EXTENSIONS OF ERGODIC GROUP ACTIONS

BY

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In this paper we shall study extensions in the theory of ergodic actions of a locally compact group. If G is a locally compact group, by an ergodic G-space we mean a Lebesgue space (X, μ) together with a Borel action of G on X, under which μ is invariant and ergodic. If (X, μ) and (Y, ν) are ergodic G-spaces, (X, μ) is called an extension of (Y, ν) and (Y, ν) a factor of (X, μ) if there is a Borel function $p: X \to Y$, commuting with the G-actions, such that $p_*(\mu) = \nu$. Various properties that one considers for a fixed ergodic G-space have as natural analogues properties of the triple (X, p, Y) in such a way as to reduce to the usual ones in case Y is a point. This is the idea of "relativizing" concepts, which is a popular theme in the study of extensions in topological dynamics. In ergodic theory, relativization is a natural idea from the point of view of Mackey's theory of virtual groups [16]. Although familiarity with virtual groups is not essential for a reading of this paper, this idea does provide motivation for some of the concepts introduced below, and a good framework for understanding our results. We shall therefore briefly review the notion of virtual group and indicate its relevance.

If X is an ergodic G-space, one of two mutually exclusive statements holds:

(i) There is an orbit whose complement is a null set. In this case, X is called essentially transitive.

(ii) Every orbit is a null set. X is then called properly ergodic.

In the first case, the action of G on X is essentially equivalent to the action defined by translation on G/H, where H is a closed subgroup of G; furthermore, this action is determined up to equivalence by the conjugacy class of H in G. In the second case, no such simple description of the action is available, but it is often useful to think of the action as being defined by a "virtual subgroup" of G. Many concepts defined for a subgroup H, can be expressed in terms of the action of G on G/H; frequently, this leads to a natural extension of the concept to the case of an arbitrary virtual subgroup, i.e., to the case of an ergodic G-action that is not necessarily essentially transitive. Perhaps the most fundamental notions that can be extended in this way are those of a homomorphism, and the concomitant ideas of kernel and range. These and other related matters are discussed in [16].

From this point of view, the notion of an extension of an ergodic G-space has a simple interpretation. A measure preserving G-map $\phi: X \to Y$ can be viewed as an embedding of the virtual subgroup defined by X into the virtual subgroup defined by Y. Thus, it is reasonable to hope that many of the concepts that one considers for a given ergodic G-space, i.e., a virtual subgroup of G,

Received October 1, 1975.

can also be defined for extensions, i.e., one virtual subgroup considered as a sub-virtual subgroup of another. We now turn to consideration of one such concept which admits a very fruitful relativization.

For any ergodic G-space X, there is always a naturally defined unitary representation of G on $L^2(X)$, and it is natural to ask what the algebraic structure of the representation implies about the geometric structure of the action. One of the earliest results obtained in this direction, when the group in question is the integers, is the now classical von Neumann–Halmos theory of actions with discrete spectrum [6], [17]. For the integers, a unitary representation is determined by a single unitary operator, and von Neumann and Halmos were able to completely describe those actions for which this operator has discrete (i.e., pure point) spectrum. Their most important results are contained in the uniqueness theorem (asserting that the spectrum is then a complete invariant of the action), the existence theorem (describing what subsets of the circle can appear as the spectrum) and the structure theorem. This last result asserts that every ergodic action of the integers with discrete spectrum is equivalent to a translation on a compact abelian group. As indicated by Mackey in [15], the methods of von Neumann and Halmos enable one to obtain an equally complete theory, for actions with discrete spectrum, of an arbitrary locally compact abelian group. When the group is not abelian, the situation is somewhat more complicated. Using techniques different from those of von Neumann and Halmos, Mackey was able to prove a generalization of the structure theorem for actions of nonabelian groups [15]. He pointed out, however, that the natural analogue of the uniqueness theorem fails to hold. Nevertheless, we shall see that for a suitably restricted class of actions (which includes all actions of abelian groups with discrete spectrum), the uniqueness and existence theorems have natural extensions, even for G nonabelian. These actions are the normal actions, so called because they are the virtual subgroup analogue of normal subgroups. Thus, restricting consideration to normal actions, one has a complete generalization of the von Neumann–Halmos theory.

With this in hand, one can now ask to what extent the whole theory generalizes to the case of extensions. The highly satisfactory answer is that it extends intact, providing a significant new generalization of the von Neumann–Halmos theory even for the group of integers and the real line. If X is an extension of Y, one can define the notion of X having "relatively discrete spectrum" over Y. This can be loosely described as follows. Decompose the measure μ as a direct integral over the fibers of p, with respect to ν. This defines a Hilbert space on each fiber, and these Hilbert spaces exhibit $L^2(X)$ as a Borel G-Hilbert bundle over Y. If $L^2(X)$ is the direct sum of finite dimensional G-invariant subbundles over Y, we say that X has relatively discrete spectrum over Y. The structure theorem below, generalizing Mackey's theorem, describes the geometric structure of the extension when this "algebraic" condition is satisfied. It asserts that X can be written as a certain type of skew-product; these are factors of Mackey's "kernel" actions, and are a generalization of Anzai's skew products.

Similarly, the notion of normality can be relativized and it is meaningful to say that X is a normal extension of Y. In virtual group terms, this of course means that X defines a normal subvirtual subgroup of Y. For normal extensions, we prove analogues of the uniqueness and existence theorems. We remark that for a properly ergodic action of the integers, there always exist nonnormal extensions with relatively discrete spectrum. Thus, even for abelian groups the question of normality is highly relevant in these considerations.

In a subsequent paper [23], we use this theory to develop the notion of actions with generalized discrete spectrum. This makes contact with Furstenberg's work in topological dynamics on minimal distal flows, Parry's notion of separating sieves, the theory of affine actions, and quasi-discrete spectrum. It promises other applications as well.

The entire theory sketched above depends in an essential way on the concept of a cocycle of an ergodic G-space. Cocycles have appeared in various considerations in ergodic theory [16], [18], [9], and from the virtual group point of view are the analogue of homomorphisms for subgroups. This "analogy" rests on the fact that for transitive G-spaces, there is a correspondence between cocycles and homomorphisms of the stability groups. This is, in fact, an essential part of Mackey's well-known imprimitivity theorem. (See [20] for an account of the imprimitivity theorem from this point of view.) For properly ergodic ations, the fundamentals of a general theory of cocycles were sketched by Mackey in [16]; some aspects of this theory are worked out in detail by Ramsay in [19]. We have continued the detailed development of certain areas within the theory, particularly the study of cocycles into compact groups. These results are basic to the rest of the paper.

The organization of this paper is as follows. Part I is preparatory, and the material is used throughout this paper as well as [23]. Aside from establishing notation and recalling various results, there are three main features. One is a general existence theorem for factors of a Lebesgue space; this appears in Section 1, and in equivariant form in Section 2. The latter section also discusses the basic connections between extensions and cocycles. This includes the notions of restriction and induction of cocycles, analogous to those for group representations, and a version of the Frobenius reciprocity theorem. Lastly, the general theory of cocycles into compact groups is developed in Section 3. Part II contains the relativized version of the von Neumann–Halmos–Mackey theory. The structure theorem is proved in Section 4, as well as a generalization that subsumes another theorem of Mackey on induced representations (which is in fact a generalization of his own structure theorem.) The virtual subgroup analogue of normality is discussed in Section 5, and is then used to complete the extension theory with the uniqueness-existence theorems in Section 6.

Some of the results of this paper were announced in [22].

The author would like to express his thanks to Professor G. W. Mackey for many helpful discussions and suggestions made during the preparation of this paper.

I. G-spaces, factors, and cocycles

1. Factors of Lebesgue spaces. We begin by recalling and discussing some facts about Borel spaces. By a Lebesgue space we mean a standard Borel space X, together with a probability measure μ. (See [12] or [19] for detailed definitions of these and other related concepts to follow.) Associated with any Lebesgue space, we have the Boolean σ-algebra $B(X, \mu)$, which consists of the Borel sets of X, any two being identified if their symmetric difference is a null set. If Y is a standard Borel space and $\phi: X \to Y$ is a Borel function, then we have a measure $\phi_*(\mu)$ defined on Y by $\phi_*(\mu)(B) = \mu(\phi^{-1}(B))$, for $B \subset Y$ a Borel set. If (Y, ν) is a Lebesgue space, we will call a Borel map $\phi: X \to Y$ a factor map if $\phi_*(\mu) = \nu$. We will call Y a factor of X or X an extension of Y. Now if $\phi: X \to Y$ is a factor map, we have an induced map $\phi^*: B(Y, \nu) \to B(X, \mu)$ that is injective. Conversely, it is well known that if $A \subset B(X, \mu)$ is a σ-subalgebra, then there exists a Lebesgue space (Y, ν) and a factor map $\phi: X \to Y$ such that $A = \phi^*(B(Y))$ [19, Theorem 2.1].

Since $\phi_*(\mu) = \nu$, the map $f \to f \circ \phi$ induces an isometric embedding of $L^2(Y)$ as a subspace of $L^2(X)$. It is easy to see that this subspace can be characterized as $\{f \in L^2(X) \mid f$ is measurable with respect to the σ-field of Borel sets in X whose equivalence class in $B(X)$ belongs to $\phi^*(B(Y))\}$. We shall on various occasions need criteria for determining when a given subspace of $L^2(X)$ is of the form $L^2(Y)$ for some factor Y of X. A useful result in this direction is the following theorem.

LEMMA 1.1. *Let A be a collection of subsets of a given set and suppose A is closed under complements. Let B be the set of finite intersections of elements of A, and C be the set of disjoint finite unions of elements of B. Then C is the field generated by A.*

THEOREM 1.2. *Let X be a Lebesgue space and $A \subset L^\infty(X)$ a *-subalgebra (not necessarily closed). Let B be the σ-field of Borel sets in X generated by the functions of A. Then as subspaces of $L^2(X)$,*

$$\bar{A} = L^2(X, B) = \{f \in L^2(X) \mid f \text{ measurable with respect to } B\}.$$

Proof. (i) If $f \in \bar{A}$, then $f = \lim f_n$, $f_n \in A$, where the limit is in $L^2(X)$. Now it follows from the proof of the Riesz-Fisher theorem that there exists a subsequence f_{n_j} such that $f_{n_j} \to f$ pointwise on a conull set. Since each f_{n_j} is measurable with respect to B, so is f. Hence $\bar{A} \subset L^2(X, B)$.

(ii) We now claim that $L^2(X, B) \subset \bar{A}$. Since A is closed under complex conjugation, it is easy to see that B is the σ-field generated by

$$D = \{f^{-1}(M) \mid f \in A, f \in L^\infty(X, \mathbf{R}), M \subset \mathbf{R} \text{ Borel}\}.$$

Let B_0 be the field generated by D. As every element of $L^2(X, B)$ is an L^2-limit of linear combinations of characteristic functions of elements of B, it suffices to see that $\chi_S \in \bar{A}$ for $S \in B$. Since B_0 generates B as a σ-field, it suffices to see that $\chi_S \in \bar{A}$ for $S \in B_0$ [1, p. 21]. By Lemma 1.1, it suffices to see that $\chi_S \in \bar{A}$ whenever S is the finite intersection of sets of D. Suppose $f_i \in A \cap L^\infty(X; \mathbf{R})$, and let $R_i = \|f_i\|_\infty$, $i = 1, \ldots, k$. Choose $M_i \subset [-R_i, R_i]$ to be a Borel set.

For each positive integer n, suppose $p_{in}, \ldots, p_{kn}$ are polynomials. Then

$$\prod_{i=1}^{k} p_{in} \circ f_i \in A \quad \text{for all } n,$$

since $f_i \in L^\infty(X; \mathbf{R})$ and A is an algebra. Now suppose that g_{in}, $i = 1, \ldots, k$, $n = 1, 2, \ldots$, are bounded Borel functions such that

$$\prod_{i=1}^{k} g_{in} \circ f_i \in \bar{A} \quad \text{and} \quad \lim_{n \to \infty} g_{in} = g_i$$

in bounded pointwise convergence. Then

$$\prod_{i=1}^{k} g_{in} \circ f_i \to \prod_{i=1}^{k} g_i \circ f_i$$

in bounded pointwise convergence, and hence the limit also holds in $L^2(X)$. Thus we also have $\prod_{i=1}^{k} g_i \circ f_i \in \bar{A}$. For each i, the smallest set of functions on $[-R_i, R_i]$ closed under bounded pointwise convergence and containing the polynomials is the set of bounded Borel functions. Hence $\prod_{i=1}^{k} g_i \circ f_i \in \bar{A}$ for all bounded Borel g_i defined on $[-R_i, R_i]$. Letting $g_i = \chi_{M_i}$, we obtain

$$\chi_{\bigcap_{i=1}^{k} f_i^{-1}(M_i)} = \prod_{i=1}^{k} \chi_{f_i^{-1}(M_i)} = \prod_{i=1}^{k} \chi_{M_i} \circ f_i \in \bar{A}.$$

This completes the proof.

Combining this theorem with the preceding remarks, we have:

COROLLARY 1.3. *Let X be a Lebesgue space and A a *-subalgebra of $L^\infty(X)$. Then $\bar{A} = L^2(Y)$ for some factor Y of X.*

We remark that techniques similar to those of the proof above appear in [10, Theorem 2.2].

If $\phi: X \to Y$ is a factor map, the measure μ may be decomposed over the fibers of ϕ. More precisely, for each $y \in Y$, let $F_y = \phi^{-1}(y)$. Then for each y, there exists a measure μ_y on X, that is supported on F_y, such that for each Borel function f on X, $y \mapsto \int f \, d\mu_y$ is Borel on Y, and

$$\int_X f \, d\mu = \int_Y \left(\int f \, d\mu_y \right) d\nu(y).$$

If $\{\mu_y\}$ is such a collection of measures, we write $\mu = \int^\oplus \mu_y \, d\nu$. This decomposition of μ is almost unique: If $\mu = \int^\oplus \mu_y \, d\nu = \int^\oplus \mu_y' \, d\nu$, then $\mu_y = \mu_y'$ almost everywhere. A decomposition of μ yields a decomposition of $L^2(X)$ as a Hilbert bundle over Y:

$$L^2(X) = \int^\oplus L^2(F_y, d\mu_y) \, d\nu \qquad \text{(see [19])}.$$

We now consider a construction which proves to be of much use when studying factors. Suppose $p: (X, \mu) \to (Z, \alpha)$ and $q: (Y, \nu) \to (Z, \alpha)$ are factors. Define

$$X \times_Z Y = \{(x, y) \in X \times Y \mid p(x) = q(y)\}.$$

This is called the fibered product of X and Y over Z, and is a Borel subset of $X \times Y$. There is a natural Borel map $t: X \times_Z Y \to Z$, given by $t(x, y) = p(x)$ $(=q(y))$, so $t^{-1}(z) = p^{-1}(z) \times q^{-1}(z)$. Suppose

$$\mu = \int^{\oplus} \mu_z \, d\alpha \qquad \nu = \int^{\oplus} \nu_z \, d\alpha.$$

Then it is easy to check that for $A \subset X \times_Z Y$ Borel,

$$(\mu \times_Z \nu)(A) = \int_Z (\mu_z \times \nu_z)(A) \, d\alpha(z)$$

defines a measure on $X \times_Z Y$, and that $\mu \times_Z \nu = \int^{\oplus} (\mu_z \times \nu_z) \, d\alpha$. In the case that Z is one point, $(X \times_Z Y, \mu \times_Z \nu)$ reduces to the usual Cartesian product, with the product measure.

