$M$-PROJECTIVE AND STRONGLY $M$-PROJECTIVE MODULES

BY

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Introduction

Given a module $M$ over a ring $R$, G. Azumaya [1] introduced the dual notions of $M$-projective and $M$-injective modules. These concepts have actually led M. S. Shrikhande to a study of hereditary and cohereditary modules [5]. More recently Azumaya, Mbuntum and the present author obtained necessary and sufficient conditions for the direct sum $\bigoplus_{\sigma \in J} A_{\sigma}$ of a family of modules to be $M$-injective [2]. While $R$-injective modules are the same as injective modules over $R$, the class of $R$-projective modules in the sense of Azumaya in general is larger than the class of projective $R$-modules. In this paper we introduce the notion of a strongly $M$-projective module and the associated notion of a strong $M$-projective cover. Next we investigate strong $M$-projective covers. We show that if every module possesses a strong $M$-projective cover then $R/\mathfrak{A}(M)$ is (left) perfect, where $\mathfrak{A}(M)$ is the annihilator of $M$. If $R/\mathfrak{A}(M)$ is perfect, we show that every $R$-module $A$ with $t_M(A) = 0$ possesses a strong $M$-projective cover, where

$$t_M(A) = \{x \in A \mid f(x) = 0 \text{ for all } f \in \text{Hom}(A, M)\}.$$

Another application of the ideas here is the result that if $\mathfrak{A}(M) = 0$, then an $R$-module $B$ is strongly $M$-projective iff $B$ is projective. In particular if $R$ is (left) perfect and $\mathfrak{A}(M) = 0$, then an $R$-module $B$ is $M$-projective iff $B$ is actually projective. Since $\mathfrak{A}(R) = 0$, we can regard this result as a generalization of the "known" result that when $R$ is perfect an $R$-module is $R$-projective iff it is projective. It will be interesting to characterise the rings with the property that $R$-projective modules are the same as the projective modules over $R$.

1. Preliminaries

Throughout this paper $R$ denotes a ring with $1 \neq 0$, $R$-mod the category of unital left modules. All the modules we deal with are unital left modules. $M$ denotes a fixed object in $R$-mod. We recall briefly the concepts of $M$-projective and $M$-injective modules introduced by G. Azumaya and state two results due to him [1].

Definition 1.1. A module $P$ is called $M$-projective if given any epimorphism $\phi: M \to N$ and any $f: P \to N$, there exists a $g: P \to M$ such that $\phi \circ g = f$. 

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An $M$-injective module is defined dually.

**DEFINITION 1.2.** An epimorphism $\psi: A \to B$ is called an $M$-epimorphism if there exists a map $h: A \to M$ such that $\ker \psi \cap \ker h = 0$.

$M$-monomorphisms are defined dually.

**PROPOSITION 1.3.** [1] Let $P \in R$-mod. Then the following statements are equivalent.

1. $P$ is $M$-projective.
2. Given any $M$-epimorphism $\psi: A \to B$ and any $f: P \to B$, there exists a $g: P \to A$ such that $\psi \circ g = f$.
3. Every $M$-epimorphism onto $P$ splits.

The dual of this proposition characterises $M$-injective modules.

**DEFINITION 1.4.** $C_p(M)$ is the class of all $M$-projective modules, $C_i(M)$ is the class of all $M$-injective modules. For any $A \in R$-mod,

$$C^p(A) = \{ M \in R$-mod $| A$ is $M$-projective $\}$$

and

$$C^i(A) = \{ M \in R$-mod $| A$ is $M$-injective $\}$$

**PROPOSITION 1.5.** [1] (1) $C_p(M)$ is closed under the formation of direct sums and direct summands.

2. $C_i(M)$ is closed under the formation of direct products and direct factors.

3. $C^p(A)$ is closed under submodules, homomorphic images and formation of finite direct sums. If $A$ has a projective cover, $C^p(A)$ is closed under the formation of arbitrary direct products (and hence arbitrary direct sums as well).

4. $C^i(A)$ is closed under submodules, homomorphic images and arbitrary direct sums.

In this paper the term $R$-projective module will be used to denote a module which is $R$-projective in the sense of Definition 1.1. As has already been pointed out in [2] the class of $R$-projective modules in general is larger than the class of projective $R$-modules.

