in $E(X, T)$ such that $\phi^{-1}(z)u$ is dense in $\phi^{-1}(zu)$, then $D_n(\phi)$ is dense in $R_n(\phi)$, for all positive integers n. In the case that Z is a singleton the above assumption reduces to Xu is dense in X which plays a key role in [4].

Also note that in the proof of 1.2, x_0 is proximal to every point of X_0u. Indeed if $x_0 \in X_0$ is proximal to a dense subset A of X_0u, then for any y in X_0u and neighborhood U of y, take $d \in A \cap U$ and note that since there is an idempotent u_d in $E(X, T)$ such that $uu_d = u$ and $x_0u_d = d$, we can take $t \in T$ such that $yut = yt \in U$ and $x_0t \in U$; thus x_0 and y are proximal. We make some further observations on this idea in the following two propositions.

(1.6) Proposition. *Suppose (X, T) is a minimal set. If T is abelian and xu is proximal to every element of Xv, then every element of Xu is proximal to every element of Xv, where u and v are idempotents in some minimal right ideal of $E(X, T)$.*

Proof. Note $xut = xtu$ is proximal to every element of $Xvt = Xtv = Xv$. Thus each element $x'v \in Xv$ is proximal to each element of xuT which is a dense subset of Xu. So as in 1.5, $x'v$ is proximal to every element of Xu. The proof is complete.

For the next proposition suppose (X, T) is a minimal set and I is a minimal right ideal in the enveloping semigroup $E = E(X, T)$ of (X, T). Fix an idempotent u in I, let $G = Iu$, and take $x_0 \in X$ with $x_0u = x_0$. Suppose ϕ is a homomorphism of (X, T) onto (Y, T) such that $\{g \in G : x_0g = x_0\}$ is a normal subgroup of the group $\{f \in G : y_0f = y_0\}$ where $y_0 = \phi(x_0)$. Note this is a property of the map ϕ and is independent of x_0 and u. Let $X_0 = \phi^{-1}(y_0)$ and v be an idempotent in I.

(1.7) Proposition. *Given the above, if x_0 is proximal to every point of X_0v, then each point of X_0u is proximal to every point of X_0v.*

Proof. Consider an arbitrary element x_0hv of X_0v. We wish to show that it is proximal to an arbitrary element x_0gu of X_0u. Now x_0u is proximal to $x_0g^{-1}hv$. As in 1.2, take an idempotent v^* in E such that $vv^* = v$, $v^*v = v^*$, $x_0uv^* = x_0g^{-1}hv$; so $uv^* = fg^{-1}hv$ for some f with $x_0f = x_0$. So $guv^* = gfg^{-1}hv$, and $x_0guv^* = x_0(gfg^{-1})hv = x_0hv$. Also $x_0hvv^* = x_0hv$. Thus they are proximal.

(1.8) *Example.* Let K be the unit circle and T be the free group on two elements a, b. Consider the transformation group (Y, T) where $Y = K$ and $ya = y\alpha$, α an irrational rotation, $yb = \exp(2\pi i r^2)$, if $y = \exp(2\pi i r)$. Then (Y, T) is minimal and proximal. Let $\phi: K \to K$ be defined by $\phi(k) = k^4$ and (X, T) be the minimal set with $X = K$ and $xa = x\alpha^{1/4}$,

$$xb = \exp\{2\pi i[4(r - (n/4))^2 + (n/4)]\},$$

if $x = \exp(2\pi i r)$ and $n/4 \leq r \leq (n + 1)/4$, $n = 0, 1, 2, 3$. Then $Q_X(x) = X \backslash \{x^{-1}\}$ where x^{-1} is the antipodal point to x on the circle X, $S_X = X \times X$, and (X, T) is not weakly mixing ($(X \times X, T)$ does not contain a transitive point). (Note that if x and x' are an arc of length $\pi/2$ apart, then xt and $x't$ will be an arc of length $\pi/2$ apart.)