There is a useful universal characterization of the fibered product. To state this, we first consider the concept of relative independence.

PROPOSITION 1.4. *Consider a commutative diagram of factor maps of Lebesgue spaces:*

$$\begin{array}{ccc} X_0 & \xrightarrow{\psi} & Y \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & Z \end{array}$$

Let m denote the measure on X_0, and $m = \int^{\oplus} m_z \, d\alpha$. We consider all the L^2-spaces as subspaces of $L^2(X_0)$. Then the following are equivalent:

(i) $(L^2(X) \ominus L^2(Z)) \perp (L^2(Y) \ominus L^2(Z))$

(ii) *If $f \in L^2(X)$, $g \in L^2(Y)$, then $E(f \cdot g \mid Z) = E(f \mid Z)E(g \mid Z)$. (Here $E(\cdot \mid Z)$ is a conditional expectation.)*

(iii) *If $A \subset X$ and $B \subset Y$, then for almost all $z \in Z$, $\phi^{-1}(A)$ and $\psi^{-1}(B)$ are independent sets in (X_0, m_z).*

Proof. (i) $\Rightarrow$ (ii) Let $f \in L^2(X)$, $g \in L^2(Y)$. Condition (i) implies $E(g \mid X) = E(g \mid Z)$. Hence $E(f \cdot g \mid X) = fE(g \mid X) = fE(g \mid Z)$. Now take $E(\ \mid Z)$ of this equation; we get $E(f \cdot g \mid Z) = E(f \mid Z)E(g \mid Z)$.

(ii) $\Rightarrow$ (i) If $f \in L^2(X) \ominus L^2(Z)$, $g \in L^2(Y) \ominus L^2(Z)$. Then $E(f \mid Z) = E(\bar{g} \mid Z) = 0$. Thus $E(f \cdot \bar{g} \mid Z) = 0$ by (ii) which implies $f \perp g$.

(ii) $\Rightarrow$ (iii) This is immediate when one notes that for a set $S \subset X_0$ $E(\chi_S \mid Z)$ is just the function $z \mapsto m_z(S)$.

(iii) $\Rightarrow$ (ii) We know $E(f \cdot g \mid Z) = E(f \mid Z)E(g \mid Z)$ when f and g are characteristic functions, and the general result follows by the usual approximation arguments.

We remark that when Z is one point, these conditions are equivalent to the σ-fields in X_0 determined by X and Y being independent. Hence, we shall say that X and Y are relatively independent over Z if the conditions of the proposition hold.

We now characterize the fibered product in terms of relative independence.

PROPOSITION 1.5. *Given a commutative diagram as in Proposition* 1.4, *X and Y are relatively independent over Z if and only if there exists a factor map $h: X_0 \to X \times_Z Y$ such that the following diagram commutes:*

$$\begin{tikzcd}
X_0 \arrow[rr, "\psi"] \arrow[dr, "h"] \arrow[dd, "\phi"'] & & Y \arrow[dd, "q"] \\
& X \times_Z Y \arrow[ur] \arrow[dl] \arrow[dr] & \\
X \arrow[rr, "p"] & & Z
\end{tikzcd}$$

Proof. Define h by $h(x_0) = (\phi(x_0), \psi(x_0))$. If $A \subset X$, $B \subset Y$ are Borel, let $A \times_Z B = (A \times B) \cap (X \times_Z Y)$. To see that h is a factor map, it clearly suffices to show that

$$m(h^{-1}(A \times_Z B)) = (\mu \times_Z \nu)(A \times_Z B).$$

Now

$$\begin{aligned} m(h^{-1}(A \times_Z B)) &= \int_Z m_z(\phi^{-1}(A) \cap \psi^{-1}(B))\, dz \\ &= \int_Z m_z(\phi^{-1}(A)) m_z(\psi^{-1}(B))\, dz. \end{aligned}$$

The uniqueness of decomposition for measures implies $\phi_*(m_z) = \mu_z$ and $\psi_*(m_z) = \nu_z$ almost everywhere. Thus the integral becomes

$$\int_Z \mu_z(A)\nu_z(B)\, dz = \int_Z (\mu_z \times \nu_z)(A \times_Z B) = (\mu \times_Z \nu)(A \times_Z B).$$

The converse assertion is more or less immediate.

2. G-spaces: Introductory remarks. Let X be a standard Borel space, and G a standard Borel group. We call X a Borel G-space if there is a (right) action of G on X such that the map $X \times G \to X$ is Borel. If X is a Lebesgue space and G is a locally compact group, we shall call X a Lebesgue G-space if it is a Borel G-space and if G preserves the measure. (We shall throughout take "locally compact" to mean locally compact and second countable.) If $X' \subset X$, we will call X' an essential subset if it is Borel, conull, and G-invariant. A factor map $\phi: X \to Y$ between G-spaces will be called a G-map if $\phi(xg) = \phi(x)g$ for all $(x, g) \in X \times G$. We will call Y a factor of X if there exists a factor G-map $X' \to Y$ where $X' \subset X$ is essential. Now G acts on $B(X)$, and if Y is a factor of X, $B(Y)$ can be identified with a G-invariant σ-subalgebra of $B(X)$. We now show that every G-invariant subalgebra of $B(X)$ arises in this way.

PROPOSITION 2.1. *Let (X, μ) be a Lebesgue G-space and $A \subset B(X)$ a G-invariant sub σ-algebra. Then there is a factor Y of X such that $B(Y) = A$.*

Proof. This argument is a small modification of the proof of [14, Theorem 2]. A is a Boolean G-space [14] and by [14, Theorem 1], there is a Borel G-space Y and a quasi-invariant measure ν such that $B(Y, \nu) \cong A$ as Boolean G-spaces. Since A has an invariant measure inherited from the measure on $B(X)$, we can assume that ν is invariant. Now let $\theta: X \to Y$ be a Borel map such that $\theta^*: B(Y) \to B(X)$ defines the isomorphism $B(Y) \cong A$ [19, Theorem 2.1]. Y is standard, so we can choose a Borel isomorphism $i: Y \to I$ where I is a subset of the unit interval. Let F_G be the universal Borel G-space as defined by Mackey in [14]. Define $\phi: Y \to F_G$ by $\phi(y)(g) = i(yg)$ and $\psi: X \to F_G$ by $\psi(x)(g) = i(\theta(xg))$.

By the proof of [14, Lemma 2], ϕ and ψ are Borel G-maps, and ϕ is a Borel isomorphism onto an invariant Borel subset of F_G. Since θ^* is a Boolean G-map, it follows that for each g, $\theta(xg) = \theta(x)g$ for almost all x. Thus, by Fubini's theorem,

$$[(\phi \cdot \theta)(x)](g) = i(\theta(x) \cdot g) = i(\theta(xg)) = \psi(x)(g)$$

for almost all (x, g); i.e., $\psi = \phi \circ \theta$ almost everywhere. Thus $X' = \psi^{-1}$ (range ϕ) is conull, Borel and G-invariant. The map $\phi^{-1} \circ \psi: X' \to Y$ is a G-map, and since it agrees with θ almost everywhere, it induces the given Boolean G-isomorphism $\theta^*: B(Y) \to A$.

When it is convenient, we shall apply (often without explicit mention) various definitions and constructions that we have given for factor maps to factors in general. By this we understand that we have passed to an essential subset for which there is a factor map, and that the notion at hand is independent (at least up to some obvious isomorphism) of the choice of such a set.

Using the correspondence between factors and σ-subalgebras, it is easy to deduce the following equivariant version of Corollary 1.3.

COROLLARY 2.2. *Let X be a Lebesgue G-space and A a G-invariant $*$-subalgebra of $L^\infty(X)$. Then in $L^2(X)$, we have $\bar{A} = L^2(Y)$ for some G-factor Y of X.*

A Lebesgue G-space X is called ergodic if the action of G on $B(X)$ is irreducible; i.e., there are no elements in $B(X)$ left fixed (by all elements of G) except ϕ and X. Mackey has shown that an equivalent condition is that for any Borel function f on X, $f \cdot g = f$ everywhere (for each $g \in G$) implies that f is constant on a conull set [14, Theorem 3]. It is trivial that a factor of an ergodic G-space is also ergodic.

If X and Y are transitive G-spaces, then X and Y are essentially isomorphic to G/H and G/K respectively, where H and K are closed subgroups of G. X will be an extension of Y if and only if H is contained in a conjugate of K. The map $X = G/H \to G/K = Y$ is determined by the embedding of H in this conjugate of K. Thus, in terms of Mackey's notion of virtual groups [16], a factor map $X \to Y$ where X and Y are ergodic but not necessarily transitive G-spaces corresponds to an embedding of the virtual subgroup defined by X into the virtual subgroup defined by Y.

We now turn our attention to cocycles of ergodic G-spaces, a concept central to this paper. The reader is referred to [19], [20], [16] as general references for cocycles, and particularly the latter for an explanation of why cocycles are the virtual subgroup analogue of homomorphism (and representation). Most of the remainder of Section 2 is devoted to setting out examples and results for cocycle representations of ergodic G-spaces that have well-known analogues for representations of locally compact groups.

Let S be an ergodic Lebesgue G-space and K a standard Borel group. We call a Borel function $\alpha: S \times G \to K$ a cocycle if for $g, h \in G$, $\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$ for almost all $s \in S$. A useful extension of this notion arises in the context of Hilbert bundles. Let $\{H_s\}$ be a Hilbert bundle on S, and suppose that for each $(s, g) \in S \times G$, we have a bounded linear map $\alpha(s, g): H_{sg} \to H_s$ such that:

(i) For each g, $\alpha(s, g)$ is unitary for almost all s.

(ii) For each pair of bounded Borel sections of the bundle $f = \{f_s\}$, $f' = \{f'_s\}$, the function $(s, g) \mapsto \langle \alpha(s, g)f_{sg} \mid f'_s \rangle_s$ is Borel.

(iii) For each $g, h \in G$, $\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$ for almost all s.

We then call α a cocycle representation of (S, G) on the Hilbert bundle $\{H_s\}$.

If $\alpha, \beta: S \times G \to K$ are cocycles, we call them cohomologous, or equivalent, if there is a Borel map $\phi: S \to K$ such that for each g,

$$\phi(s)\alpha(s, g)\phi(sg)^{-1} = \beta(s, g) \quad \text{for almost all } s \in S.$$

Similarly, if α is a cocycle representation on the bundle $\{H_s\}$ and β a cocycle representation on the bundle $\{H'_s\}$, we call them equivalent if there exists a Borel field of bounded linear maps $U(s): H_s \to H'_s$ such that:

(i) $U(s)$ is unitary for almost all s.

(ii) For each g, $U(s)\alpha(s, g)U(sg)^{-1} = \beta(s, g)$ for almost all $s \in S$.

Suppose α is a cocycle representation in the product bundle $H = S \times H_0$. From condition (ii) in the definition of cocycle representation, the map $(s, g) \mapsto \alpha(s, g)$ is a Borel map from $S \times G$ into $L(H_0)$, the bounded linear operators on H_0, where the latter is given the weak topology. $L(H_0)$ is standard under the weak Borel structure, and the unitary group $U(H_0)$ is a Borel subset [2]. Hence $\{(s, g) \mid \alpha(s, g) \in U(H_0)\}$ is Borel, and so by changing α on a conull Borel set, we obtain an equivalent cocycle representation β on $S \times H_0$ such that:

(i) $\beta(s, g)$ is unitary for all $(s, g) \in S \times G$.

(ii) For each g, $\alpha(s, g) = \beta(s, g)$ for almost all s.

Thus, up to equivalence, any cocycle representation on the product bundle can be considered as a cocycle into the unitary group $U(H_0)$.

As pointed out by Ramsay [19, p. 264], if $\{H_s\}$ is a Hilbert bundle on S, there exists a decomposition of S into disjoint Borel sets $\{S_\infty, S_0, S_1, \ldots\}$ such that for each n, there exists a Borel field of unitary operators on S_n, $U(s)\colon H_s \to H_n$, where H_n is a fixed Hilbert space of dimension n. Thus

$$\int_{S_n}^{\oplus} H_s \cong S_n \times H_n.$$

If some S_n is conull, we say that $\{H_s\}$ is of uniform multiplicity n. If S is ergodic and $\alpha(s, g)$ is a cocycle representation on the Hilbert bundle $\{H_s\}$, then $\{H_s\}$ is of uniform multiplicity [20, Lemma 9.10]. Thus, every cocycle representation of an ergodic G-space is equivalent to one on a constant field of Hilbert spaces, and hence equivalent to a cocycle into a unitary group.

Example 2.3. We now describe a general method of constructing cocycle representations. Suppose $\phi\colon X \to Y$ is a G-factor map of ergodic G-spaces. Write $\mu = \int^{\oplus} \mu_y \, d\nu$. For a fixed $g \in G$, $\mu \cdot g = \mu$, and hence by the uniqueness of decompositions, we have $\mu_y \cdot g = \mu_{yg}$ for almost all $y \in Y$.

LEMMA 2.4. *The set* $A = \{(y, g) \in Y \times G \mid \mu_y \cdot g = \mu_{yg}\}$ *is Borel.*

Proof. Let $M(X)$ be the space of measures on X with the usual Borel structure (see e.g., [5] or [11]). Then the map $y \mapsto \mu_y$ is Borel and thus $(y, g) \to yg \to \mu_{yg}$ is Borel. Thus to see that A is Borel, it suffices to see that $(y, g) \to \mu_y \cdot g$ is Borel. But it follows from [11, Theorem 5.2] and [20, Theorem 8.7], that the action of G on $M(X)$ is a Borel action, and hence that $(y, g) \to (\mu_y, g) \to \mu_y \cdot g$ is Borel.

Now

$$L^2(X) = \int_Y^{\oplus} L^2(F_y, \mu_y) \, d\nu$$

is a Hilbert bundle. If $(y, g) \in A$, then g maps (F_y, μ_y) onto (F_{yg}, μ_{yg}) in a measure preserving way. Let $\alpha(y, g)\colon L^2(F_{yg}) \to L^2(F_y)$ be the induced unitary map. It is straightforward, in light of Lemma 2.4, that α can be extended to a (Borel) cocycle representation (which we also denote by α) of (Y, G) on the Hilbert bundle $L^2(X)$. We call α the natural cocycle representation of the factor map ϕ. We remark that up to equivalence, (in fact up to equality on conull sets) α is independent of the various choices made in its construction.