**LEMMA 1.6.** Let $A \in C_p(M)$, $K \subseteq A$ and $i: K \to A$ the inclusion. If

$$i^*: \text{Hom}(A, M) \to \text{Hom}(K, M)$$

is the zero map then $A/K \in C_p(M)$.

**Proof.** Write $B$ for $A/K$ and let $\eta: A \to B$ denote the canonical quotient map. Let $\phi: M \to N$ be any epimorphism and $f: B \to N$ any map. Since $A \in C_p(M)$, there exists a map $g: A \to M$ such that $\phi \circ g = f \circ \eta$. Now, $g \circ i = i^*(g) = 0$.

Hence $g$ induces a map $\bar{g}: B \to M$ satisfying $\bar{g} \circ \eta = g$. It is clear that $\phi \circ \bar{g} = f$.

Recall that an epimorphism $\alpha: A \to B$ is called minimal if $\text{Ker} \, \alpha$ is small in $A$. 

LEMMA 1.7. Any minimal M-epimorphism \( \alpha: A \to B \) with \( B \in C_p(M) \) is an isomorphism.

Proof. By (3) of Proposition 1.3, \( \alpha \) splits. Thus \( \ker \alpha \) is a direct summand of \( A \). Since \( \ker \alpha \) is small in \( A \) we see that \( \ker \alpha = 0 \).

LEMMA 1.8. Let

\[ 0 \to K \xrightarrow{i} A \xrightarrow{\phi} B \to 0 \]

be exact with \( i(K) \) small in \( A \). If \( B \in C_p(M) \), then \( i^*: \text{Hom}(A,M) \to \text{Hom}(K,M) \) is the zero map.

Proof. Let \( f \in \text{Hom}(A,M) \). Writing \( L \) for \( K \cap \ker f \) we get an exact sequence

\[ 0 \to K/L \xrightarrow{i} A/L \xrightarrow{\phi} B \to 0 \]

where \( i \) and \( \phi \) are induced by \( i \) and \( \phi \) respectively. If \( \overline{f}: A/L \to M \) is induced by \( f \), it is clear that \( \ker \overline{f} \cap \ker \phi = 0 \). Thus \( \overline{\phi}: A/L \to B \) is an M-epimorphism. Moreover \( i(K/L) \) is small in \( A/L \). Lemma 1.7 now implies that \( \overline{\phi} \) is an isomorphism and hence \( K/L = 0 \). Thus, \( L = K \) and \( i^*(f) = f \circ i = f/K = 0 \).

2. Strongly M-projective modules

Given any set \( J \) and any \( A \in R\text{-mod} \), we write \( A^J \) for the direct product \( \prod_{\alpha \in J} A_\alpha \) and \( A^{(J)} \) for the direct sum \( \oplus_{\alpha \in J} A_\alpha \), where \( A_\alpha = A \) for each \( \alpha \in J \). The annihilator of \( A \) will be denoted by \( \mathcal{A}(A) \).

DEFINITION 2.1 A module \( A \) is called strongly M-projective if \( A \in C_p(M^J) \) for every indexing set \( J \).

Trivially every projective module is strongly M-projective for every \( M \in R\text{-mod} \). From the second half of (3) of Proposition 1.5 we get the following as an immediate consequence.

LEMMA 2.2 Let \( A \in C_p(M) \). If \( A \) possesses a projective cover, then \( A \) is strongly M-projective.

DEFINITION 2.3. A submodule \( K \) of \( A \) is said to be M-independent in \( A \) if given any \( x \neq 0 \) in \( K \), there exists an \( f \in \text{Hom}(A,M) \) such that \( f(x) \neq 0 \).

If \( K = 0 \), the condition stated in Definition 2.3 is emptyly satisfied. Also if \( L \subset K \subset B \subset A \) and \( K \) is M-independent in \( A \), then trivially \( L \) is seen to be M-independent in \( B \).

DEFINITION 2.4. A homomorphism \( f: A \to B \) is called M-independent if \( \ker f \) is M-independent in \( A \).

