POLYNOMIAL AND LINEAR FRACTIONAL FACTORS OF AUTOMORPHY

BY

DAVID A. JAMES

1. Introduction

Suppose $f(z)$, not identically zero, is a function meromorphic on the open upper half plane Π^+, and Γ is a group of unimodular two by two matrices with real entries. If

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is in Γ, let Mz denote $(az + b)/(cz + d)$. We wish to find possible forms for $f(Mz)/f(z)$ as M ranges over Γ. If $v(z, M) = f(Mz)/f(z)$, we have the *consistency condition*

$$v(z, MN) = v(z, N)v(Nz, M) \quad \forall M, N \in \Gamma, \forall z \in \Pi^+.$$

Bochner [2] named a v which satisfies this equation a *factor of automorphy for f on* Γ, (a "Transformationsfaktor" in Petersson's terminology). We relax Bochner's stipulation that v be analytic in z by permitting meromorphic v.

Two distinguished factors of automorphy are

$$v(z, M) \equiv 1 \quad \text{and} \quad v(z, M) = u(M)(cz + d)^r$$

where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is in $\Gamma \subset SL(2, R)$, and $u(M)$ is a scalar representation of Γ. For the first, $f(Mz) = f(z)$, and f is called an *unrestricted automorphic function on* Γ. For the second, $f(Mz) = u(M)(cz + d)^r f(z)$, and f is an *unrestricted automorphic form of degree* $-r$ (weight $r/2$) *on* Γ *with multiplier system* u. "Unrestricted" here indicates the omission of the usual condition that automorphic functions and forms satisfy certain growth conditions at parabolic vertices of a fundamental region [10]. Poincaré [14] showed that if Γ is discontinuous and r is integral, there is a nontrivial meromorphic function which has $(dMz/dz)^r = (cz + d)^{-2r}$ as a factor of automorphy. For this reason Gunning calls dMz/dz the *Poincaré factor of automorphy*. For discontinuous Γ, Petersson and others studied functions which have such factors of automorphy for the cases r real [11] and complex [12] with multiplier systems. More recently, Knopp [9] gave a classification for these functions (automorphic forms) which correspond to nondiscontinuous Γ.

Received July 29, 1975.

An outstanding question mentioned by Siegel [15, p. 39] is that of the determination of the factors of automorphy other than powers of the Poincaré factor. Some early results of Appell concerned discontinuous Γ containing only translations. For such a cyclic $\Gamma = \langle T \rangle$, he found [1, Chapter 1, Section 4] that for any entire nonvanishing $v(z)$, there corresponds an entire nonvanishing $f(z)$ such that $f(Tz)/f(z) = v(z)$. Thus the nonvanishing entire factor of automorphy defined by $v(z, T^l) = f(T^l z)/f(z)$ has in some sense arbitrary form. Particular examples such as the Weierstrass σ-function and the Jacobi functions have been well studied [3], [8, p. 158], [15, p. 41]. On the other hand, Gunning [4], [5, Section II] has studied discontinuous Γ such that Π^+/Γ is compact, and found through potential theoretic techniques that any analytic nonvanishing (on Π^+) factor of automorphy has the form

$$v(z, M) = u(M)(cz + d)^r h(Mz)/h(z)$$

where r is rational and depends on Γ, and $h(z)$ is analytic and nonzero on Π^+. Conversely, any such $v(z, M)$ is a factor of automorphy.

More generally, for C^∞ differentiable manifolds M in place of Π^+, Gunning [5], [6], with minor qualifications, classified into cohomology classes, nonvanishing analytic factors of automorphy which act on the full Lie group of C^∞ automorphisms of M, by making use of the fact that such a factor of automorphy is a one-cycle with coefficients in the abelian group of all analytic nowhere vanishing functions.

In Petersson's paper [13], $D = \mathbf{C}$ and $\Gamma = GL(2, \mathbf{C})$, and rational factors of automorphy on Γ are considered; that is, in lowest terms,

$$v(z, M) = \frac{a(M)z^m + a_1(M)z^{m-1} + \cdots + a_m(M)}{b(M)z^n + b_1(M)z^{n-1} + \cdots + b_n(M)}$$

with $a(M) \neq 0$, $b(M) \neq 0$, and m, n fixed integers. The consistency condition $v(z, MN) = v(z, N)v(Nz, M)$ is considered an equation in nine complex dimensions, four for each matrix, and z, and is assumed to hold everywhere except at most on a manifold of lower dimension. It is proved that

$$v(z, M) = u(M)(cz + d)^n Q(Mz)/Q(z)$$

for some rational function $Q(z)$ and integer n, both independent of M.