(1.9) PROPOSITION. *If X is metric and $P(x)$ residual in X for every $x \in X$, then D is either dense in $X \times X$ or D is nowhere dense. Indeed any open invariant set which intersects D is dense.*

Proof. Let $B = \{(x_0, y_0) \in X \times X: D \subseteq N = \text{cls}((x_0, y_0)T)\}$. From the proof of 1.2 we see that B is dense in $X \times X$ since the assumption that $D(\phi)$ is dense is used only in the last line of the proof and the condition that $P(x)$ is residual may be used as in Lemma 1.1 to insure that for any idempotent u in J and any $y_0 \in Xu$ (thus effectively for any $y_0 \in X = (Xu)J$) there is a residual set of points x_0 with $D \subseteq \text{cls}((x_0, y_0)T)$.

Now if U is an open invariant set in $X \times X$ which intersects D, then for each b in B, $bt \in U$ for some t, so $b \in U$ and thus U is dense.

If D is not a nowhere dense set, then cls D contains an open set and thus contains an open invariant set since D is invariant. Thus D would be dense.

(1.10) *Remark.* The proof of the structure theorem in [6] proceeds through a series of steps so that if X is PD, it is not necessary that $X^* = X$, the following is an example of how this can occur; note that by 3.9 of [6], if X is metric and PD, then $X^* = Y$.

(1.11) *Example.* We will construct the desired minimal set from a less complicated one following the approach taken in [11].

We now present that approach. Let (X, T) be a minimal set with T a discrete group. Fix x_0 in X, let B be the Stone-Cech compactification of x_0T and let p be the extension of the inclusion map of x_0T into X. Form the transformation group (B, T) where the action is the extension to B of the action on x_0T, and note that $p^{-1}(x)$ is a singleton for x in x_0T and so p is a proximal homomorphism. Now if $x \in x_0T$ and $b \in B$ with $p(b) = x$, then b is an almost periodic point since for any open set V containing b, there exists an open set U containing x with $p^{-1}(U) \subseteq V$ and there is a syndetic set $S \subseteq T$ with $xS \subseteq U$, and so $bS \subseteq p^{-1}(U) \subseteq V$. Also clearly bT is dense in B for $p(b) \in x_0T$, so (B, T) is minimal.

Now suppose f is a bounded real-valued function that is continuous on x_0T in X and f^* is its unique continuous extension to B. Then the set

$$R_f = \{(b, d) \in B \times B: p(b) = p(d) \text{ and } f^*(bt) = f^*(dt) \text{ for every } t \text{ in } T\}$$

is a closed invariant equivalence relation. Let $(X_f, T) = (B/R_f, T)$.

Let K be the unit circle in the complex plane and a and b be the homeomorphism defined by $ka = k\alpha$ where $\alpha \in K$ such that α' is a transcendental number, where $\alpha = \exp(2\pi i\alpha')$, and

$$kb = \exp(2\pi i[2(r - (n/2))^2 + (n/2)])$$

if $k = \exp(2\pi ir)$ and $n/2 \leq r < (n + 1)/2$, $n = 0, 1$ (compare this with b's action on X in Example 1.8; points a distance of π apart on an arc preserve that distance). Let T be the group of homeomorphisms of K generated by a and b. Note the requirement that α' be transcendental implies (K, T) is minimal and also that $-1 \notin 1T$ (since for t in T, $1t = \exp(2\pi i\gamma)$ where γ is some polynomial in α' with rational coefficients). Define f by $f(k) = r$ where $k = \exp(2\pi ir)$, $0 \leq r \leq 1$. Then f is continuous except at 1. Let $k_0 = -1$ and consider (K_f, T) as constructed above. Let $(X, T) = (K_f, T)$. Then in the structure theorem of [6], we first consider Xu for some idempotent u in the minimal right ideal I of the enveloping semigroup of (X, T). Xu consists of two points x, x' with $p(x)$ and $p(x')$ being antipodal points of the circle K. Then $Z = \overline{Xu}\, u = Xu$ in S (the set of closed subsets of X endowed with the Hausdorff topology). Then $(Y, T) = (ZI, T)$ is (W_g, T) where (W, T) is the factor of (K, T) under the map $k \to k^2$ (W then is K) and $g(w) = r$ where $w = \exp(2\pi ir)$, $0 \leq r \leq 1$.