LEMMA 2.5. Let \( \phi: A \to B \) be an M-independent epimorphism and \( L = \ker \phi \). Then \( \phi \) is an \( M^L \)-epimorphism.
Proof. For any $x \neq 0$ in $L$ let $f_x: A \to M$ be such that $f_x(x) \neq 0$. Let $f_0: A \to M$ be the zero map. Let $h: A \to M^L$ be defined by $h(a) = (f_x(a))_{x \in L}$. Then $\ker h \cap \ker \phi = 0$.

For any $A \in R$-mod, let $t_M(A) = \{ x \in A \mid f(x) = 0 \text{ for all } f \in \text{Hom}(A, M) \}$. Then $t_M(R) = \mathfrak{M}(M)$. It is clear that $A$ is $M$-independent in itself if and only if $t_M(A) = 0$.

**Definition 2.6.** An object $A \in R$-mod is called $M$-independent if $t_M(A) = 0$.

**Remark 2.7.** (a) Given $x \in A$ with $x \notin t_M(A)$, there exists an $f: A \to M$ with $f(x) \neq 0$. Since $f/t_M(A) = 0$, we get an induced map $\tilde{f}: A/t_M(A) \to M$. Clearly $\tilde{f}(x + t_M(A)) \neq 0$. Thus $A/t_M(A)$ is $M$-independent in itself. In other words $t_M(A/t_M(A)) = 0$. For any $g: A \to B$ it is clear that $g(t_M(A)) \subseteq t_M(B)$. Thus $t_M$ is a radical on $R$-mod in the sense of Bo-Stenström [6, Chap 1]. However, $t_M$ is neither left exact, nor idempotent. For instance consider $t = t_{Z_p}$ on $Z$-mod, where $Z_p = Z/pZ$. Then $t(Z) = pZ$, $t(pZ) = p^2Z$. Thus

$$t(Z) \cap pZ = pZ \neq p^2Z = t(pZ).$$

Also $t(t(Z)) = p^2Z \neq t(Z)$. This is just to impress upon the reader that $M$-projectivity and $M$-injectivity can not in general be “subsumed” under “torsion theories”.

(b) When $M$ is injective $t_M$ is the radical associated to a hereditary torsion theory on $R$-mod.

It is easily seen that every $A \in R$-mod is $M$-projective iff $M$ is semi-simple iff every $A \in R$-mod is $M$-injective. The next theorem gives conditions under which every $A \in R$-mod is strongly $M$-projective.

**Theorem 2.8.** The following statements are equivalent.

1. Every $R$-module is strongly $M$-projective.
2. Every cyclic $R$-module is strongly $M$-projective.
3. $R/\mathfrak{M}(M)$ is a semisimple Artinian ring.
4. $M^J$ is a semisimple $R$-module for every indexing set $J$.

**Proof:** (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3). Any left ideal of $R/\mathfrak{M}(M)$ is of the form $I/\mathfrak{M}(M)$ with $I$ a left ideal of $R$ satisfying $I \supseteq \mathfrak{M}(M)$. Let $\eta: R/\mathfrak{M}(M) \to R/I$ denote the quotient map. Then $\ker \eta = I/\mathfrak{M}(M)$. Since $R/\mathfrak{M}(M)$ is $M$-independent in itself it follows that $I/\mathfrak{M}(M)$ is $M$-independent in $R/\mathfrak{M}(M)$. If we write $K$ for $I/\mathfrak{M}(M)$, from Lemma 2.5 it follows that $\eta$ is an $M^K$-epimorphism. Assumption (2) implies that $R/I \in C_s(M^K)$. An application of (3), Proposition 1.3 shows that $\eta: R/\mathfrak{M}(M) \to R/I$ splits in $R$-mod and hence in $R/\mathfrak{M}(M)$-mod. Thus $R/\mathfrak{M}(M)$ is a direct summand of $R/\mathfrak{M}(M)$ as an $R/\mathfrak{M}(M)$-module.

(3) $\Rightarrow$ (4). Since $\mathfrak{M}(M)M^J = 0$ (for any indexing set $J$) we can regard $M^J$ as an $R/\mathfrak{M}(M)$-module. The $R$-submodules of $M^J$ are the same as the $R/\mathfrak{M}(M)$-submodules.
submodules of \( M^J \). The semisimplicity of \( R/\mathfrak{M}(M) \) implies that \( M^J \) is semi-
simple as an \( R/\mathfrak{M}(M) \)-module and hence as an \( R \)-module also.