Unfortunately, little of the above cited work imparts any information about the function $f(z)$ if it is known to have a given type of factor of automorphy for a certain Γ. The reason is that except in [5], both Petersson and Gunning assume the factors of automorphy act on very large groups, groups in fact which at each point fail to be discontinuous, a situation not calculated to allow a corresponding $f(z)$. Even in [5], Π^+/Γ is assumed compact, also undesirable; e.g., the modular group. I therefore present two results valid for Γ arbitrary: (1) for polynomial, and (2) for linear fractional factors of automorphy. I thank M. Knopp for his encouragement and suggestions in this work.

2. Polynomial factors of automorphy

A factor of automorphy on the group $\Gamma \subset GL(2, \mathbf{C})$ is a function, not identically zero, which satisfies the consistency condition $v(z, MN) = v(z, N)v(Nz, M)$ for all M, N in Γ and all z in D, an open Γ-invariant subset of the Riemann sphere. Many results cited in the introduction required $\Gamma \subset SL(2, R)$ and $D = \Pi^+$, and are easily transferable to any Fuchsian group. However, the theorems below require no such restrictions on Γ and D.

A useful consequence of the consistency condition is that

$$v(z, M^n) = v(z, M)v(Mz, M) \cdots v(M^{n-1}z, M).$$

THEOREM 1. *If a factor of automorphy $v(z, M)$ is a polynomial in z of degree r or less for every $M \in \Gamma$, where the coefficients depend on M, and if r is minimal, then*

$$v(z, M) = u(M)(cz + d)^r, \quad \forall M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Thus if a function $f(z)$ has such a polynomial factor of automorphy, it is an unrestricted automorphic form.

Proof. This follows from a result of Gunning [5, Section II] in the case that Γ is discontinuous and the planar closure of its fundamental region is compact. We present an elementary proof for all Γ.

Let N be the matrix in Γ such that

$$v(z, N) = a_r(N)z^r + \cdots + a_1(N)z + a_0(N) \quad \text{where } a_r(N) \neq 0.$$

Let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

belong to Γ. Then

$$\begin{aligned} v(z, NM) &= v(z, M)v(Mz, N) \\ &= v(z, M)\left[a_r(N)\left(\frac{az+b}{cz+d}\right)^r + \cdots + a_1(N)\left(\frac{az+b}{cz+d}\right) + a_0(N)\right] \\ &= v(z, M)[a_r(N)(az+b)^r + a_{r-1}(N)(az+b)^{r-1}(cz+d) + \cdots \\ &\quad + a_1(N)(az+b)(cz+d)^{r-1} + a_0(N)(cz+d)^r]/(cz+d)^r. \end{aligned}$$

But NM is in Γ so $v(z, NM)$ is a polynomial of degree r or less. One of three cases occur:

(i) $c = 0$, so that $a \neq 0$ since $ad - bc \neq 0$. This implies the degree of $v(z, M)$ is zero. $u(M)$ is then defined as this constant polynomial divided by d^r, so that $v(z, M) = u(M)(0z + d)^r$.

(ii) $(cz + d)^r$ divides $v(z, M)$ where $c \neq 0$. Thus $v(z, M) = u(M)(cz + d)^r$ for some $u(M)$ independent of z.

(iii) $(cz + d)$ divides

$$a_r(N)(az + b)^r + \cdots + a_1(N)(az + b)(cz + d)^{r-1} + a_0(N)(cz + d)^r$$

where $c \neq 0$, so that $cz + d$ divides $(az + b)^r$ and hence $az + b$. Thus $d/c = b/a$ which contradicts $ad - bc \neq 0$.

This completes the proof.

We introduce a definition of Gunning [5]. A *summand of automorphy* is a function, analytic in z, which satisfies $\sigma(z, MN) = \sigma(z, N) + \sigma(Nz, M)$ for all M, N in Γ and all z in D.

Suppose $\Gamma \subset SL(2, R)$ is discontinuous (for nondiscontinuous Γ, see [9]). We now ask what type of $f(z)$ corresponds to a given factor of automorphy of the type in Theorem 1. For Γ with only translations, there is an entire nonvanishing $f(z)$ [1]. For other Γ, $\log u(M)(cz + d)^r$ is well defined on $D = \Pi^+$, and is a summand of automorphy. If Π^+/Γ is compact, then by Lemma 1 of [5, Section II], there is a $C^\infty g(z)$ on Π^+ such that

$$g(Mz) = g(z) + \log u(M)(cz + d)^r,$$

so if we set $f(z) = \exp g(z)$, then $f(Mz) = u(M)(cz + d)^r f(z)$. Of course for r even, Poincaré showed a $f(z)$, meromorphic on Π^+, can be constructed for any discontinuous Γ. If r is odd, this is still possible for certain Γ [11]. For domains of meromorphicity larger than Π^+, a complete classification is known [7] which relates the three variables: domain of meromorphicity, Γ, and type of $f(z)$.