Now (X^*, T) is the minimal set $((x_0, Z)I, T)$ in $(X \times S, T)$ and is the distal extension of (Y, T) via the map $k \to k^2$. (X^*, T) is the proximal extension of (X, T) which has singleton fiber above all points of X except the points on the orbit of $-1T$ over which the fibers have two elements.

Section 2

The following examples provide a study of various conditions that are useful in studying the relativized problems and in generalizing Abelian.

(2.0) LEMMA. *If (X, T) is minimal, $\phi: (X, T) \to (Y, T)$, and, for each y in Y, $\phi^{-1}(y)$ is dense in $\phi^{-1}(y)$ for every $v \in J$ with $yv = y$, then ϕ is open.*

Proof. Let V be an open set and $x \in V$. We wish to show that $\phi(V)$ is a neighborhood of $\phi(x) = y$. Let U be an open neighborhood of x whose closure is contained in V. Suppose $\phi(V)$ is not a neighborhood of y. Then there is a net y_n in Y with $y_n \to y$ and $y_n \notin \phi(V) \supseteq \phi(\text{cls } U)$. So there exists a net t_n in T with $yt_n \notin \phi(\text{cls } U)$. Now let (M, T) be the universal minimal set and J be the set of idempotents in M. Let $u \in J$ with $yu = y$ and let t_m be a subset of t_n with ut_m converging in M. Suppose $ut_m \to pv$, $v \in J$. Note $yut_m \to y$, so $ypv = y$.

Now since $\phi^{-1}(y)v$ is dense in $\phi^{-1}(y)$, there exists

$$xv \in \phi^{-1}(y)v \cap U.$$

Then $(xv)p^{-1}ut_m \in \phi^{-1}(y)vut_m = \phi^{-1}(y)ut_m \subseteq \phi^{-1}(yt_m)$ and converges to xv (where p^{-1} the inverse of p in the group Mv). So for some m, $(xv)p^{-1}ut_m \in U$, but this is a contradiction since $yt_m = xvp^{-1}ut_m$ and $yt_m \notin \phi(U)$.

The homomorphism $\phi_{f'}$ of 2.1.1 shows that the condition of 2.0 cannot be reduced to just one idempotent and ϕ_f shows that the converse that with ϕ_f open, $\phi^{-1}(y)v$ dense in $\phi^{-1}(y)$ for some v implies it is dense for every v, does not hold.

The assumption that $\phi^{-1}(y)v$ is dense in $\phi^{-1}(y)$ plays an important role in relativized disjointness as (3.12) of [13] illustrates.

PROPOSITION (3.12 of [13]). *Suppose the homomorphism $\phi: (X, T) \to (Z, T)$ is PD and open and $\phi^{-1}(z)v$ is dense in $\phi^{-1}(z)$ for some z in Z and some idempotent v. Let (Y, T) be any extension of (Z, T). Then $X \perp^Z Y$ iff $X_1 \perp Y_1$ where (X_1, T) and (Y_1, T) are respectively the maximal distal factors of (X, T) and (Y, T) relative to (Z, T).*

(2.1) *Example.* Let (Y, S) be the equicontinuous minimal set consisting of the Cantor set given the discrete topology acting on the Cantor set by right multiplication. Let (W, S') be a *POD* flow such that S' is the group of integers and acts freely on W and W is homeomorphic to the Cantor set. Let $(X, T) = (Y \times W, S \times S')$ where the action is defined by $(y, w)(s, s') = (ys, ws')$. Note that (X, T) is a minimal set with $xT \neq X$ for any x in X and with T a discrete Abelian group acting freely on X. Consider (Y, T) with the action defined by $y(s, s') = ys$ for $y \in Y$ and $(s, s') \in T$. Let q be the homomorphism of (X, T) onto (Y, T) taking (y, w) to y. Given a function that is continuous except at one point x_1, we take a point x_0 not on the orbit of x_1 and construct (X_f, T) as in 1.11; then we will remark on the properties of $\phi_f = q \circ p_f$ where

$$p_f: (X_f, T) \to (X, T)$$

is the homomorphism induced by $p: (B, T) \to (X, T)$.