(4) \( \Rightarrow \) (1) is trivial.

Remark 2.9. \( M = \bigoplus_p \mathbb{Z}_p \) (direct sum over all the primes \( p \)) is an example
of a semisimple \( \mathbb{Z} \)-module for which \( \mathbb{Z}/\mathfrak{M}(M) = \mathbb{Z} \) is not semisimple.

Proposition 2.10. If every \( M \)-independent \( R \)-module is injective then \( R/\mathfrak{M}(M) \)
is a semisimple ring.

Proof. Since \( R/\mathfrak{M}(M) \) is \( M \)-independent, any left ideal of \( R/\mathfrak{M}(M) \) being a
submodule of \( R/\mathfrak{M}(M) \) is \( M \)-independent, and hence injective as an \( R \)-module.
Thus every left ideal of \( R/\mathfrak{M}(M) \) is an \( R \)-direct summand and hence an \( R/\mathfrak{M}(M) \)
direct summand of \( R/\mathfrak{M}(M) \).

Lemma 2.11. For any \( A \in R \text{-mod} \) we have \( \mathfrak{M}(M)A \subset t_M(A) \).

Proof. Trivial.

Remark 2.12. If \( A \) is any \( M \)-independent \( R \)-module, from Lemma 2.11 we see
that \( \mathfrak{M}(M)A = 0 \). Hence \( A \) can be regarded as an \( R/\mathfrak{M}(M) \)-module in a natural
way. If \( R/\mathfrak{M}(M) \) is semisimple Artin (as a ring) then \( A \) is injective as an \( R/\mathfrak{M}(M) \)-
module. But in general \( A \) need not be injective as an \( R \)-module. Thus the con-
verse of Proposition 2.10 is not true. For instance let \( M = \mathbb{Z}_p \) in \( \mathbb{Z} \text{-mod} \) and
\( A = \mathbb{Z}_p \). Then \( \mathfrak{M}(M) = p\mathbb{Z} \) and \( \mathbb{Z}/\mathfrak{M}(M) = \mathbb{Z}_p \) is a field. Also \( t_M(\mathbb{Z}_p) =
t\mathbb{Z}_p(\mathbb{Z}_p) = 0 \). However \( \mathbb{Z}_p \) is not injective as a \( \mathbb{Z} \)-module.

When \( M \) is an injective \( R \)-module the converse of Proposition 2.10 is valid.

Proposition 2.13. Let \( M \) be an injective \( R \)-module such that \( R/\mathfrak{M}(M) \) is a
semisimple ring. Then any \( M \)-independent \( R \)-module is injective

Proof. Let \( A \) be any \( M \)-independent \( R \)-module. Let \( I \) be any left ideal in
\( R \) and \( f: I \to A \) any map. We will show that \( f(I \cap \mathfrak{M}(M)) = 0 \) using the fact
that \( M \) is an injective \( R \)-module. Suppose on the contrary \( f(\lambda) \neq 0 \) for some
\( \lambda \in I \cap \mathfrak{M}(M) \). Since \( t_M(A) = 0 \) we can find a \( g: A \to M \) with \( g(f(\lambda)) \neq 0 \).
Since \( M \) is injective, there exists an \( h: R \to M \) such that \( h|I = g \circ f \). Then
\( 0 \neq g(f(\lambda)) = h(\lambda) = h(\lambda \cdot 1) = \lambda h(1) = 0 \) since \( \lambda \in \mathfrak{M}(M) \) and \( h(1) \in M \).
This contradiction shows that \( f(I \cap \mathfrak{M}(M)) = 0 \).

Thus \( f \) induces a map \( \bar{f}: I/I \cap \mathfrak{M}(M) \to A \). Clearly \( \bar{f} \) is an \( R/\mathfrak{M}(M) \)-map.
The semisimplicity of \( R/\mathfrak{M}(M) \) implies that \( \bar{f} \) can be extended to an \( R/\mathfrak{M}(M) \)
homomorphism \( \theta: R/\mathfrak{M}(M) \to A \). If \( \eta: R \to R/\mathfrak{M}(M) \) is the canonical quotient
map, then it is clear that \( \theta \circ \eta: R \to A \) is an \( R \)-homomorphism extending
\( f: I \to A \). Thus \( A \) is an injective \( R \)-module.