Example 2.5. The preceding example admits a natural generalization. Suppose $\beta(x, g)$ is a cocycle representation of (X, G) on a Hilbert bundle $\{V_x\}$. We define an associated cocycle representation α of (Y, G) which we call the induced cocycle representation of β. For each y, let

$$H_y = \int_{F_y}^{\oplus} V_x \, d\mu_y(x).$$

Then $\{H_y\}$ is a Hilbert bundle on Y. Furthermore, for $(y, g) \in A$, the map

$$\alpha(y, g)\colon H_{yg} \to H_y, \quad \alpha(y, g) = \int_{x \in F_y}^{\oplus} \beta(x, g),$$

is defined, and for each g, will be unitary for almost all $y \in Y$. Again, α can be extended to a cocycle representation on the Hilbert bundle $\{H_y\}$. In the case where $V_x = \mathbf{C}$ for each $x \in X$, and $\beta(x, g) = 1$ for all (x, g), α is just the cocycle of Example 2.3.

If $X = G/H$ and $Y = G/K$ with $H \subset K$, then a cocycle representation β of $X \times G$ corresponds to a representation Π_β of H [20, Theorem 8.27], and the induced cocycle α will correspond to a representation Π_α of K. One can check that Π_α is the representation induced by Π_β. Thus, in the general case, regarding β as a representation of the virtual subgroup defined by X, we can regard α as the representation of the larger virtual subgroup defined by Y that is induced by β.

In case $Y = \{e\}$, a (Y, G) cocycle representation is simply a unitary representation of the group G. Since $\{e\}$ is a factor of any G-space, the above construction yields, for any cocycle representation α of (X, G), a unitary representation of G, called the representation induced by α, and which we denote by U^α. We recall, for later use, one well-known fact about the relationship between α and U^α.

THEOREM 2.6. *Let α, β be cocycle representations of (X, G) on the Hilbert bundles $\{H_x\}$ and $\{H'_x\}$ respectively. For each $E \subset X$, let $P_E(P'_E)$ be the associated projection operator in $\int^\oplus H_x(\int^\oplus H'_x)$. Then every intertwining operator T of U^α and U^β with the additional property that $P'_E T = TP_E$ can be written as a bounded Borel field of operators*

$$T = \int^{\oplus} T_x, \quad T_x\colon H_x \to H'_x$$

such that

(*) *for each $g \in G$, $T_x\alpha(x, g) = \beta(x, g)T_{xg}$ for almost all $x \in X$.*

Conversely, any bounded Borel field satisfying (*) defines an intertwining operator T of U^α and U^β satisfying $P'_E T = TP_E$. We will call a field satisfying (*) an intertwining field for the cocycles α and β.

Proof. Suppose $P'_E T = TP_E$ and $TU^\alpha = U^\beta T$. From the first condition, we have $T = \int^\oplus T_x$, $T_x\colon H_x \to H'_x$, where T is a bounded Borel field. If $f = \int^\oplus f_x \in \int^\oplus H_x$, then

$$(**) \qquad (TU^\alpha_g f)_x = T_x\alpha(x, g)f_{xg} \quad \text{and} \quad (U^\beta_g Tf)_x = \beta(x, g)T_{xg}f_{xg}.$$

Choose $f^i \in \int^\oplus H_x$ such that $\{f^i_x\}$ is dense in H_x for each $x \in X$. Then for each i, and each $g \in G$, (**) implies

$$T_x\alpha(x, g)f^i_{xg} = \beta(x, g)T_{xg}f^i_{xg}$$

for all x in a conull set $N_{i,g}$. Then for $x \in \bigcap_i N_{i,g}$, we have

$$T_x\alpha(x, g) = \beta(x, g)T_{xg}.$$

The converse is immediate from (**).

Example 2.7. The notion of induced cocycle is an analogue of the notion of induced representation for groups. We now consider an analogue of restriction of representations.

Suppose $\phi: X \to Y$ is a factor G-map and $\{H_y\}$ a Hilbert bundle on Y. Then the assignment $x \mapsto V_x = H_{\phi(x)}$ is a Hilbert bundle on X, with a fundamental sequence [3] defined as follows. If $\{f_y^i\} \subset \int^{\oplus} H_y$ is a fundamental sequence for the Hilbert bundle $\{H_y\}$, $g^n \in L^\infty(X)$ is such that $\{g_y^n\}$ is a fundamental sequence for $L^2(X)$ as a Hilbert bundle over Y, and

$$h_{i,n}(x) = g^n(x)f^i_{\phi(x)} \in V_x,$$

then finite linear combinations of $\{h_{i,n}(x)\}$ form a fundamental sequence for the bundle $\{V_x\}$.

Suppose β is a cocycle representation of $Y \times G$ on the Hilbert bundle $\{H_y\}$. Define $\alpha(x, g): V_{xg} \to V_x$ by $\alpha(x, g) = \beta(\phi(x), g)$. Then α is called the restriction of β to (X, G).

If $X = G/H$ and $Y = G/K$ with $H \subset K$, and β is a cocycle corresponding to a representation π of K, then α will be a cocycle corresponding to the restricted representation $\pi \mid H$ of H. Hence, in general, restriction of cocycles can be thought of as the virtual subgroup analogue of restriction of representations.

In further analogy with group representations, we now discuss the algebraic operations of direct sum, tensor product, and conjugation for cocycle representations of ergodic G-spaces. If α and β are cocycle representations of (S, G) on the Hilbert bundles $\{H_s\}$ and $\{H'_s\}$ respectively, then one can form the cocycle $\alpha \oplus \beta$ on the bundle $\{H_s \oplus H'_s\}$, by defining

$$(\alpha \oplus \beta)(s, g) = \alpha(s, g) \oplus \beta(s, g): H_{sg} \oplus H'_{sg} \to H_s \oplus H'_s.$$

Similarly, one can define countable direct sums of cocycles. Given a cocycle representation α, one can ask when it is cohomologous to a representation of the form $\alpha_1 \oplus \alpha_2$. If it is, α will be called reducible, and α_i sub-cocycle representations of α. Otherwise, α is called irreducible. An alternate phrasing of this is made possible by the following easily checked result.

PROPOSITION 2.8. *If $\{V_s\}$ is a sub-Hilbert bundle of $\{H_s\}$, then the following are equivalent:*

(i) *$\int^{\oplus} V_s\, ds$ is a U^α-invariant subspace.*

(ii) *For each g, $\alpha(s, g)(V_{sg}) = V_s$ for almost all s.*

(iii) *α is cohomologous to $\alpha_1 \oplus \alpha_2$, where α_i are cocycles into $U(H_i)$ and $\int^{\oplus} V_s$ is unitarily equivalent to $S \times H_1$, $\int^{\oplus} V_s^{\perp}$ is unitarily equivalent to $S \times H_2$ (as Hilbert bundles).*

Thus, saying α is irreducible is equivalent to saying that there are no U^{α}-invariant sub-Hilbert bundles of $\int^{\oplus} H_s$.

Suppose $\{T_s\}$ is a nontrivial intertwining field for α and β. Let $U(s)$ be the unitary part of the polar decomposition of T_s. Then $U = \int^{\oplus} U(s)$ intertwines U^{α} and U^{β}, and gives a unitary equivalence of

$$U^{\alpha} \Big| \int^{\oplus} \ker (T_s)^{\perp} \quad \text{and} \quad U^{\beta} \Big| \int^{\oplus} \overline{\text{range } (T_s)}.$$

From this it follows that α and β have equivalent subcocycle representations. In particular, if α (or β) is irreducible, and $T_s \neq 0$ on a set of positive measure, then α is a subcocycle representation of β (or vice-versa).

If $\{H_s\}$ and $\{H_s'\}$ are Hilbert bundles, then $\{H_s \otimes H_s'\}$ is also, and one can form the cocycle representation $\alpha \otimes \beta$ on this bundle. Similarly, if α is a cocycle representation on the constant field $S \times H_0$, one can define the conjugate cocycle $\bar{\alpha}$, as follows: Choose a conjugation $f \mapsto \bar{f}$ in H_0 [8, p. 15], and for any $A\colon H_0 \to H_0$, let $\bar{A}$ be the operator defined by $\langle \bar{A}f \mid g\rangle = \langle A\bar{f} \mid \bar{g}\rangle$, i.e., $\bar{A}(f) = \overline{A(\bar{f})}$. Let $\bar{\alpha}(s, g) = \overline{\alpha(s, g)}$. It is clear that $\bar{\alpha}$ is a cocycle and one can check as in [8, p. 16] that the equivalence class of $\bar{\alpha}$ is independent of the choice of conjugation.

For group representations, a useful relation between these various algebraic concepts is the Frobenius reciprocity theorem. We prove a version of this theorem in the context of induced cocycle representations of ergodic actions. Our theorem is modeled after the group theoretic version given by Mackey [13, Theorem 8.2].

DEFINITION 2.9. If α and β are cocycle representations of $Y \times G$, let $S(\alpha, \beta)$ be the set of intertwining fields $T = \int^{\oplus} T_y$ such that each T_y is a Hilbert–Schmidt operator. $S(\alpha, \beta)$ is a vector space, and $\dim S(\alpha, \beta) = j(\alpha, \beta)$ is called the strong intertwining number of α and β.

THEOREM 2.10. (Frobenius reciprocity). *Let $\phi\colon X \to Y$ be a factor G-map, α a cocycle representation of $X \times G$, and β a cocycle representation of $Y \times G$. Let* ind (α) *and* res (β) *be the induced and restricted cocycles. Then* $j(\text{ind } \alpha, \beta) = j(\alpha, \text{res } \beta)$.

We begin the proof with several lemmas.

LEMMA 2.11. *Suppose β is a $Y \times G$ cocycle representation on $\{H_y\}$ and I is the* 1*-dimensional identity cocycle. Then $j(\beta, I)$ is the dimension of the space of G-invariant elements in $\int_Y^{\oplus} H_y$.*

Proof. If $T = \int^{\oplus} T_y\colon \int^{\oplus} H_y \to \int_Y^{\oplus} \mathbf{C}$, then each $T_y\colon H_y \to \mathbf{C}$ and hence there is an element $v = \int^{\oplus} v_y \in \int^{\oplus} H_y$ such that for any $f = \int^{\oplus} f_y$, $(Tf)_y = \langle f_y \mid v_y\rangle$. It is straightforward to see that $T \in S(\beta, I)$ if and only if v is G-invariant. Thus $T \leftrightarrow v$ defines a vector space isomorphism (conjugate linear) between $S(\beta, I)$ and the G-invariant elements.

LEMMA 2.12. *If α is an $X \times G$ cocycle, then* $j(\alpha, I) = j(\text{ind}(\alpha), I)$.

Proof. If α is a cocycle representation then $U^{\alpha} \cong U^{\text{ind}(\alpha)}$, so the dimensions of the spaces of G-invariant elements are equal. The result now follows by Lemma 2.11.

LEMMA 2.13. *Suppose α and β are $S \times G$ cocycles. Then* $j(\alpha, \beta) = j(\alpha \otimes \bar{\beta}, I)$.

Proof. We recall first that for Hilbert spaces H_1 and H_2, there is an isomorphism of $(H_1 \otimes H_2)^*$ with $L_2(H_1; H_2)$ the Hilbert–Schmidt maps from H_1 to H_2, defined by

$$L_2(H_1, H_2) \to (H_1 \otimes H_2)^*, \quad T \to A,$$

where A is given by $A(v \otimes w) = \langle Tv \mid \bar{w} \rangle$.

To prove the lemma, we can suppose that α and β are cocycle representations on the product bundles $S \times H_1$ and $S \times H_2$ respectively. Via the above correspondence, there is a vector space isomorphism between bounded Borel fields of Hilbert–Schmidt operators $T_s\colon H_1 \to H_2$ and bounded Borel fields $A_s\colon H_1 \otimes H_2 \to \mathbf{C}$. To prove the lemma it suffices to show that

$$\{T_s\} \in S(\alpha, \beta) \Leftrightarrow \{A_s\} \in S(\alpha \otimes \bar{\beta}, I).$$

Now

$$\begin{aligned} A_s(\alpha \otimes \bar{\beta})(s, g)(v \otimes w) &= A_s(\alpha(s, g)v \otimes \bar{\beta}(s, g)w) \\ &= A_s(\alpha(s, g)v \otimes \overline{\beta(s, g)\bar{w}}) \\ &= \langle T_s\alpha(s, g)v \mid \beta(s, g)\bar{w} \rangle \\ &= \langle \beta(s, g)^{-1} T_s\alpha(s, g)v \mid \bar{w} \rangle. \end{aligned}$$

But $A_{sg}(v \otimes w) = \langle T_{sg}v \mid \bar{w} \rangle$. Thus

$$A_s(\alpha \otimes \bar{\beta})(s, g) = A_{sg} \quad \text{if and only if} \quad \beta(s, g)^{-1} T_s\alpha(s, g) = T_{sg}.$$

The result now follows.

LEMMA 2.14. *Let $\phi\colon X \to Y$ a factor G-map, α an $X \times G$ cocycle representation on the Hilbert bundle $\{H_x\}$ and β a $Y \times G$ cocycle representation on $\{W_y\}$. Let* ind (α) *and* res (β) *be the induced and restricted cocycles. Then as $Y \times G$ cocycle representations,* $\text{ind}(\alpha) \otimes \beta \cong \text{ind}(\alpha \otimes \text{res}(\beta))$.

Proof. Let $\gamma_1 = \text{ind}(\alpha) \otimes \beta$ and $\gamma_2 = \text{ind}(\alpha \otimes \text{res}(\beta))$. Let

$$V_y = \int_{x \in \phi^{-1}(y)} H_x \, d\mu_y(x).$$

Then γ_1 is a cocycle on the Hilbert bundle $y \to V_y \otimes W_y$ and

$$\gamma_1(y, g): V_{yg} \otimes W_{yg} \to V_y \otimes W_y$$

is defined by

$$\gamma_1(y, g) = \operatorname{ind}(\alpha)(y, g) \otimes \beta(y, g) = \left[\int_{x \in \phi^{-1}(y)}^{\oplus} \alpha(x, g)\right] \otimes \beta(y, g),$$

(for each g, almost all y).