3. Linear fractional factors of automorphy

We return to the hypotheses $\Gamma \subset GL(2, \mathbf{C})$ and D an open Γ-invariant subset of the Riemann sphere. $F(z, M) \sim G(z, M)$ will mean there is a $c = c(M) \neq 0$ independent of z such that for each M in Γ, $F(z, M) = cG(z, M)$. To abbreviate proofs using proportionality $\sim$, we adopt the convention that $(z - A(M))^*$ denotes $(z - A(M))$ unless $A(M)$ is infinite, in which case it denotes 1. Further,

$$\left(\frac{z - A(M)}{z - B(M)}\right)^*$$

denotes $(z - A(M))^*/(z - B(M))^*$. This of course is projective arithmetic.

THEOREM 2. *If a factor of automorphy $v(z, M)$ is*

$$\frac{\alpha(M)z + \beta(M)}{\gamma(M)z + \delta(M)}$$

for each

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

then for each $M \in \Gamma$, $v(z, M)$ *is proportional* ($\sim$) *to one of the seven types:*

(1) 1

(2) $(cz + d)$; $c \neq 0$

(3) $(cz + d)^{-1}$; $c \neq 0$

(4) $\left(z - \frac{a}{c}\right)$; $c \neq 0$

(5) $\left(z - \frac{a}{c}\right)^{-1}$; $c \neq 0$

(6) $(z - z_1)/(z - M^{-1}z_1)$; $z_1 \neq M^{-1}z_1$, $M^{-1}z_1 \neq \infty$

(7) $(z - M^{-1}z_1)/(z - z_1)$; $z_1 \neq M^{-1}z_1$, $M^{-1}z_1 \neq \infty$

Proof. Neither $\alpha = \beta = 0$ nor $\gamma = \delta = 0$ since neither $f \equiv 0$ nor $f \equiv \infty$. If $\alpha(M) = \gamma(M) = 0$, then (1) follows.

If $\gamma(M) = 0$ and $\alpha(M) \neq 0$, then

$$v(z, M) \sim (z - z_1) \quad \text{where } z_1 = -\beta(M)/\delta(M)\alpha(M).$$

Thus

$$\begin{aligned} v(z, M^2) &= v(z, M)v(Mz, M) \\ &\sim (z - z_1)\left(\frac{az + b}{cz + d} - z_1\right) \\ &= (z - z_1)\left(\frac{z(a - cz_1) + (b - dz_1)}{cz + d}\right). \end{aligned}$$

If $a - cz_1 = 0$, then $c \neq 0$ ($c = 0$ implies $a = 0$ which contradicts $ad - bc \neq 0$) and so $z_1 = a/c$ and (4) follows. If $a - cz_1 \neq 0$, then one of the numerator terms must cancel with $cz + d$ where $c \neq 0$. So $-d/c$ is either z_1 or $-(b - dz_1)/(a - cz_1)$. From $-d/c = z_1$, (2) follows. From

$$-d/c = -(b - dz_1)/(a - cz_1) = M^{-1}z_1,$$

it follows $z_1 = M(-d/c) = \infty$, a contradiction.

If $\alpha(M) = 0$ and $\gamma(M) \neq 0$, we note $1/v(z, M)$ is also a factor of automorphy, and the above shows $1/v(z, M)$ satisfies (2) or (4), so $v(z, M)$ must satisfy (3) or (5).

If $\alpha(M) \neq 0$ and $\gamma(M) \neq 0$, then $v(z, M) \sim (z - z_1)/(z - z_2)$. If $z_1 = z_2$, (1) holds. Otherwise,

$$\begin{aligned} v(z, M^2) &= v(z, M)v(Mz, M) \\ &\sim \left(\frac{z - z_1}{z - z_2}\right)\left(\frac{Mz - z_1}{Mz - z_2}\right) \\ &= \left(\frac{z - z_1}{z - z_2}\right)\left[\frac{z(a - cz_1) + (b - dz_1)}{z(a - cz_2) + (b - dz_2)}\right]. \end{aligned}$$

This must reduce to the quotient of two terms, each at most linear. There are four cases. The first is $a - cz_1 = a - cz_2 = 0$ so that $z_1 = z_2$. Here $c \neq 0$ for the same reason as above. The second is that neither $a - cz_1$ nor $a - cz_2$ is zero, and that cancellation takes place within the square brackets, so that

$$\left(\frac{b - dz_1}{a - cz_1}\right) = \left(\frac{b - dz_2}{a - cz_2}\right);$$

i.e., $M^{-1}z_1 = M^{-1}z_2$ and thus $z_1 = z_2$. The third case is

$$z_2 = -\left(\frac{b - dz_1}{a - cz_1}\right) = M^{-1}z_1,$$

from which (6) follows. The fourth is

$$z_1 = -\left(\frac{b - dz_2}{a - cz_2}\right) = M^{-1}z_2$$

and (7) follows.