We will now show that (X, T) and (X_f, T) are minimal right ideals. Suppose (y, w) and (y', w') are in X, then there exist s in S and s' in S' such that $ys = y'$ and ws' is proximal to w' and so we have a homomorphism (s, s') of (X, T) onto itself taking (y, w) proximal to (y', w'); by [1] this implies that (X, T) is a minimal right ideal. Now suppose x and x' are in X_f and take t in T such that $p_f(x)t$ is proximal to $p_f(x')$, then since p_f is a proximal homomorphism xt is proximal to x' and (X_f, T) is a minimal right ideal.

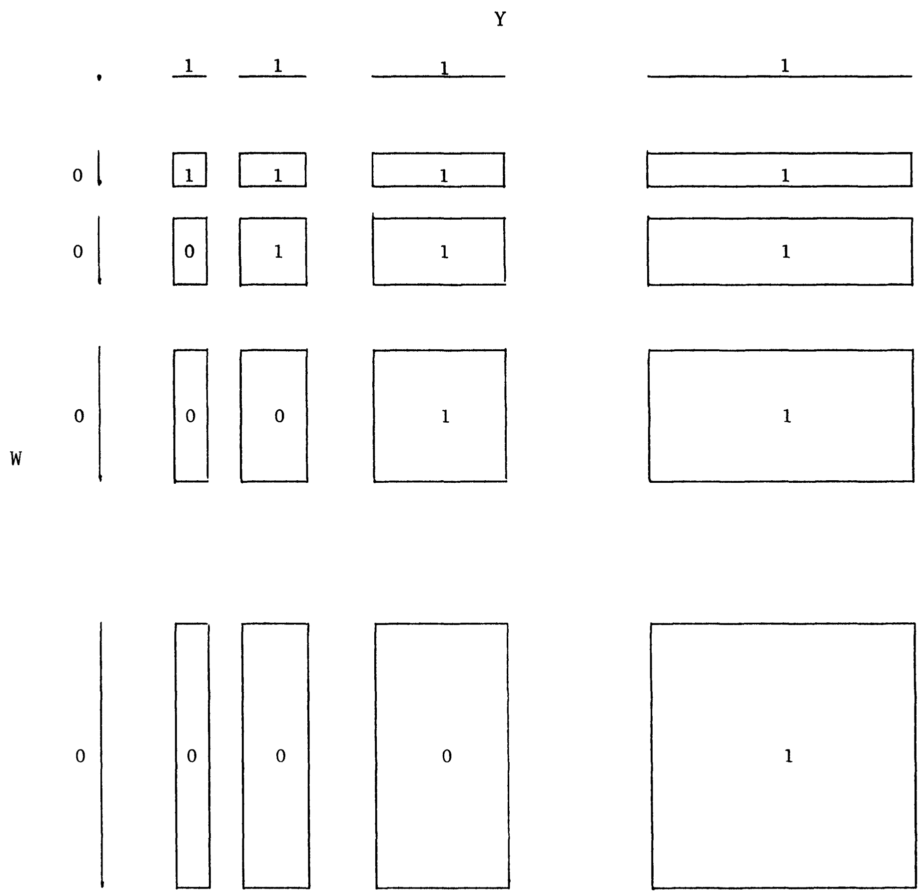
Fix a point d in the Cantor set, C. Then $C \backslash \{d\}$ is a disjoint union of a countable number of open-closed sets, C_i. For b in $C \backslash \{d\}$ define $h(b) = i$, if $b \in C_i$ and $h(d) = \infty$. Pictorially

d $\quad h(b) = 4 \quad h(b) = 3 \quad h(b) = 2 \quad h(b) = 1$

Define

$$f(y, w) = \begin{cases} 0 & \text{if } h(y) > h(w). \\ 1 & \text{if } h(y) \le h(w) \end{cases} \tag{1}$$

Pictorially f takes on the values



Then for some u in J, X_0u is dense in X_0, X_0v is not dense in X_0 for v in J with $v \neq u$, $D(\phi_f)$ is dense in $R(\phi_f)$, and ϕ_f is open.