On the other hand, $\alpha \otimes \operatorname{res}(\beta)$ is a cocycle representation on the bundle $x \mapsto H_x \otimes W_{\varphi(x)}$ and

$$\gamma_2(y, g): \int_{x \in \phi^{-1}(yg)}^{\oplus} [H_x \otimes W_{yg}]\, d\mu_{yg}(x) \to \int_{x \in \phi^{-1}(y)}^{\oplus} (H_x \otimes W_y)\, d\mu_y(x)$$

is given by (for each g, almost all y),

$$\gamma_2(y, g) = \int_{x \in \phi^{-1}(y)}^{\oplus} [\alpha(x, g) \otimes \beta(y, g)].$$

Now up to ismorphism, tensor products commute with direct integrals. Hence

$$\int_{\phi^{-1}(y)}^{\oplus} (H_x \otimes W_y)\, d\mu_y(x) \cong V_y \otimes W_y,$$

and under this isomorphism, γ_1 corresponds to γ_2.

We are now ready to prove the reciprocity theorem.

Proof of Theorem 2.10. We can suppose α and β are representations on product bundles. By Lemma 2.13, $j(\operatorname{ind}(\alpha), \beta) = j(\operatorname{ind}(\alpha) \otimes \bar{\beta}, I)$. By Lemma 2.14, this is $j(\operatorname{ind}(\alpha \otimes \operatorname{res}(\bar{\beta})), I)$ and by Lemma 2.12, equals $j(\alpha \otimes \operatorname{res}(\bar{\beta}), I)$. Lemma 2.13 now implies that this is $j(\alpha, \operatorname{res}(\beta))$.

COROLLARY 2.15. *Suppose $\alpha_i: S \times G \to U(H_i)$ are equivalent, $i = 1, 2$; $\dim H_i < \infty$. Then $\alpha_1 \otimes \bar{\alpha}_2$ contains the identity as a subcocycle representation.*

Proof. This follows immediately from Lemma 2.13.

3. Cocycles into Compact Groups. We now turn to a consideration of some basic facts about cocycles into compact (second countable) groups. Many of the results in this section have natural interpretations in terms of Mackey's definitions of kernel and range for homomorphisms of virtual subgroups [16]. Some of these results have been indicated (without complete proof) by Mackey in [16].

We will find useful a slight weakening of the notion of a G-space.

DEFINITION 3.1. Let S be a Lebesgue space. By a near-action of G on S, we mean a Borel map $S \times G \to S$, $(s, g) \mapsto sg$ such that:

(i) For each $g, h \in G$, $(sg)h = s(gh)$ for almost all s.

(ii) $s \cdot e = s$ for almost all s.

(iii) Each $g \in G$ preserves the measure on S.

Each near-action of G on S determines an essentially unique action of G on S. More precisely, we have the following result:

PROPOSITION 3.2. *If S is a near G-space, then there is a natural induced action of G on $B(S)$. Furthermore, there is a Borel action of G on S, which we denote $s \circ g$ such that:*

(i) *For each g, $sg = s \circ g$ for almost all s.*

(ii) *The induced actions on $B(S)$ are equal.*

Proof. The first statement is [19, Lemma 3.1]. The second statement follows from [19, Lemma 3.2] by defining $s \circ g = \psi^{-1}(\psi(s) \cdot g)$ where ψ is as in that lemma.

We shall assume for the remainder of this section that S is an ergodic G-space.

Given a cocycle $\alpha: S \times G \to K$, where K is compact, we define a near-action of G on $S \times K$ by $(s, x)g = (sg, x\alpha(s, g))$. If the cocycle identity holds for $(s, g, h) \in S \times G \times G$, then $(s, x)(gh) = [(s, x)g]h$, so that (i) and (ii) of Definition 3.1 hold, and it is straightforward to check that each $g \in G$ preserves the product measure on $S \times K$. Thus, for a cocycle α, there is, by Proposition 3.2, an essentially unique action of G defined on $S \times K$ agreeing almost everywhere with the near action. When necessary, we shall write $S \times_\alpha K$ to specify the action of G. K also acts on $S \times K$ by $(s, x) \cdot k = (s, k^{-1}x)$. This action commutes with the near-action of G, but not necessarily with the G-action it defines. However, for each $(g, k) \in G \times K$, we have, for almost all $y \in S \times K$, $(y \circ g)k = (yg)k = (yk)g = (yk) \circ g$. Thus we have a naturally induced action of $G \times K$ on $B(S \times K)$.

LEMMA 3.3. *If α and β are cohomologous, then $B(S \times_\alpha K)$ and $B(S \times_\beta K)$ are isomorphic Boolean $G \times K$ spaces.*

Proof. Let $\phi: S \to K$ such that for each $g \in G$,

$$\phi(s)\beta(s, g)\phi(sg)^{-1} = \alpha(s, g) \quad \text{for almost all } s \in S.$$

Define a map $\theta: S \times_\alpha K \to S \times_\beta K$ by $\theta(s, x) = (s, x \cdot \phi(s))$. Then θ is an isomorphism of Lebesgue spaces, and for each $(g, k) \in G \times K$,

$$\theta((s, x) \cdot (g, k)) = \theta(s, x) \cdot (g, k) \quad \text{for almost all } (s, x).$$

Hence $\theta^*: B(S \times_\beta K) \to B(S \times_\alpha K)$ is a Boolean $G \times K$-isomorphism.

COROLLARY 3.4. *If α and β are cohomologous, then $S \times_\alpha K$ and $S \times_\beta K$ are essentially isomorphic G-spaces.*

An important question which we now consider about cocycles into compact groups is to determine when they are equivalent to cocycles into proper closed subgroups. This question is related to the properties of the Boolean $G \times K$-spaces defined above, as well as to Mackey's definition of the kernel and range of α. The reader is again referred to [16] for an explanation of these concepts, and their relation to the $G \times K$-space $S \times_\alpha K$.

Consider the σ-subalgebra B of $B(S \times_\alpha K)$ consisting of elements left fixed by the action of $G \times \{e\}$. Since the actions of $G \times \{e\}$ and $\{e\} \times K$ commute, B is a Boolean K-space. Since the action of $G \times K$ on $B(S \times K)$ is irreducible, the action of K on B must also be irreducible. Hence there is an ergodic K-space Y such that $B(Y) \cong B$ as Boolean K-spaces [14]. Since K is compact, we can choose $Y = K/H_0$ for some closed subgroup $H_0 \subset K$. The following result was indicated without proof by Mackey [16], and motivates the definition of the K-space Y as the "range-closure" of α.

THEOREM 3.5. *α is cohomologous to a cocycle into a subgroup $H \subset K$ if and only if K/H is a factor of the K-space Y.*

Proof. (i) Suppose $\alpha \sim \beta$ and $\beta(s, g) \in H$ for every (s, g). It follows from Lemma 3.3 that the Boolean K-space B is K-isomorphic to the Boolean K-space B' of elements of $B(S \times_\beta K)$ that are left fixed by $G \times \{e\}$. Now let K/H be the space of left cosets, and $p: S \times K \to K/H$ be $p(s, k) = [k]$. Then p commutes with the K-actions and hence $p^*: B(K/H) \to B(S \times_\beta K)$ is an injective map of Boolean K-spaces. Thus it suffices to see that $p^*(B(K/H)) \subset B'$. Now

$$p((s, k)g) = p(sg, k\beta(s, g)) = [k\beta(s, g)] = [k]$$

since $\beta(s, g) \in H$. Therefore $p((s, k)g) = p(s, k)$. It follows readily that for $A \in B(K/H)$, $p^*(A) \cdot g \subset p^*(A)$ for each $g \in G$. Since this also holds for g^{-1}, we have $p^*(A) \cdot g = p^*(A)$, i.e., $p^*(B(K/H)) \subset B'$.

(ii) Now suppose that K/H is a factor of $Y = K/H_0$. Then H_0 is contained in a conjugate of H, so it suffices to see that α is cohomologous to a cocycle into H_0. Let $q: B(Y) \to B \subset B(S \times K)$ be an isomorphism of Boolean K-spaces. Then there exists a K-invariant Borel null set $N \subset S \times K$ and a K-map $\lambda: (S \times K) - N \to Y$ such that $\lambda^* = q$. As a K-invariant null set, N must clearly be of the form $A \times K$ where $A \subset S$ is Borel and null. For each $g \in G$, we must have

$$\lambda((s, k) \cdot g) = \lambda(s, k) \quad \text{for almost all } (s, k) \in [(S - A) \cap (S - A) \cdot g^{-1}] \times K.$$

(This last condition so that the equation is well defined.) This follows since $\lambda^* = q$ and g leaves elements of B fixed. Thus

$$D = \{(s, g, k) \mid s \in (S - A) \cap (S - A) \cdot g^{-1}; \;\; \lambda((s, k) \cdot g) = \lambda(s, k)\}$$

is a Borel conull set by Fubini's theorem. Thus, there exists $k_0 \in K$ such that

$$D_0 = \{(s, g) \mid (s, g, k_0) \in D\}$$

is conull. We phrase this condition in a form we will need: If $(s, g) \in D_0$, then

$$\lambda(s, k_0) = \lambda(sg, k_0\alpha(s, g)).$$

Since λ is a K-map, we can write this as

$$\lambda(s, e)k_0^{-1} = \lambda(sg, e)\alpha(s, g)^{-1}k_0^{-1}.$$

Equivalently for all $(s, g) \in D_0$,

$$(*) \qquad \lambda(s, e)\alpha(s, g) = \lambda(sg, e).$$

To construct a cocycle equivalent to α with values in H, we first construct a cocycle β such that $\beta(s, g) \in H$ for almost all (s, g). Choose a point $y_0 \in Y$ and let

$$H_0 = \{k \in K \mid y_0 \cdot k = y_0\}.$$

By [13, Lemma 1.1], there exists a Borel map θ: $Y \to K$ such that $y_0 \cdot \theta(y) = y$ for all $y \in Y$. Define u: $S \to K$ by $u(s) = \theta(\lambda(s, e))$. Now define

$$\beta(s, g) = u(s)\alpha(s, g)u(sg)^{-1}.$$

We claim that for $(s, g) \in D_0$, $\beta(s, g) \in H_0$, i.e., $y_0 \cdot u(s)\alpha(s, g)u(sg)^{-1} = y_0$. We have

$$y_0 \cdot u(s)\alpha(s, g) = y_0 \cdot \theta(\lambda(s, e))\alpha(s, g) = \lambda(s, e)\alpha(s, g).$$

Similarly,

$$y_0 \cdot u(sg) = y_0 \cdot \theta(\lambda(sg, e)) = \lambda(sg, e).$$

Thus for $(s, g) \in D_0$, $y_0 \cdot \beta(y, g) = y_0$ follows from $(*)$. To complete the proof of the theorem, it suffices to prove the following lemma:

LEMMA 3.6. *Suppose β: $S \times G \to K$ is a cocycle, and that $\beta(s, g) \in H$ for almost all (s, g), where $H \subset K$ is a closed subgroup. Then β is cohomologous to a cocycle β_0 such that $\beta_0(s, g) \in H$ for all (s, g).*

Proof. By changing β on a Borel null set, we can obtain a function β': $S \times G \to H$, and using Fubini's theorem, we can see that

$$\beta'(s, g)\beta'(sg, h) = \beta'(s, gh) \quad \text{for almost all } (s, g, h) \in S \times G \times G.$$

By [19, Theorem 5.1], we can change β' on a null set to obtain a Borel function β_0: $S \times G \to H$ such that there is a conull set $S_0 \subset S$ with the following property: if $s \in S_0$, $sg \in S_0$, $sgh \in S_0$, then $\beta_0(s, g)\beta_0(sg, h) = \beta_0(s, gh)$. For each $(g, h) \in G \times G$, $S_0 \cap S_0 g^{-1} \cap S_0(gh)^{-1}$ is conull and hence β_0 is a cocycle that differs from β on a null set. We claim that for each $g \in G$, $\beta_0(s, g) = \beta(s, g)$ for almost all $s \in S$. This suffices to prove the lemma. Let

$$G_0 = \{g \in G \mid \beta_0(s, g) = \beta(s, g) \text{ for almost all } s \in S\}.$$

By Fubini's theorem, G_0 is conull. For $g, h \in G_0$, let

$$S_1 = \{s \mid \beta_0(s, g) = \beta(s, g)\}, \quad S_2 = \{s \mid \beta_0(s, h) = \beta(s, h)\},$$

$$S_3 = \{s \mid \beta_0(s, g)\beta_0(sg, h) = \beta_0(s, gh)\}$$

and

$$S_4 = \{s \mid \beta(s, g)\beta(sg, h) = \beta(s, gh)\}.$$

Then if $s \in S_1 \cap S_2 g^{-1} \cap S_3 \cap S_4$, we have

$$\beta_0(s, gh) = \beta_0(s, g)\beta_0(sg, h) = \beta(s, g)\beta(sg, h) = \beta(s, gh).$$

Thus, $gh \in G_0$; therefore G_0 is closed under multiplication and conull, and hence must equal G.

DEFINITION 3.7. If $\alpha: S \times G \to K$ is a cocycle, let K_α be the closed subgroup of K generated by $\{\alpha(s, g)\}$. We will call α a minimal cocycle if there is no cocycle β cohomologous to α such that $K_\beta \subset_{\neq} K_\alpha$.

As a consequence of Theorem 3.5, we have:

COROLLARY 3.8. (i) *Any cocycle (into a compact group) is equivalent to a minimal cocycle.*

(ii) *α is minimal and $K_\alpha = K$ if and only if the action of G on $S \times_\alpha K$ is ergodic.*

(iii) *If α and β are equivalent minimal cocycles, then K_α and K_β are conjugate subgroups of K.*

Proof. From the theorem we see that a subgroup H will be of the form K_α for a minimal α if and only if H is a subgroup conjugate to H_0, so (i) and (iii) are clear. Part (ii) follows since the action of G on $B(S \times_\alpha K)$ is irreducible if and only if $H_0 = K$.

We will need an auxiliary result which asserts that minimality is independent of embedding.

THEOREM 3.9. *Let $\alpha: S \times G \to K$ be a minimal cocycle with $K_\alpha = K$. Suppose there is a compact group K' with $K \subset K'$. Then as a cocycle into K', α is still minimal.*

Proof. Let K'/K be the space of left cosets, and $p: K' \to K'/K$ the projection. Let $t: K'/K \to K'$ be a section of p and define

$$\phi: K' \to K \times K'/K$$

by $\phi(x) = (t(p(x))^{-1}x, p(x))$. Then ϕ is a measure-preserving Borel isomorphism, and the near action of G on $S \times_\alpha K'$ is carried over to the near action of G on $S \times K \times K'/K$ given by $(s, k, [k']) \cdot g = (sg, k\alpha(s, g), [k'])$. Since the action of G on $S \times_\alpha K$ is ergodic, any G-invariant element of the Boolean algebra $B(S \times K \times K'/K)$ must be of the form $S \times K \times A$ where $A \subset K'/K$. Therefore, the σ-algebra of G-invariant elements of $B(S \times K \times K'/K)$ is K'-isomorphic to the Boolean K'-space $B(K'/K)$. It follows from Theorem 3.5 that $\alpha: S \times G \to K'$ is minimal with $K'_\alpha = K$.