Combining Propositions 2.10 and 2.13 we get the following:

Corollary 2.14. When \( M \) is injective, each of the statements (1), (2), (3), (4)
of Theorem 2.8 is equivalent to (5) stated below:

(5) Every \( M \)-independent \( R \)-module is injective.
3. Strong \( M \)-projective covers

**Definition 3.1.** A minimal epimorphism \( \alpha: A \to B \) is called a strong \( M \)-projective cover if

1. \( A \) is strongly \( M \)-projective and
2. \( \alpha \) is \( M \)-independent (in the sense of Definition 2.4)

As in the case of projective covers, strong \( M \)-projective covers do not exist in general. Conditions for existence will be investigated presently. But before that we will prove the essential uniqueness of a strong \( M \)-projective cover when it exists.

**Lemma 3.2.** Suppose \( \alpha: A \to B \) is a strong \( M \)-projective cover and \( \pi: P \to B \) an epimorphism with \( P \) strongly \( M \)-projective. Then there exists an epimorphism \( h: P \to A \) satisfying \( \pi \circ h = \alpha \).

**Proof.** Let \( L = \ker \alpha \). Since \( \alpha \) is \( M \)-independent, from Lemma 2.5 we see that \( \alpha \) is an \( M' \)-epimorphism. Since \( P \in \mathcal{C}_p(M^J) \), by (2) of Proposition 1.3 we get a map \( h: P \to A \) satisfying \( \alpha \circ h = \pi \). Since \( \pi \) is onto, we get \( \text{Im} h + L = A \).

The smallness of \( L \) in \( A \) gives \( \text{Im} h A \).

**Proposition 3.3.** Suppose \( \alpha_1: A_1 \to B \), \( \alpha_2: A_2 \to B \) are any two strong \( M \)-projective covers of \( B \). Then there exists an isomorphism \( h: A_1 \to A_2 \) such that \( \alpha_2 \circ h = \alpha_1 \).

**Proof.** By Lemma 3.2, there exists an epimorphism \( h: A_1 \to A_2 \) satisfying \( \alpha_2 \circ h = \alpha_1 \). If \( K_1 = \ker \alpha_1 \), \( K = \ker h \) from \( \alpha_2 \circ h = \alpha_1 \) we immediately get \( K \subset K_1 \). Hence \( K \) is \( M \)-independent in \( A_1 \) and is also small in \( A_1 \). Lemma 2.5 now implies that \( h \) is a minimal \( M^K \)-epimorphism. Since \( A_2 \in \mathcal{C}_p(M^K) \), an application of Lemma 1.7 yields that \( h \) is an isomorphism.

We next show that any \( B \in \text{R-mod} \) which possesses a projective cover automatically admits a strong \( M \)-projective cover. We will actually indicate a method of constructing a strong \( M \)-projective cover of \( B \) from a given projective cover of \( B \).

**Theorem 3.4.** Suppose \( B \) has a projective cover \( \pi: P \to B \). Let \( L = \ker \pi \) and

\[ T = \{ x \in L \mid f(x) = 0 \text{ for all } f \in \text{Hom} (P, M) \}. \]

Let \( \alpha: P/T \to B \) be the map induced by \( \pi \). Then \( \alpha: P/T \to B \) is a strong \( M \)-projective cover of \( B \).

**Proof.** If \( i: T \to P \) denotes the inclusion of \( T \) in \( P \), from the very definition of \( T \) we have \( i^* : \text{Hom} (P, M) \to \text{Hom} (T, M) \) to be the zero homomorphism. By Lemma 1.6 we see that \( P/T \in \mathcal{C}_p(M) \). Clearly \( T \) is small in \( P \). Hence the canonical quotient map \( \eta: P \to P/T \) is a projective cover of \( P/T \). Lemma 2.2 now yields \( P/T \in \mathcal{C}_p(M^J) \) for every set \( J \). It is easily seen that \( L/T \) is \( M \)-inde-
pendent in $P/T$. In addition $L/T$ is small in $P/T$. This proves that $\alpha: P/T \to B$ is a strong $M$-projective cover of $B$.