Also note that ϕ_f is weakly mixing and illustrates 2.11 of [6] referred to in Lemma 1.1 above, that is for x in X_0, $P(\phi_f)(x)$ is not residual in X_0 but for some closed subset $\hat{X}$ of X_0, $P(\phi_f)(x) \cap \hat{X}$ is residual in $\hat{X}$.

We can obtain the same results except that ϕ_f is not open by defining

$$f'(b, c) = \begin{cases} 0 & \text{if } h(y) > h(w) \text{ or } h(y) \text{ is odd} \\ 1 & \text{if } h(y) \leq h(w) \text{ and } h(y) \text{ is even} \end{cases}$$

Define

$$(2) \qquad g(y, w) = \begin{matrix} 0 & \text{if } h(y) > h(w) & \\ 1 & \text{if } h(y) = h(w) \text{ and } h(y) = 0 & (3) \\ 2 & \text{if } h(y) < h(w) \text{ and } h(y) = 0 & (3) \\ 1 & \text{if } h(y) = h(w) \text{ and } h(y) = 1 & (3) \\ 3 & \text{if } h(y) < h(w) \text{ and } h(y) = 1 & (3) \\ 2 & \text{if } h(y) = h(w) \text{ and } h(y) = 2 & (3) \\ 3 & \text{if } h(y) < h(w) \text{ and } h(y) = 2 & (3) \end{matrix}$$

where $m = n$ (3) means m is congruent to n module 3. Then $D_2(\phi_g)$ is dense in $R_2(\phi_g)$ but $D_3(\phi_g)$ is not dense in $R_3(\phi_g)$. Also ϕ_g is weakly mixing, but $(R_3(\phi_g), T)$ is not point-transitive.

(2.2) *Example.* The following example has D dense in $X \times X$ and X is proximally equicontinuous but not locally almost periodic. Let $X = K \times K$ be the torus and let T be the free group on three elements a, b, c. Let T act on X as follows: $(x, y)a = (x\alpha, y\beta)$, $\alpha, \beta \in K$ such that $(X, \{a^n\})$ is minimal; $(x, y)b = (x, \exp(2\pi i r^2))$ if $y = \exp(2\pi i r)$; $(x, y)c = (x, xy)$. Note the purpose of b is to make (x, K) a proximal cell. The purpose of c is to make D dense in $X \times X$. It works this way: we will show that $((1, y), (x, z)) \in \bar{D}$. Fix $(1, y)$ and (x, z). Given $\varepsilon > 0$, take x' such that $d(x', x) < \varepsilon/2$ and $(K, \{(x')^n\})$ is minimal and take y' such that $((1, y), (x', y'))$ is an almost periodic point in $(X \times X, T)$. Note y' exists since the first projection is an equivariant homomorphism to an equicontinuous transformation group. Now $(x', y')c^m = (x', y'(x')^m)$ so since $(K, \{(x')^n\})$ is minimal, we can choose m such that $y'(x')^m$ is within $\varepsilon/2$ of z. Then

$$d((x, z), (x', y')c^m) < \varepsilon.$$

Also $((1, y), (x', y')c^m)$ is an almost periodic point of $(X \times X, T)$ since $((1, y), (x', y'))$ is. (Note $(1, y)c^m = (1, y)$. In general, if u is an idempotent in $(\beta(T), T)$ fixing $(x_1, x_2) \in X \times X$, then $t^{-1}ut$ is an idempotent fixing (x_1t, x_2t) and by 3.7 of [1], (x_1t, x_2t) is an almost periodic point in $(X \times X, T)$.) Thus $((1, y), (x, z)) \in \bar{D}$. And so clearly $\bar{D} = X \times X$. Note (X, T) is proximally equicontinuous. It is not locally almost periodic since there are no distal points [6].

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