Remark. In terms of virtual subgroups, $\alpha: S \times G \to K$ is minimal with $K_\alpha = K$ if and only if "the range of α is dense in K".

We now consider the cocycle representations obtained by composing the representations of a compact group with a minimal cocycle into the group. The result we are aiming for is Theorem 3.14 which asserts that if $\alpha: S \times G \to K$ is minimal, with $K_\alpha = K$, then $\pi \to \pi \circ \alpha$ sets up a one-to-one correspondence between $\hat{K}$ (the dual object of K) and a collection of equivalence classes of irreducible cocycle representations of (S, G). In light of the above remark, this result is highly plausible from the virtual subgroup viewpoint.

PROPOSITION 3.10. *Suppose $\alpha: S \times G \to K$ is minimal, with $K_\alpha = K$. If $\rho: K \to H$ is a surjective homomorphism, then $\rho \circ \alpha: S \times G \to H$ is a minimal cocycle with $H_{\rho \circ \alpha} = H$.*

Proof. The map $(1, \rho): S \times K \to S \times H$ is a surjective measure-preserving map. Since $S \times_\alpha K$ is an ergodic G-space and

$$(1, \rho)^*: B(S \times_{\rho \circ \alpha} H) \to B(S \times_\alpha K)$$

is a G-map, $S \times_{\rho \circ \alpha} H$ is ergodic, which implies by Corollary 3.8 that $\rho \circ \alpha$ is minimal, and $H_{\rho \circ \alpha} = H$.

LEMMA 3.11. *Let $\beta: S \times G \to U(n)$ be a minimal cocycle. If β is reducible, then $U(n)_\beta$ is a reducible group of matrices.*

Proof. If β is reducible, then β is equivalent to a cocycle into a subgroup of $U(n)$ of the form $U(p) \times U(n-p)$ for some $1 \leq p < n$. Thus there exists a minimal cocycle α equivalent to β such that $U(n)_\alpha$ is a reducible group. Since α and β are both minimal, Corollary 3.8 implies that $U(n)_\alpha$ and $U(n)_\beta$ are conjugate subgroups of $U(n)$. Since $U(n)_\alpha$ is reducible, so is $U(n)_\beta$.

PROPOSITION 3.12. *Let $\alpha: S \times G \to K$ be minimal, $K_\alpha = K$, and let π be a finite dimensional representation (unitary) of K. Then π is irreducible if and only if $\pi \circ \alpha$ is irreducible.*

Proof. (i) If π is reducible, clearly $\pi \circ \alpha$ is also.

(ii) Let $\beta = \pi \circ \alpha$. By Proposition 3.10 and Theorem 3.9, β is minimal with $U(n)_\beta = \pi(K)$. (Here $n = \dim \pi$). If β is reducible, then Lemma 3.11 implies $\pi(k)$ is reducible.

LEMMA 3.13. *Let $\alpha: S \times G \to K$ minimal, $K_\alpha = K$. Suppose π is a finite dimensional unitary representation of K such that $\pi \circ \alpha$ contains the one-dimensional identity cocycle representation. Then π contains the identity representation.*

Proof. Let $\pi = \sum^\oplus \pi_i$, where π_i are irreducible. Then $\pi \circ \alpha = \sum^\oplus \pi_i \circ \alpha$, and each $\pi_i \circ \alpha$ is an irreducible cocycle by Proposition 3.12. Now consider a G-invariant field of one-dimensional subspaces in $L^2(S, H(\pi))$ under the representation $U^{\pi \circ \alpha}$, say $V = \int^\oplus V_s \, ds$. The projection of V onto $L^2(S, H(\pi_i))$ must be nonzero for some π_i, and this projection will be an intertwining field for $\pi \circ \alpha \mid V$ and $\pi_i \circ \alpha$. Since $\pi \circ \alpha \mid V$ is one dimensional and hence irreducible, and $\pi_i \circ \alpha$ is irreducible, we have $\pi \circ \alpha \mid V$ is equivalent to $\pi_i \circ \alpha$. So π_i is a character χ of K, and $\chi \circ \alpha$ is equivalent to the identity. We claim that this implies $\chi = 1$. Now $\chi \circ \alpha$ equivalent to 1 means there exists a function $u: S \to \mathbf{C}$ such that:

(i) $|u(s)| = 1$
(ii) For each g, $u(s)\chi(\alpha(s, g))u(sg)^{-1} = 1$ for almost all s.

Equation (ii) can be rewritten

$$\frac{\chi(\alpha(s, g))}{u(sg)} = \frac{1}{u(s)}.$$

Now let $f: S \times K \to \mathbf{C}$ be defined by $f(s, k) = \chi(k)/u(s)$. For each $g \in G$, and almost all (s, k), we have

$$f((s, k)g) = f(sg, k\alpha(s, g)) = [\chi(k)\chi(\alpha(s, g))/u(sg)] = \chi(k)/u(s) = f(s, k).$$

Since α is minimal and $K_\alpha = K$, $S \times_\alpha K$ is ergodic, so this implies f is constant on a conull set, which shows $\chi = 1$.

THEOREM 3.14. *Let $\alpha: S \times G \to K$ be a minimal cocycle, $K_\alpha = K$. Let π_1, π_2 be irreducible unitary representations of K. Then $\pi_1 \circ \alpha$ and $\pi_2 \circ \alpha$ are irreducible; they are equivalent if and only if π_1 and π_2 are unitarily equivalent representations.*

Proof. That $\pi_i \circ \alpha$ is irreducible is just Proposition 3.12. If π_1 and π_2 are equivalent, it is clear that $\pi_1 \circ \alpha$ and $\pi_2 \circ \alpha$ are also. If $\pi_1 \circ \alpha$ and $\pi_2 \circ \alpha$ are equivalent, by Corollary 2.15, $(\pi_1 \circ \alpha) \otimes \overline{(\pi_2 \circ \alpha)}$ contains the identity cocycle. But $(\pi_1 \circ \alpha) \otimes \overline{(\pi_2 \circ \alpha)} = (\pi_1 \otimes \bar{\pi}_2) \circ \alpha$. By Lemma 3.13, $\pi_1 \otimes \bar{\pi}_2$ contains the identity representation. Since π_1 and π_2 are irreducible, they must be unitarily equivalent.

II. Extensions with relatively discrete spectrum

4. The Structure Theorem and a generalization. Given a cocycle representation $\alpha(s, g)$ one can try to decompose α into irreducible components. In case α is equivalent to $\sum^{\oplus} \alpha_i$, where α_i are finite dimensional irreducible cocycle representations, we say that α has discrete spectrum. Equivalently, the Hilbert bundle on which α is defined is a direct sum of finite dimensional G-invariant Hilbert subbundles. If $\phi: X \to Y$ is a factor map of ergodic G-spaces, we have a natural $Y \times G$ cocycle representation on the Hilbert bundle $\int^{\oplus} L^2(F_y)$ (Example 2.3). One question that presents itself is what the "spectral" structure of the cocycle α implies about the geometric structure of the triple (X, ϕ, Y). In the case where Y is a point, α just becomes the natural unitary representation of G on $L^2(X)$. The main result of Mackey's paper [15] is a description of the geometric consequences of the representation α of G having discrete spectrum. This generalized a classic result of Halmos and von Neumann in the special case when $G = Z$, the group of integers [6]. The main result of this section is to generalize Mackey's theorem to the case where Y is not necessarily one point.

In virtual group terms, what Mackey does is to describe those virtual subgroups of G for which the representation of G induced by the identity has discrete spectrum. We describe here, for a virtual-subgroup (Y, G), those sub-virtual subgroups for which the representation of (Y, G) induced by the identity has discrete spectrum.

Before stating the theorem, we produce a class of examples where the natural $Y \times G$ cocycle will have discrete spectrum.

Example 4.1. Let K be a compact group and $H \subset K$ a closed subgroup. Suppose $\alpha\colon Y \times G \to K$ is a minimal cocycle, with $K_\alpha = K$ (and Y is ergodic). Define a near-action of G on $Y \times K/H$ by

$$(y, [k]) \cdot g = (yg, [k]\alpha(y, g)).$$

By Proposition 3.2, $X = Y \times K/H$ becomes a G-space, and Y is a factor of X. Since α is minimal, $K_\alpha = K$, and X is a factor of $Y \times_\alpha K$, it follows from Corollary 3.8 that X is ergodic. We claim that the natural (Y, G) cocycle, β, has discrete spectrum. Let σ be the natural representation of K on $L^2(K/H)$ given by right translation. Then $U^\beta = U^{\sigma\circ\alpha}$ since both are just the natural representation of G on $L^2(X)$. Theorem 2.6 shows that β and $\sigma \circ \alpha$ are equivalent. Now $\sigma = \sum^\oplus \sigma_i$ where σ_i are finite dimensional and irreducible, so $\sigma \circ \alpha = \sum^\oplus \sigma_i \circ \alpha$ has discrete spectrum, and hence so does β.

DEFINITION 4.2. Suppose X and Z are G-extensions of a Lebesgue G-space Y, with $q_1\colon B(Y) \to B(X)$, $q_2\colon B(Y) \to B(Z)$ the designated embeddings. We say that X and Z are essentially isomorphic extensions of Y if there exists a Boolean G-isomorphism $p\colon B(X) \to B(Z)$ such that $p \cdot q_1 = q_2$.

Equivalently, there exist essential sets $X' \subset X$, $Z' \subset Z$, a G-isomorphism $f\colon X' \to Z'$, and factor G-maps $\phi_1\colon X' \to Y$, $\phi_2\colon Z' \to Y$ such that:

(i) $\phi_2 \circ f = \phi_1$,
(ii) $\phi_i^* = q_i$, $i = 1, 2$.

Remark. To see the equivalence, choose $X_0 \subset X$ and $Z_0 \subset Z$ essential, and factor G-maps $\lambda_1\colon X_0 \to Y$, $\lambda_2\colon Z_0 \to Y$ with $\lambda_i^* = q_i$. This can be done by Proposition 2.1. Now $p\colon B(X_0) \to B(Z_0)$ is an isomorphism, so by [14, Theorem 2], there exist essential $X_1 \subset X_0$, $Z_1 \subset Z_0$ and a Borel isomorphism $f_1\colon Z_1 \to X_1$ such that $f_1^* = p$. Now $\lambda_2^* = f_1^*\lambda_1^*$, so $\lambda_2 = \lambda_1 f_1$ almost everywhere on Z_1. Since λ_2 and $\lambda_1 f_1$ are both G-maps, they agree on an essential set $Z' \subset Z_1$. Then $X' = f_1(Z')$ is essential, since f_1 is an isomorphism, and we can take $f = (f_1 \mid Z')^{-1}$, $\phi_1 = \lambda_1 \mid X'$, $\phi_2 = \lambda_2 \mid Z'$.

The theorem alluded to above is the following:

THEOREM 4.3 (Structure Theorem). *If $\phi\colon X \to Y$ is a factor map of ergodic G-spaces and the natural (Y, G) cocycle representation on $\{L^2(F_y)\}$ has discrete spectrum, then there is a compact group K, a closed subgroup $H \subset K$, and a* *minimal cocycle* $\alpha: Y \times G \to K$ *with* $K_\alpha = K$ *such that* X *is essentially isomorphic as an extension of* Y *to the* G*-space* $Y \times K/H$ *of Example* 4.1.

In the case when $Y = \{e\}$, this is exactly Mackey's theorem [15, Theorem 1].

Proof (*of Theorem* 4.3). We begin the proof by summarizing some facts we will need about the Effros Borel structure. Given a separable Hilbert space H_0, let E be the set of von Neumann algebras on H_0. Effros has shown the following [4, p. 1161]: There exists a standard Borel structure on E such that if S is any standard Borel space, a map $\mathscr{A}: S \to E$ is Borel if and only if there exist countably many Borel functions $A_i: S \to L(H)$ such that for each s, $\{A_i(s)\}$ generate $\mathscr{A}(s)$ as a von Neumann algebra. The unitary group $U(H_0)$ is a standard Borel group with the weak Borel structure [2, Lemma 4] and acts on E by $\mathscr{A} \cdot U = U^{-1}\mathscr{A}U$. In [5, Lemma 2.1], Effros shows that this is a Borel action, i.e., $(\mathscr{A}, U) \mapsto \mathscr{A} \cdot U$ is Borel.

LEMMA 4.4. Let $A = \{\mathscr{A} \in E \mid \mathscr{A}$ *is abelian*$\}$. *Then* A *is a* $U(H_0)$*-invariant Borel set in* E.

Proof. By [4, Theorem 3], $\mathscr{A} \to \mathscr{A}'$ is Borel and from [4, Corollary 2], $(\mathscr{A}, B) \to \mathscr{A} \cap B$ is Borel. Since $A = \{\mathscr{A} \in E \mid \mathscr{A} \cap \mathscr{A}' = \mathscr{A}\}$, A is Borel. $U(H_0)$-invariance is trivial.

If $\mathscr{A}$ is an abelian von Neumann algebra, let $B(\mathscr{A})$ be the set of projection operators in $\mathscr{A}$. Then it is well known that $\mathscr{A} \to B(\mathscr{A})$ sets up a bijection between abelian von Neumann algebras and Boolean σ-algebras of projections on H_0. Thus we will identify A above with the set $\mathscr{M}$ of such Boolean algebras. Further, for the Borel structure on $\mathscr{M}$ defined by the Effros Borel structure on A, $\mathscr{M}$ is standard, and Borel maps into $\mathscr{M}$ can be identified by the following.

LEMMA 4.5. *If* $P: Y \to \mathscr{M}$, *with* Y *standard, then* P *is Borel if and only if there exist countably many Borel fields,* $P_i(y)$, *of projections on* H_0, *such that for each* $y \in Y$, $\{P_i(y)\}$ *generates* $P(y)$ *as a Boolean* σ*-algebra.*

Proof. Since any abelian von Neumann algebra is generated by its projections, the "only if" statement is clear. Conversely, let us suppose we are given bounded Borel fields $\{A_i(y)\}$ of operators on H_0, such that for each y, $\{A_i(y)\}$ generates a von Neumann algebra $D(y)$, where $D: Y \to E$ is Borel. We can assume $A_i(y)$ is self-adjoint for each i and y.

If A is a self-adjoint operator on H_0, the operators $\chi_{[a,b]}(A)$ are projection operators, where $\chi_{[a,b]}$ is the characteristic function of the interval $[a, b]$ of real numbers. It follows from the spectral theorem (see [7, C 40, C 41] for example) that the von Neumann algebra generated by A is equal to the von Neumann algebra generated by $\{\chi_{[a,b]}(A) \mid a, b \text{ rational}\}$. Thus, given a sequence of operators A_i, the von Neumann algebra $W^*(A_1, A_2, \ldots)$ generated by $\{A_1, A_2, \ldots\}$ is equal to the von Neumann algebra generated by

$$Q = \{\chi_{[a,b]}(A_i) \mid a, b \text{ rational, all } i\}.$$

Hence, the Boolean σ-algebra generated by Q is equal to the Boolean σ-algebra associated to $W^*(A_1, A_2, \ldots)$.