**Corollary 3.5.** If $R$ is left perfect (resp. semiperfect) every module (resp. cyclic module) over $R$ possesses a strong $M$-projective cover.

**Proposition 3.6.** Suppose $M \in R\text{-mod}$ satisfies $\mathfrak{U}(M) = 0$. Then $B \in R\text{-mod}$ is strongly $M$-projective iff $B$ is projective.

**Proof.** The implication $\Leftarrow$ is trivial. As for the implication $\Rightarrow$, let $B$ be strongly $M$-projective. Let

$$0 \longrightarrow K \xrightarrow{i} F \xrightarrow{\phi} B \longrightarrow 0$$

be an exact sequence in $R\text{-mod}$ with $F$ free. Let $\{e_x\}_{x \in J}$ be as basis for $F$. Suppose $0 \neq x \in k$. Then $x = \sum \lambda_x e_x$ with at least one $\lambda_x \neq 0$. Since $\mathfrak{U}(M) = 0$ there exists a $g_x: R \to M$ with $g_x(\lambda_x) \neq 0$. Then $h: F \to M$ given by $h | Re = g_x$, $h | Re_\beta = 0$ for $\beta \neq \alpha$ clearly satisfies $h(x) \neq 0$. Thus $K$ is $M$-independent in $F$. By Lemma 2.5, $\phi$ is an $M^K$-epimorphism. Since $B \in C_p(M^K)$, by (3) of Proposition 1.3 we see that $\phi$ splits. Hence $B$ is projective.

**Corollary 3.7.** Let $M \in R\text{-mod}$ be such that $\mathfrak{U}(M) = 0$. Suppose $B$ is an $R$-module possessing a projective cover. Then $B$ is projective iff $B$ is $M$-projective.

**Proof.** We have only to prove the implication $\Leftarrow$. This is immediate from Lemma 2.2 and Proposition 3.6.

Any $R$-module $B$ satisfying $\mathfrak{U}(M)B = 0$ can be regarded as an $R/\mathfrak{U}(M)$-module. In particular this is the case if $t_M(B) = 0$ by Lemma 2.11.

**Lemma 3.8.** Suppose $B \in R\text{-mod}$ satisfies $\mathfrak{U}(M)B = 0$. Then $B$ is strongly $M$-projective iff as an $R/\mathfrak{U}(M)$-module $B$ is projective.

**Proof.** From $\mathfrak{U}(M)M^J = 0$ we see that $M^J$ is an $R/\mathfrak{U}(M)$-module, (whatever be the indexing set $J$). Also it is clear that for any $A \in R\text{-mod}$ satisfying $\mathfrak{U}(M) = 0$, the $R$-submodules of $A$ are the same as the $R/\mathfrak{U}(M)$-submodules of $A$. It follows from this comment that $B$ is strongly $M$-projective in $R\text{-mod}$ iff $B$ is strongly $M$-projective in $R/\mathfrak{U}(M)$-mod. The annihilator $\mathfrak{U}_{R/\mathfrak{U}(M)}(M)$ of $M$ as an $R/\mathfrak{U}(M)$-module is clearly seen to be zero. Lemma 3.8 now follows from Proposition 3.6.

**Theorem 3.9.** The following statements are equivalent.

1. Every $B \in R\text{-mod}$ satisfying $\mathfrak{U}(M)B = 0$, possesses a strong $M$-projective cover (in $R\text{-mod}$).
2. $R/\mathfrak{U}(M)$ is left perfect.