In light of these remarks, to complete the proof it suffices to show that if $A(y)$ is a bounded Borel field of self-adjoint operators on H_0, so is $\chi_{[a,b]}(A(y))$. Choose M so that $\|A(y)\| \leq M$ for all y. Then there are polynomials $p_n(x)$ such that $\lim_{n\to\infty} p_n(x) = \chi_{[a,b]}(x)$ in bounded pointwise convergence on $[-2M, 2M]$, and hence

$$\lim_{n\to\infty} p_n(A(y)) = \chi_{[a,b]}(A(y))$$

weakly for each i and y. Since $p_n(A(y))$ is Borel in y, so is $\chi_{[a,b]}(A(y))$.

Now suppose Y is an ergodic G-space and α a cocycle representation on a Hilbert bundle $\{H_y\}$ with discrete spectrum. Thus, there exist G-invariant fields V_i of finite dimensional subspaces such that $\int^{\oplus} H_y = H = \sum^{\oplus} V_i$. Using the comments preceding example 2.3, it is not hard to see the following: There exists a conull set $Y_0 \subset Y$, Y_0 Borel, Hilbert spaces H_i with $\dim H_i < \infty$, and a Borel field of maps $U(y)\colon H_y \to H_0 = \sum^{\oplus} H_i$ such that:

(i) $U(y)$ is unitary for every $y \in Y_0$.
(ii) $U(y)(V_i(y)) = H_i$ for every $y \in Y_0$.

Let us consider further what happens when α is in addition the natural cocycle corresponding to a factor map $\phi\colon X \to Y$. For each $y \in Y$, we have a Boolean σ-algebra of projections B_y on $L^2(F_y)$, namely the set of multiplication operators corresponding to the elements of $B(F_y)$. Let $\{E_i\}$ be a countable generating sequence of sets for the Borel structure on X and let $P_i(y)$ be multiplication by $\chi_{E_i \cap F_y}$ in $L^2(F_y)$. Then for each i, $\{P_i(y)\}_y$ is a Borel field of operators on $\int^{\oplus} L^2(F_y)$, and for each y, $\{P_i(y)\}_i$ generates B_y as a Boolean σ-algebra. Now let Y_0 and $\{U(y)\}$ be as above. For $y \in Y_0$, let $A_y = U(y)B_y \cdot U(y)^{-1}$. Then A_y is a Boolean σ-algebra on H_0, so we have a map

$$A\colon Y_0 \to \mathcal{M}, \quad y \mapsto A_y.$$

If $y \in Y_0$, $\{U(y)P_i(y)U(y)^{-1}\}_i$ generates A_y as a Boolean σ-algebra, and hence A is a Borel function. We can extend A to a Borel function $Y \to \mathcal{M}$, which we continue to denote with the same letter.

If α is the natural cocycle, for each (y, g) such that $y \in Y_0 \cap Y_0 g^{-1}$, let

$$\tilde{\alpha}(y, g) = U(y)\alpha(y, g)U(yg)^{-1}.$$

Then $\tilde{\alpha}$ is Borel and for each g and almost all y, $\tilde{\alpha}(y, g)H_i = H_i$ (see Proposition 2.8). By changing $\tilde{\alpha}$ on a set of measure 0, and extending it to all of $Y \times G$, we can obtain a Borel cocycle $\beta\colon Y \times G \to U(H_0)$ such that:

(i) $\beta(y, g)(H_i) = H_i$ for every y, g.
(ii) For each g, $\beta(y, g) = U(y)\alpha(y, g)U(yg)^{-1}$ for almost all y.

Now for each g and almost all y, we have $\alpha(y, g)^{-1}B_y\alpha(y, g) = B_{yg}$. Thus, for all g, the following hold for almost all y:

$$\begin{aligned}\beta(y, g)^{-1}A_y\beta(y, g) &= U(yg)\alpha(y, g)^{-1}U(y)^{-1}A_yU(y)\alpha(y, g)U(yg)^{-1} \\ &= U(yg)\alpha(y, g)^{-1}B_y\alpha(y, g)U(yg)^{-1} \\ &= U(yg)B_{yg}U(yg)^{-1}.\end{aligned}$$

In other words,

$$(*) \qquad \beta(y, g)^{-1}A_y\beta(y, g) = A_{yg}.$$

Now let $K = \{U \in U(H_0) \mid U(H_i) = H_i\}$. Then K is a compact subgroup of $U(H_0)$ (weak topology). Condition (i) on β above just says that $\beta(y, g) \in K$, and (*) says that $A_y \cdot \beta(y, g) = A_{yg}$, (where the expression on the left is the action of $U(H_0)$ on $\mathscr{M}$), the equation holding for almost all y, given any g. By restricting the action of $U(H_0)$, we obtain an action of K on $\mathscr{M}$, and we let $\hat{\mathscr{M}}$ be the space of orbits in $\mathscr{M}$ under K. Since K is compact, $\hat{\mathscr{M}}$ is standard under the quotient Borel structure. Let $p\colon \mathscr{M} \to \hat{\mathscr{M}}$ be the natural projection and $\lambda\colon Y \to \hat{\mathscr{M}}$, $\lambda = p \circ A$. Now equation (*) implies that for each g, $p(A_y) = p(A_{yg})$ for almost all y, i.e., $\lambda(y) = \lambda(yg)$ for almost all y. Since $\hat{\mathscr{M}}$ is Borel isomorphic to a subset of $[0, 1]$, and Y is an ergodic G-space, we conclude that λ is constant almost everywhere; i.e., there exists $A_0 \in \mathscr{M}$ such that $p(A_0) = p(A_y)$ for almost all $y \in Y$. Equivalently, if we let $\mathscr{M}_0$ be the orbit of A_0 in $\mathscr{M}$ under K, then $A_y \in \mathscr{M}_0$ for y in a conull Borel set. By changing A on a null set, we obtain a function (Borel) $\tilde{A}\colon Y \to \mathscr{M}_0$ such that $\tilde{A}_y = A_y$ almost everywhere.

Let $K_0 = \{U \in K \mid A_0 \cdot U = A_0\}$. Then $\mathscr{M}_0$ is Borel isomorphic to K/K_0, and we can choose a Borel section $\theta\colon \mathscr{M}_0 \to K$. So for each $M \in \mathscr{M}_0$, we have a unitary operator $\theta(M) \in K$ such that $M = A_0 \cdot \theta(M)$, i.e., $M = \theta(M)^{-1} \cdot A_0\theta(M)$. We now use the function $\theta \circ \tilde{A}\colon Y \to K$ to define a new cocycle, namely

$$\beta_0(y, g) = \theta(\tilde{A}_y)\beta(y, g)\theta(\tilde{A}_{yg})^{-1}.$$

We now claim that for each g, $\beta_0(y, g) \in K_0$ for almost all y. To see $\beta_0(y, g) \in K_0$, it suffices to see $\beta_0(y, g)^{-1}A_0\beta_0(y, g) = A_0$. But the left side is just, for almost all y,

$$\begin{aligned}&\theta(A_{yg})\beta(y, g)^{-1}\theta(A_y)^{-1}A_0\theta(A_y)\beta(y, g)\theta(A_{yg})^{-1} \\ &\qquad = \theta(A_{yg})\beta(y, g)^{-1}A_y\beta(y, g)\theta(A_{yg})^{-1} \\ &\qquad = \theta(A_{yg})A_{yg}\theta(A_{yg})^{-1} \\ &\qquad = A_0.\end{aligned}$$

Thus by changing β_0 on a set of measure 0, we can assume it is a cocycle with values in K_0. As K_0 is compact, β_0 has an equivalent minimal cocycle γ (Corollary 3.8). So there exists a Borel map $S\colon Y \to K_0$ such that for each g,

$$\gamma(y, g) = S(y)\beta_0(y, g)S(yg)^{-1} \quad \text{for almost all } y.$$

We now summarize what we have done so far by collecting some relevant facts about γ:

(1) $\gamma: Y \times G \to K_0$ is a minimal cocycle; K_0 is a compact group in $U(H_0)$, and leaves invariant a Boolean σ-algebra of projections A_0 on H_0.

(2) γ is cohomologous to the natural cocycle representation α; more exactly, if $T(y) = S(y)\theta(\tilde{A}_y)U(y)$, then $T(y): H(y) \to H_0$ is a Borel field of operators such that

(i) $T(y)$ is unitary for almost all y.

(ii) For each g, $T(y)\alpha(y, g)T(yg)^{-1} = \gamma(y, g)$ for almost all y.

(3) For almost all y, $T(y)B_yT(y)^{-1} = A_0$.

We now show the way in which these properties imply the structure theorem. By (1), A_0 is a K_γ-space and thus [14, Theorem 1], there exists a K_γ-Lebesgue space Z and an isomorphism of Boolean K_γ-spaces $p: B(Z) \to A_0$. Form the product space $Y \times Z$; this has a G-action defined by γ, namely $(y, z)g = (yg, z \cdot \gamma(y, g))$. We claim that this action is essentially isomorphic to the action of G on X. Now $B(X) \cong \int^\oplus B_y\, dy$ and the G-action is given as follows: If $S \in B(X)$ and $S \cong \int^\oplus S_y\, dy$, $S_y \in B_y$, then

$$(S \cdot g)_{yg} = \alpha(y, g)^{-1}S_y\alpha(y, g).$$

Similarly, if $E \in B(Y \times Z)$, and $E = \int^\oplus E_y\, dy$, then $(E \cdot g)_{yg} = E_y \cdot g = E_y \cdot \gamma(y, g)$. Since P: $B(Z) \to A_0$ is a K_γ-isomorphism, we have

$$P((E \cdot g)_{yg}) = P(E_y) \cdot \gamma(y, g) = \gamma(y, g)^{-1}P(E_y)\gamma(y, g).$$

Define a Boolean isomorphism $\tilde{P}: B(Y \times Z) \to \int_Y^\oplus A_0$ by $\tilde{P}(\int^\oplus E_y) = \int^\oplus P(E_y)$. Then the G-action on $B(Y \times Z)$ is carried over to the G-action on $\int^\oplus A_0\, dy$ given by $F = \int^\oplus F_y$, then $(F \cdot g)_{yg} = \gamma(y, g)^{-1}F_y\gamma(y, g)$. We thus have Boolean isomorphisms

$$B(X) \cong \int_Y^\oplus B_y \xrightarrow{Q} \int_Y^\oplus A_0 \xleftarrow{\tilde{P}} B(Y \times Z)$$

where Q is defined by $Q(\int^\oplus S_y) = \int^\oplus T(y)S_yT(y)^{-1}$, and is an isomorphism by property (3).

The outside 2 maps defined G-space structures on the inside 2 algebras as shown above. To show that $B(X)$ and $B(Y \times Z)$ are isomorphic, it suffices to show that Q is a G-map. Now

$$\begin{aligned} Q\left(\left(\int^\oplus S_y\right)\cdot g\right)_{yg} &= T(yg)(S \cdot g)_{yg}T(yg)^{-1} \\ &= T(yg)\alpha(y, g)^{-1}S_y\alpha(y, g)T(yg)^{-1} \\ &= \gamma(y, g)^{-1}T(y)S_yT(y)^{-1}\gamma(y, g) \qquad \text{(by (2)(ii) above)} \\ &= \left(\left[Q\left(\int^\oplus S_y\right)\right]\cdot g\right)_{yg}, \end{aligned}$$

all these holding for almost all y.

It is clear that under the isomorphism $B(X) \leftrightarrow B(Y \times Z)$, $B(Y)$ is mapped to itself by the identity. Thus, X and $Y \times Z$ are essentially isomorphic extensions of Y. To prove the structure theorem, it remains only to clarify the structure of the K_γ-space Z. Since $B(X)$ is an irreducible Boolean G-space, so is $B(Y \times Z)$. If $F \in B(Z)$ is K_γ-invariant then clearly $Y \times F$ is G-invariant in $B(Y \times Z)$, and hence F must be θ or Z; i.e., $B(Z)$ is an irreducible K_γ-space. Since K_γ is compact, we can choose Z to be K_γ/H for some closed subgroup H of K_γ [15, Lemma 2], and this completes the proof.

If K is a compact abelian group, and $\alpha: S \times G \to K$ is minimal, $K_\alpha = K$, then the natural (Y, G) cocycle of the extension $Y \times_\alpha K$ of Y has discrete spectrum, and each summand of the direct sum of irreducible cocycles is one-dimensional. As a corollary of the structure theorem, we have the following.

COROLLARY 4.6. *Suppose $\phi: X \to Y$ is a factor G-map and the natural (Y, G) cocycle has discrete spectrum with each summand one-dimensional. Then there exists a compact abelian group K and a minimal cocycle $\alpha: Y \times G \to K$, $K_\alpha = K$ such that X and $Y \times_\alpha K$ are essentially isomorphic extensions of Y.*

Proof. In the proof of the structure theorem, the compact group

$$\{U \in U(H_0) \mid U(H_i) = H_i\}$$

was constructed. Under the hypothesis of the corollary, each H_i is one-dimensional, and hence this group is abelian. Thus in the proof above, K_γ is abelian. If we let $p: K_\gamma \to K_\gamma/H$ the natural homomorphism, then $Y \times_\gamma K_\gamma/H \cong Y \times_{p \circ \gamma} K_\gamma/H$, and since $p \circ \gamma$ is minimal by Proposition 3.10, the corollary is proven.

The techniques of the proof of Theorem 4.3 actually enable us to prove a stronger result. In [15, Theorem 2], Mackey shows that if S is an ergodic G-space and the representation of G induced by any cocycle representation of $S \times G$ has discrete spectrum, then the representation induced by the identity cocycle also has discrete spectrum (i.e., S can be characterized by the conclusion of his structure theorem). Furthermore, one can give an explicit description of the given $S \times G$ cocycle. We will prove a generalization of this result to the case of extensions. We preface a statement of the theorem with a class of examples.

Example 4.7. Let Y, K, H, α, and X be as in Example 4.1. We construct $X \times G$ cocycles such that the induced $Y \times G$ cocycles have discrete spectrum. Let π be a representation of H on a Hilbert space H_0, and

$$\gamma: K/H \times K \to U(H_0)$$

a cocycle corresponding to this representation [20, Theorem 8.27]. Define a cocycle

$$\beta_\pi: X \times G \to U(H_0)$$

by $\beta_\pi((y, [k]), g) = \gamma([k], \alpha(y, g))$. One sees that the (Y, G) cocycle ρ induced by β_π,

$$\rho\colon Y \times G \to U(L^2(K/H; H_0)),$$

is given by $\rho(y, g) = U^\pi(\alpha(y, g))$, where U^π is the representation of K on $L^2(K/H; H_0)$ induced by π. (In case π is the one-dimensional identity representation, $U^\pi = \sigma =$ natural representation of K on $L^2(K/H)$. Then $\rho = \sigma \circ \alpha$ which is the cocycle considered in Example 4.1.) As before, $U^\pi = \sum^\oplus \sigma_i$, so $\rho = \sum^\oplus \sigma_i \circ \alpha$, with $\dim \sigma_i < \infty$. Thus ρ has discrete spectrum.

Loosely, our generalization of the structure theorem says that whenever $\phi\colon X \to Y$ is a factor map of ergodic G-spaces, and there is a cocycle β on $X \times G$ such that the induced (Y, G) cocycle of β has discrete spectrum, then there exist K, H, α, and π as above so that β is essentially the cocycle β_π constructed above.

THEOREM 4.8. *Let $\phi\colon X \to Y$ be a G-factor map. Suppose β is a cocycle representation of (X, G) such that the induced cocycle representation α of (Y, G) has discrete spectrum. Then there exists a compact group K, a closed subgroup $H \subset K$, a unitary representation π of H, a minimal cocycle $\gamma\colon Y \times G \to K$, $K_\gamma = K$, such that the pair $(U^\beta, B(X))$ is unitarily equivalent to the pair $(U^{\beta_\pi}, B(Y \times_\gamma K/H))$. Thus, under the identification of the G-spaces X and $Y \times_\gamma K/H$, β is cohomologous to the cocycle β_π.*

Before considering the proof, we examine what this says in the case $Y = \{e\}$. Then γ will be a homomorphism $G \to K$ with dense range, and

$$(U^{\beta_\pi}, B(Y \times K/H)) = (U^\pi \circ \gamma, B(K/H)).$$

Thus, Theorem 4.8 asserts a unitary equivalence of $(U^\beta, B(X))$ with $(U^\pi \circ \gamma, B(K/H))$, which is exactly the content of Mackey's Theorem 2 of [15].

Proof (*of Theorem* 4.8). We construct γ, $T(y)$, and A_0 exactly as in the proof of the structure theorem. Since A_0 is a K_γ-invariant Boolean algebra on H_0, we can write $H_0 \cong L^2(Z; H_1)$ where H_1 is a Hilbert space, Z is a K_γ-space, and $A_0 \cong B(Z)$ as Boolean K_γ-spaces. Since $K_\gamma \subset U(H_0)$, we have (tautologously) a unitary representation σ of K_γ that leaves $B(Z)$ invariant. As in the proof of the structure theorem, one can now check that $T = \int^\oplus T(y)$ will yield a unitary equivalence of the pairs of $(U^\alpha, B(X))$ and $(U^{\sigma\circ\gamma}, B(Y \times Z))$. Reasoning as before, Z must be of the form K_γ/H for a closed subgroup $H \subset K_\gamma$. It then follows from Mackey's imprimitivity theorem that $\sigma = U^\pi$ for some representation π of H. But $U^\pi \circ \gamma = \operatorname{ind}(\beta_\pi)$, and we have an equivalence of

$$(U^{U^\pi\circ\gamma}, B(Y \times K_\gamma/H))$$

with $(U^{\beta_\pi}, B(Y \times K_\gamma/H))$. Hence, the equivalence of $(U^\alpha, B(X))$ with $(U^{\beta_\pi}, B(Y \times K_\gamma/H))$ is established. The last statement of the theorem follows from Theorem 2.6.

5. Normal actions and extensions. In their study of ergodic transformations with discrete spectrum [6], [17], von Neumann and Halmos were able not only to describe the structure of such transformations, but to prove an existence-uniqueness theorem. This theorem says that if two ergodic transformations with discrete spectrum have the same spectrum, they are essentially isomorphic. Furthermore, this (discrete) spectrum is always a countable subgroup of the circle, and conversely, for any countable subgroup S of the circle, there exists an ergodic transformation with S as its spectrum. As Mackey points out in [15], there is an equally complete theorem for actions of a locally compact abelian group (see [21] for a detailed proof). However, as also indicated in [15], the result fails when the hypothesis that the group is abelian is dropped. Our aim in this and the following section is two-fold. First, a natural class of ergodic actions of a group, called normal actions, is defined. We show that this includes all ergodic actions of a locally compact abelian group with discrete spectrum, and that for the class of normal actions, an analogue of the existence-uniqueness theorem is valid. Secondly, and more interestingly, we extend the notion of normality to the relative situation of extensions of arbitrary G-spaces, and prove an existence-uniqueness theorem for normal extensions. Even in the case where $G = Z$, this provides a new direction of generalization of the von Neumann–Halmos theorem.

It will be convenient to establish some notation.

DEFINITION 5.1. If Y is a factor of an ergodic G-space X, and the natural (Y, G) cocycle representation has discrete spectrum, we shall say that X has relatively discrete spectrum over Y. If the cocycle can be written as a direct sum of one-dimensional cocycle representations, we will say that X has relatively elementary spectrum over Y.

Thus Theorem 4.13 and Corollary 4.6 describe the structure of extensions with relatively discrete and relatively elementary spectrum, respectively.

DEFINITION 5.2. Let S be an ergodic Lebesgue G-space and let

$$\beta\colon G \to U(L^2(S))$$

be the natural representation. Define the cocycle $\alpha\colon S \times G \to U(L^2(S))$ by $\alpha(s, g) = \beta(g)$. S is called a normal G-space if α is cohomologous to the identity cocycle.

Remark. In terms of virtual groups, α is the "restriction" of β to the virtual subgroup (S, G) (Example 2.7).

PROPOSITION 5.3. *If S is a transitive G-space, i.e., $S = G/H$ for some closed subgroup $H \subset G$, then S is normal if and only if H is a normal subgroup.*

Proof. $\alpha\colon G/H \times G \to U(L^2(G/H))$ is a cocycle and hence corresponds to a representation of H on $L^2(G/H)$ [20, Theorem 8.27], which in this case is clearly just the restriction of β to H. Furthermore, α is cohomologous to the identity if and only if the representation of H is the identity. It is clear that $\beta \mid H$ is the identity on $L^2(G/H)$ if H is a normal subgroup. Conversely, suppose H acts by the identity on $L^2(G/H)$. Then the Boolean action of H on $B(G/H)$ is the identity and by [14, Theorem 2], almost all points of G/H are left fixed. But the set of points fixed by H is $N/H \subset G/H$ where $N \subset G$ is the normalizer of H. If N/H is conull in G/H, N must be conull in G, and being a subgroup, must equal G. Hence H is normal.

More generally, now suppose that $p\colon X \to Y$ is a factor G-map of ergodic G-spaces. As usual, let $F_y = p^{-1}(y)$. Then the assignment

$$x \to H_x = L^2(F_{p(x)})$$

defines a Hilbert bundle in a natural way. Let $\beta(y, g)\colon L^2(F_{yg}) \to L^2(F_y)$ be the natural (Y, G) cocycle representation and define a cocycle representation

$$\alpha(x, g)\colon H_{xg} \to H_x$$

by $\alpha(x, g) = \beta(p(x), g)$.

DEFINITION 5.4. X is called a normal extension of Y (or Y a normal factor of X) if the cocycle α is equivalent to the identity.

Remarks. (i) When $Y = \{e\}$, this construction reduces to that of Definition 5.2.

(ii) In terms of virtual groups, α is the "restriction" of β to the sub-virtual subgroup (X, G). (See Example 2.7.)

PROPOSITION 5.5. *If $H \subset K \subset G$ are subgroups of G, and $p\colon G/H \to G/K$ the natural map, then G/H is a normal extension of G/K if and only if H is a normal subgroup of K.*

Proof. The $G/H \times G$ cocycle representation α is defined by a representation of H on the Hilbert space $H_{[e]} = L^2(F_{[e]})$. But $F_{[e]}$ can be naturally identified with K/H, and the representation of H on $L^2(K/H)$ is the restriction of the natural representation of K. α will be equivalent to the identity if and only if this representation of H is the identity, which is true if and only if H is a normal subgroup of K.

We now consider the question of when an action with discrete spectrum or an extension with relatively discrete spectrum, is normal.

PROPOSITION 5.6. *Let $\beta\colon Y \times G \to K$ a minimal cocycle, $K_\beta = K$, $H \subset K$ a closed subgroup, and $X = Y \times_\beta K/H$. Then if H is normal, X is a normal extension of Y.*

Proof. Let $T\colon K \to K/H$ the natural homomorphism. Replacing K by K/H and β by $T \circ \beta$, we see that we can assume $H = \{e\}$. It is easy to check that the Hilbert bundle $\{H_x\}$ is unitarily equivalent to the Hilbert bundle

$x \mapsto L^2(K)$, and the (X, G) cocycle is given by $\alpha(x, g) = \pi \circ \beta(p(x), g)$, where $p: X \to Y$ is projection and π is the right regular representation of K. For each $x = (y, k)$, let $U(x): H_x \to H_x$ be $U(x) = \pi(k)$. Then

$$U(x)\alpha(x, g)U(xg)^{-1} = \pi(k)\pi(\beta(p(x), g))\pi(k\beta(p(x), g))^{-1} = I,$$

proving the proposition.

The main result of this section is the converse of Proposition 5.6.

THEOREM 5.7. *Let $\beta: Y \times G \to K$, minimal, with $K_\beta = K$. Let $H \subset K$ be a closed subgroup, and suppose that $X = Y \times_\beta K/H$ is a normal extension of Y. Then H is a normal subgroup of K.*

Proof. Let π be the natural representation of K on $L^2(K/H)$. Thus

$$\alpha: X \times G \to U(L^2(K/H))$$

is given by $\alpha(x, g) = \pi \circ \beta(p(x), g)$. Suppose now that α is cohomologous to the identity cocycle i. We will denote by α_Y and i_Y the (Y, G) cocycles induced by α and i respectively. To write α_Y and i_Y in a more convenient form, we introduce some notation. Let

$$\tilde{H} = \int_{K/H}^{\oplus} L^2(K/H) = L^2(K/H; L^2(K/H)).$$

We have 2 representations of K on $\tilde{H}$. For $f \in \tilde{H}$, $s \in K/H$, let $(U_k f)(s) = f(s \cdot k)$, $(W_k f)(s) = \pi_k(f(sk))$. Now $\alpha_Y, i_Y: Y \times G \to U(\tilde{H})$, and it is easy to check from the definition of induced cocycle that we can take $\alpha_Y(y, g) = W_{\beta(y,g)}$ and $i_Y(y, g) = U_{\beta(y,g)}$. Since α and i are cohomologous, there exists a Borel function $A: X \to U(L^2(K/H))$ such that for each g, $A(x)\alpha(x, g)A(xg)^{-1} = i(x, g)$, almost everywhere. Define $T: Y \to U(\tilde{H})$ by

$$T(y) = \int_{x \in p^{-1}(y)}^{\oplus} A(x).$$

Then T is a Borel function and for each g,

$$(*) \qquad T(y)\alpha_Y(y, g)T(yg)^{-1} = i_Y(y, g) \quad \text{for almost all } y \in Y.$$

Now K acts on $U(\tilde{H})$ by $T \cdot K = U_k^{-1} T W_k$. Under this action, $U(\tilde{H})$ is a standard Borel K-space, and since K is compact, the space of orbits, B, is standard. Let $q: U(\tilde{H}) \to B$ be the natural map. Now by (*), for each g,

$$\begin{aligned} T(yg) &= i_Y(y, g)^{-1} T(y)\alpha_Y(y, g) \\ &= U_{\beta(y,g)}^{-1} T(y) W_{\beta(y,g)} \\ &= T(y) \cdot \beta(y, g) \quad \text{for almost all } y; \end{aligned}$$

i.e., for each g, $q(T(yg)) = q(T(y))$ for almost all y. By the ergodicity of G on Y, and the fact that B is standard, we have that $q \circ T$ is constant on a conull set Y_0. That is, there exists an orbit $R \subset U(\tilde{H})$ such that $T(y) \in R$ for every $y \in Y_0$. Choose an element T_0 of the orbit R. Then there exists a Borel function $\lambda: R \to K$ such that $T \cdot \lambda(T) = T_0$, for all $T \in R$ [13, Lemma 1.1]. Let $\theta: Y_0 \to K$ be $\theta = \lambda \circ T$, and extend θ to a Borel map $Y \to K$. Thus $T(y) \cdot \theta(y) = T_0$ for almost all y. We now use θ to define a new cocycle $\gamma: Y \times G \to K$ equivalent to β, namely $\gamma(y, g) = \theta(y)^{-1}\beta(y, g)\theta(yg)$. We now claim the following, which is basically just unravelling the definitions.

LEMMA. *For each g, $T_0 W_{\gamma(y,g)} T_0^{-1} = U_{\gamma(y,g)}$ for almost all y.*

Proof.

$$\begin{aligned} T_0 W_{\gamma(y,g)} T_0^{-1} &= (T(y) \cdot \theta(y)) W_{\theta(y)^{-1}\beta(y,g)\theta(yg)} (T(yg) \cdot \theta(yg))^{-1} \\ &= U_{\theta(y)}^{-1} T(y) W_{\beta(y,g)} T(yg)^{-1} U_{\theta(yg)} \\ &= U_{\theta(y)}^{-1} U_{\beta(y,g)} U_{\theta(yg)} \\ &= U_{\gamma(y,g)}. \end{aligned}$$

Now let $K_0 = \{k \in K \mid T_0 W_k T_0^{-1} = U_k\}$. It is easy to see that K_0 is a closed subgroup of K, and the lemma says that for each g, $\gamma(y, g) \in K_0$ for almost all $y \in Y$. By changing γ on a suitable null set, we get an equivalent cocycle γ_1 such that $\gamma_1(y, g) \in K_0$ for all (y, g). Now γ_1 is equivalent to β, and since β is a minimal cocycle, we must have $K_0 = K$. Thus T_0 is an intertwining operator for U and W. We claim furthermore that T_0 commutes with the natural projection valued measure $P: B(K/H) \to L(\tilde{H})$. To see this, first note that since $T(y) = \int_{x \in F_y}^{\oplus} A(x)$, for each $E \in B(K/H)$, we have $P_E T(y) = T(y) P_E$. In addition, for all k, $U_k^{-1} P_E U_k = P_{Ek}$ and a similar relation holds for W. Combining these identities, we see that $U_k^{-1} T(y) W_k$ commutes with P_E for all E, k, y. Since $T_0 = T(y) \cdot \theta(y)$ for almost all (and hence at least one) y, T_0 commutes with P_E. It follows that the representations of H which induce U and W must be equivalent. But U is the representation induced by the identity representation of H on $L^2(K/H)$, and W is the representation induced by the restriction to H of the natural representation of K on $L^2(K/H)$. For these to be equivalent, H must be normal.