**Proof.** (1) $\Rightarrow$ (2). Let $B \in R/\mathfrak{U}(M)$-mod. Then $B$ regarded as an $R$-module satisfies $\mathfrak{U}(M)B = 0$. Let $\alpha: A \to B$ be a strong $M$-projective cover of $B$ in $R\text{-mod}$. Let $K = \ker \alpha$. From $\alpha(\mathfrak{U}(M)A) \subset \mathfrak{U}(M)B = 0$ we see that
\( \mathfrak{A}(M)A \subset K \). Hence \( \alpha \) induces a map \( \tilde{\alpha}: A/\mathfrak{A}(M)A \to B \). Now, \( A/\mathfrak{A}(M)A \) is an \( R/\mathfrak{A}(M) \)-module and \( \ker \tilde{\alpha}: K/\mathfrak{A}(M)A \) is small in \( A/\mathfrak{A}(M)A \). Thus \( \tilde{\alpha} \) is a minimal epimorphism in \( R/\mathfrak{A}(M) \)-mod. If \( i: \mathfrak{A}(M)A \to A \) denotes the inclusion, it is clear that

\[
i^* : \text{Hom}_R (A, M) \to \text{Hom}_R (\mathfrak{A}(M)A, M)
\]

is zero. Hence for any indexing set \( J \), the map \( i^* : \text{Hom}_R (A, M^J) \to \text{Hom}_R (\mathfrak{A}(M)A, M^J) \) is zero. Since \( A \) is strongly \( M \)-projective as an \( R \)-module, applying Lemma 1.6 we see that \( A/\mathfrak{A}(M)A \) is strongly \( M \)-projective in \( R \)-mod. Now Lemma 3.8 implies that \( A/\mathfrak{A}(M)A \) is a projective \( R/\mathfrak{A}(M) \)-module. Thus \( \tilde{\alpha}: A/\mathfrak{A}(M)A \to B \) is a projective cover of \( B \) in \( R/\mathfrak{A}(M) \)-mod. This proves that \( R/\mathfrak{A}(M) \) is left perfect.

(2) \( \Rightarrow \) (1). Let \( B \in R \)-mod be such that \( \mathfrak{A}(M)B = 0 \). Let \( \pi: P \to B \) be a projective cover of \( B \) in \( R/\mathfrak{A}(M) \)-mod. Then \( P \) is an \( R/\mathfrak{A}(M) \)-direct summand and hence an \( R \)-direct summand of \( \bigoplus_{x \in S} R/\mathfrak{A}(M) \) for some set \( S \). If \( i: \mathfrak{A}(M) \to R \) denotes the inclusion, clearly \( i^* : \text{Hom}_R (R, M) \to \text{Hom}_R (\mathfrak{A}(M), M) \) is zero and hence

\[
i^* : \text{Hom}_R (R, M^J) \to \text{Hom}_R (\mathfrak{A}(M), M^J)
\]

is zero for every set \( J \). Since \( R \) is free it is strongly \( M \)-projective in \( R \)-mod. By Lemma 1.6 we see that \( R/\mathfrak{A}(M) \) is strongly \( M \)-projective in \( R \)-mod. From (1) of Proposition 1.5 it follows that \( P \) is strongly \( M \)-projective in \( R \)-mod.

Now \( R/\mathfrak{A}(M) \) is \( M \)-independent. From this it follows immediately that \( \bigoplus_{x \in S} R/\mathfrak{A}(M) \) and hence \( P \) are \( M \)-independent. If \( K = \ker \alpha \), then \( K \) is \( M \)-independent in \( P \) (by the comments following Definition 2.3). Thus \( \pi: P \to B \) is a strong \( M \)-projective cover of \( B \) in \( R \)-mod.

Obvious modifications in the proof of Theorem 3.9 yield:

**Theorem 3.10.** The following statements are equivalent.

1. Every cyclic \( B \in R \)-mod satisfying \( \mathfrak{A}(M)B = 0 \) possesses a strong \( M \)-projective cover as an \( R \)-module.
2. \( R/\mathfrak{A}(M) \) is semiperfect.

**Proposition 3.11.** The following statements are equivalent.

1. The direct product \( \prod_{x \in J} B_a \) of any family \( B_a \) of strongly \( M \)-projective \( R \)-modules with \( \mathfrak{A}(M)B_a = 0 \) for all \( x \in J \) is strongly \( M \)-projective.
2. \( (R/\mathfrak{A}(M))^J \) is strongly \( M \)-projective for every indexing set \( J \).
3. \( R/\mathfrak{A}(M) \) is left perfect, and any finitely generated right ideal of \( R/\mathfrak{A}(M) \) is finitely related.

**Proof.** Immediate consequence of Theorem 3.3 of [4] and Lemma 3.8.

**References**


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