COROLLARY 5.8. *Let X be an ergodic extension of Y with relatively discrete spectrum. Let α be the natural (Y, G) cocycle representation, and let S be the set of equivalence classes of irreducible (Y, G) cocycle representations that are subcocycles of α. Then X is a normal extension of Y if and only if S satisfies the following conditions:*

(i) *$\alpha_1, \alpha_2 \in S$ implies every irreducible component of $\alpha_1 \otimes \alpha_2$ is in S.*
(ii) *$\sigma \in S$ implies $\bar{\sigma} \in S$.*
(iii) *$\alpha = \sum_{\sigma \in S} (\dim \sigma)\sigma$.*

Proof. Let $X = Y \times_\beta K/H$ where $\beta: Y \times G \to K$ is a minimal cocycle with $K_0 = K$. Let π be the natural representation of K on $L^2(K/H)$. Then α

is equivalent to $\pi \circ \beta$. Let S' be the set of equivalence classes of irreducible representations of K that are subrepresentations of π, and for each $\lambda \in S'$, n_λ the positive integer such that $\pi = \sum_{\lambda \in S'}^{\oplus} n_\lambda \lambda$. Then

$$\alpha = \pi \circ \beta = \sum_{\lambda \in S'}^{\oplus} n_\lambda (\lambda \circ \beta).$$

From this equation and Theorem 3.14, it follows that $S = S' \circ \beta$ and that S satisfies (i), (ii), (iii) above if and only if S' satisfies the analogous properties. By [8, 30. 60] and the Peter-Weyl theorem, this will be true if and only if H is normal, and by Theorem 5.7, this holds if and only if X is a normal extension of Y.

COROLLARY 5.9. *If X is an ergodic extension of Y with relatively elementary spectrum, then X is a normal extension of Y.*

Proof. By Corollary 4.6, $X = Y \times_\beta K$ where K is compact, abelian, and the result is immediate from Theorem 5.7.

COROLLARY 5.10. *Any action of a locally compact abelian group with discrete spectrum is normal.*

6. The existence-uniqueness theorems. We now prove analogues of von Neumann's and Halmos' existence-uniqueness theorems for the class of normal actions and extensions. An essential step is the following general result about cocycles into compact groups.

THEOREM 6.1. *Let $K \subset L$ be compact groups α, $\beta\colon Y \times G \to K$ minimal cocycles with $K_\alpha = K_\beta = K$. Suppose that α and β are equivalent as cocycles into L. Then there is a continuous automorphism σ of K such that $\sigma \circ \alpha$ and β are equivalent as cocycles into K.*

Proof. Let $\phi\colon Y \to L$ be a Borel function such that for each g and almost all y, $\phi(y)\alpha(y, g)\phi(yg)^{-1} = \beta(y, g)$, i.e.,

$$(*) \qquad \beta(y, g)^{-1}\phi(y)\alpha(y, g) = \phi(yg).$$

Now $K \times K$ acts on L by $l \cdot (k_1, k_2) = k_1^{-1} l k_2$. Since the action is continuous and $K \times K$ is compact, the space of orbits in L under $K \times K$ is a standard Borel space $\hat{L}$. Let $p\colon L \to \hat{L}$ be the natural (Borel) map. Since $\alpha(y, g)$, $\beta(y, g) \in K$, equation (*) implies that for each g, $p(\phi(y)) = p(\phi(yg))$ for almost all y. By the ergodicity of G on Y, $p \circ \phi$ is constant on a conull Borel set Y_0. Choose a point l_0 in the orbit $p(\phi(Y_0))$. Then [13, Lemma 1.1], there exists a Borel map

$$(k_1, k_2)\colon p(l_0) \to K \times K$$

such that for any $l \in p(l_0)$, $l = k_1(l)^{-1} l_0 k_2(l)$. Now $k_1 \circ \phi$, $k_2 \circ \phi\colon Y_0 \to K$ are Borel functions, and can be extended to Borel functions ϕ_1, $\phi_2\colon Y \to K$. Thus, for any $y \in Y_0$, $\phi(y) = \phi_1(y)^{-1} l_0 \phi_2(y)$. From (*), we have for all $g \in G$ and $y \in Y_0 \cap Y_0 g^{-1}$,

$$\phi_1(y)^{-1} l_0 \phi_2(y) \alpha(y, g) \phi_2(yg)^{-1} l_0^{-1} \phi_1(yg) = \beta(y, g).$$

With $\alpha_2(y, g) = \phi_2(y)\alpha(y, g)\phi_2(yg)^{-1}$, this becomes

$$(**) \qquad l_0 \alpha_2(y, g) l_0^{-1} = \phi_1(y)\beta(y, g)\phi_1(yg)^{-1}.$$

Now α is a minimal cocycle with $K_\alpha = K$, and since α_2 is equivalent to α, we also have $K_{\alpha_2} = K$. Consider $K_0 = \{k \in K \mid l_0 k l_0^{-1} \in K\}$. Then K_0 is a closed subgroup of K, and by (**), $\alpha_2(y, g) \in K_0$ for each g and almost all y. Changing α_2 on a suitable null set, we see that α_2 is equivalent to a cocycle with all values in K_0, and since α is minimal, $K_0 = K$. Let σ be the automorphism of K defined by $\sigma(k) = l_0 k l_0^{-1}$. Letting $\phi_0(y) = \sigma(\phi_2(y))$, we have $\phi_0(y) \in K$ for all y, and from (**), for each $g \in G$,

$$\phi_0(y)(\sigma \circ \alpha)(y, g)\phi_0(yg)^{-1} = \phi_1(y)\beta(y, g)\phi_1(yg)^{-1}$$

for almost all y. This completes the proof.

We now apply this result to prove the uniqueness theorem.

THEOREM 6.2 (Uniqueness Theorem). *Suppose X_1 and X_2 are normal ergodic extensions of Y with relatively discrete spectrum. If the corresponding natural (Y, G) cocycle representations are equivalent, then X_1 and X_2 are essentially isomorphic extensions of Y.*

Remark. In the case $G = Z$ and $Y = \{e\}$, this is just the von Neumann–Halmos uniqueness theorem.

Proof. By the Structure Theorem (4.3), and Theorem 5.7, we can write $X_i = Y \times_{\alpha_i} K_i$, where α_i are minimal cocycles with $K_{\alpha_i} = K_i$. Let π_i be the right regular representation of K_i, S_i the dual object of K_i, and form the canonical decompositions

$$L^2(K_1) = \sum_{\sigma \in S_1}^{\oplus} H_\sigma, \quad L^2(K_2) = \sum_{\tau \in S_2}^{\oplus} H_\tau.$$

The hypothesis of the theorem is that $\pi_1 \circ \alpha_1$ and $\pi_2 \circ \alpha_2$ are unitarily equivalent. By Theorem 3.14, for each $\sigma \in S_1$, $\sigma \circ \alpha_1$ will be equivalent to exactly one $\tau \circ \alpha_2$, $\tau \in S_2$. Then it is easy to see that the equivalence of the cocycle representations $\pi_1 \circ \alpha_1$ and $\pi_2 \circ \alpha_2$ can be implemented by a Borel field of unitary operators $U(y)$: $L^2(K_1) \to L^2(K_2)$ such that for σ, τ related as above, $U(y)(H_\sigma) = H_\tau$ for all $y \in Y$. Fix an operator U: $L^2(K_1) \to L^2(K_2)$ such that $U(H_\sigma) = H_\tau$. Let

$$\beta(y, g) = U\pi_1\alpha_1(y, g)U^{-1},$$

and let

$$K = \{T \in U(L^2(K_2)) \mid T(H_\tau) = H_\tau \quad \text{for all } \tau \in S_2\}.$$

Then K is a compact group and β is a cocycle with values in K. Since $U(y)U^{-1} \in K$ for all y, it follows that β is equivalent as a cocycle into K, to $\pi_2 \circ \alpha_2$. Since α_1 is minimal with $K_{\alpha_1} = K_1$, it follows from Proposition 3.10 and Theorem 3.9 that β is also minimal and $K_\beta = U\pi_1(K_1)U^{-1}$. As $\pi_2 \circ \alpha_2$ is also minimal with $K_{\pi_2 \circ \alpha_2} = \pi_2(K_2)$, Corollary 3.8 implies that there exists $W \in K$ such that

$$WU\pi_1(K_1)U^{-1}W^{-1} = \pi_2(K_2).$$

Let $\phi\colon K_1 \to K_2$ be defined by $\phi(k) = \pi_2^{-1}(WU\pi_1(k)U^{-1}W^{-1})$. It is clear that ϕ is an isomorphism (since π_i are). Let $\alpha_0\colon Y \times G \to K_2$ be the cocycle $\alpha_0 = \phi \circ \alpha_1$. Clearly $Y \times_{\alpha_0} K_2$ and $Y \times_{\alpha_1} K_1$ are essentially isomorphic extensions of Y. Thus, to prove the theorem it suffices to show that $Y \times_{\alpha_0} K_2$ and $Y \times_{\alpha_2} K_2$ are essentially isomorphic extensions. Now for all g,

$$WUU(y)^{-1}(\pi_2 \circ \alpha_2)(y, g)U(yg)U^{-1}W^{-1} = (\pi_2 \circ \alpha_0)(y, g)$$

for almost all y. As remarked above, $U \cdot U(y)^{-1} \in K$ for all y, and hence $\pi_2 \circ \alpha_0$ and $\pi_2 \circ \alpha_2$ are equivalent as cocycles into K. Since both of these cocycles take values in $\pi_2(K_2) \subset K$, by Theorem 6.1 there exists a continuous automorphism σ of $\pi_2(K_2)$ such that $\sigma \circ \pi_2 \circ \alpha_0$ and $\pi_2 \circ \alpha_2$ are cohomologous as cocycles into $\pi_2(K_2)$. Since π_2 is an isomorphism, there exists a continuous automorphism γ of K_2 such that $\gamma \circ \alpha_0$ and α_2 are cohomologous. Thus $Y \times_{\alpha_2} K_2$ and $Y \times_{\gamma \circ \alpha_0} K_2$ are essentially isomorphic extensions of Y, and the latter is clearly essentially isomorphic to the extension $Y \times_{\alpha_0} K_2$. By the remarks above, this completes the proof.

COROLLARY 6.3. *For an ergodic G-space Y, let S be the set of equivalence classes of one-dimensional cocycle representations. Then S is a group under the operation of tensor product. If X is an extension of Y with relatively elementary spectrum over Y, let S_X be the subset of S consisting of cocycles appearing in the decomposition of the natural (Y, G) cocycle representation on the Hilbert bundle $L^2(X)$. Then S_X is a countable subgroup of S. If Z is another extension of Y with relatively elementary spectrum, then X and Z are essentially isomorphic extensions of Y if and only if $S_X = S_Z$.*

Proof. By Corollary 4.6, we can take $X = Y \times_\alpha K$, K compact, abelian. The natural (Y, G) cocycle representation, β, on $L^2(X)$ is just $\pi \circ \alpha$, where π is the right regular representation of K. Thus,

$$\beta \cong \sum_{\chi \in K^*}^{\oplus} \chi \circ \alpha,$$

and by Theorem 3.14, S_X is a group isomorphic to $\{\chi \circ \alpha \mid \chi \in K^*\}$. The last assertion follows from Corollary 5.9 and Theorem 6.2.

Remarks. (i) Since S_X is isomorphic to $\{\chi \circ \alpha \mid \chi \in K^*\} \cong K^*$, we can identify the group K in Corollary 4.6 as S_X^*. When $G = Z$, and $Y = \{e\}$, this is a well-known fact in the von Neumann–Halmos theory.

(ii) By Frobenius reciprocity (Theorem 2.10), a one-dimensional cocycle representation γ of (Y, G) will be a subcocycle representation of the natural induced cocycle representation β defined by the extension X, if and only if the restriction of γ to X (Example 2.7) is equivalent to the identity. Thus, S_X can be identified as the group

$$\{\gamma \in S \mid \mathrm{res}\,(\gamma) \text{ is the class of the identity cocycle of } (X, G)\}.$$

We now proceed to the existence theorem.

THEOREM 6.4. *Let Y be an ergodic Lebesgue G-space and let $S = \{[\alpha_i]\}_{i \in I}$ be a countable set of equivalence classes of finite dimensional irreducible cocycle representations of (Y, G) where I is some countable index set. Suppose S satisfies:*

(i) *If $\alpha \in S$ then $\bar{\alpha} \in S$.*

(ii) *If $\alpha, \beta \in S$ then every irreducible component of $\alpha \otimes \beta$ is in S.*

Then there exists a normal ergodic extension X of Y with relatively discrete spectrum over Y such that the natural (Y, G) cocycle representation on $L^2(X)$ is equivalent to $\sum_{\alpha \in S}^{\oplus} (\dim \alpha)\alpha$.

Proof. Let J be a sequence of elements from I such that each $i \in I$ appears in J $\dim(\alpha_i)$ times. For each $j \in J$, we have

$$\alpha_j \colon Y \times G \to U(H_j)$$

where H_j is a finite dimensional Hilbert space. Let $H = \sum_J^{\oplus} H_j$ and $\alpha = \sum_{j \in J}^{\oplus} \alpha_j$. Let $K = \{T \in U(H) \mid T(H_j) = H_j \text{ for all } j\}$ with the weak operator topology. Then K is a compact group and $\alpha\colon Y \times G \to K$. Let $\beta\colon Y \times G \to K$ be an equivalent minimal cocycle (Corollary 3.8), and for each j, $\pi_j\colon K_\beta \to U(H_j)$ the map obtained by restricting elements of K_β to H_j. It is easy to see that $\pi_j \circ \beta$ is equivalent to α_j, and hence is irreducible. It follows that for each j, π_j is an irreducible representation of K_β. Let $S' = \{[\pi_j]\}_{j \in J} \subset \hat{K}_\beta$ (the dual object of K_β). It is straightforward to see that S' satisfies properties (i) and (ii) as well as S. Since the representation $\sum_{j \in J}^{\oplus} \pi_j$ is faithful, we have $S' = \hat{K}_\beta$ by [8, Theorem 27.39]. It follows from Theorem 3.14, the construction of J, and the Peter-Weyl theorem, that $\sum_{j \in J}^{\oplus} \pi_j$ is unitarily equivalent to the right regular representation, π, of K_β. Let $X = Y \times_\beta K_\beta$. Then the natural (Y, G) cocycle representation is $\pi \circ \beta$, which is equivalent to

$$\sum_{j \in J}^{\oplus} \pi_j \circ \beta \cong \sum_{j \in J}^{\oplus} \alpha_j.$$

By the choice of J, this is equivalent to the theorem.

COROLLARY 6.5. *Let Y be an ergodic G-space and S the group described in Corollary 6.3. Let $S' \subset S$ be a countable subgroup. Then there exists an ergodic extension X of Y with relatively elementary spectrum over Y such that $S_X = S'$.*

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