Next we ask about mixing types; for instance, may Γ contain M and N such that $v(z, M)$ is of type (4) and $v(z, N)$ is of type (5)? The answer is Theorem 3. In preparation for the proof, two procedures are formalized.

(I) For any T in $GL(2, \mathbf{C})$ a conjugate group of the group Γ is given by

$$\hat{\Gamma} = \{\hat{M} \mid \hat{M} = T^{-1}MT \text{ where } M \in \Gamma\}.$$

Set $T\zeta = z$, and $\hat{v}(\zeta, \hat{M}) = v(T\zeta, M)$. Thus

$$\begin{aligned} \hat{v}(\zeta, \hat{M}\hat{N}) &= v(T\zeta, MN) \\ &= v(T\zeta, N)v(NT\zeta, M) \\ &= v(T\zeta, N)v(T\hat{N}\zeta, M) \\ &= \hat{v}(\zeta, \hat{N})\hat{v}(\hat{N}\zeta, \hat{M}) \end{aligned}$$

so $\hat{v}$ is a factor of automorphy on $\hat{\Gamma}$. Suppose Γ is countable. T can be chosen so

$$T\infty \notin \{\infty, M\infty, M^{-1}\infty, z_1, M^{-1}z_1\}$$

for every M in Γ, where the z_1 is from Theorem 2. Elementary calculations show if $v(z, M)$ is of type (3), (4), or (6), then

$$\hat{v}(\zeta, \hat{M}) \sim \left(\frac{\zeta - \zeta_1}{\zeta - \hat{M}^{-1}\zeta_1}\right)$$

where for (3) $\zeta_1 = T^{-1}\infty$, for (4) $\zeta_1 = T^{-1}M\infty$, and for (6) $\zeta_1 = T^{-1}z_1$. If $v(z, M)$ is of type (2), (5), or (7), then

$$\hat{v}(\zeta, \hat{M}) \sim \left(\frac{\zeta - \hat{M}^{-1}\zeta_1}{\zeta - \zeta_1}\right)$$

where for (2) $\zeta_1 = T^{-1}\infty$, for (5) $\zeta_1 = T^{-1}M\infty$, and for (7) $\zeta_1 = T^{-1}z_1$. Thus $\hat{v}(\zeta, \hat{M})$ is restricted to types (1), (6), and (7), and is type (1) if and only if $v(z, M)$ is type (1).

(II) If

$$v(z, R) \sim \left(\frac{z - z_1}{z - R^{-1}z_1}\right) \neq 1,$$

then with

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$\begin{aligned} v(Sz, R) &\sim \left(\frac{Sz - z_1}{Sz - R^{-1}z_1}\right) \\ &= \left(\frac{z(a - cz_1) + (b - dz_1)}{z(a - cR^{-1}z_1) + (b - dR^{-1}z_1)}\right) \\ &\sim \left(\frac{z - S^{-1}z_1}{z - S^{-1}R^{-1}z_1}\right)^* \neq 1. \end{aligned}$$

Similarly if

$$v(z, R) \sim \left(\frac{z - R^{-1}z_1}{z - z_1}\right) \neq 1,$$

then

$$v(Sz, R) \sim \left(\frac{z - S^{-1}R^{-1}z_1}{z - S^{-1}z_1}\right)^* \neq 1.$$

If $v(z, R^2)$ is of type (1), (6), or (7) of Theorem 2, and

$$v(z, R) \sim \left(\frac{z - z_1}{z - R^{-1}z_1}\right) \neq 1,$$

then

$$v(z, R^2) = v(z, R)v(Rz, R) \sim \left(\frac{z - z_1}{z - R^{-1}z_1}\right)\left(\frac{z - R^{-1}z_1}{z - R^{-2}z_1}\right)^* = \left(\frac{z - z_1}{z - R^{-2}z_1}\right)$$

which equals 1 if and only if $R^2z_1 = z_1$. If the latter held, R^2 would have z_1 as a fixed point in addition to the points left fixed by R, so that $R^2 = \pm I$. Similarly if

$$v(z, R) \sim \left(\frac{z - R^{-1}z_1}{z - z_1}\right) \neq 1,$$

then

$$v(z, R^2) \sim \left(\frac{z - R^{-2}z_1}{z - z_1}\right)$$

which is 1 if and only if $R^2 = \pm I$.

THEOREM 3. *If a factor of automorphy $v(z, M)$ is*

$$\frac{\alpha(M)z + \beta(M)}{\gamma(M)z + \delta(M)}$$

for every M in Γ, then one of the following holds:

(1) *for all $M \in \Gamma$, $v(z, M) \sim 1$;*

(2) *for all $M \in \Gamma$, $v(z, M) \sim (cz + d)$ where*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

(3) *for all $M \in \Gamma$, $v(z, M) \sim (cz + d)^{-1}$ where*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix};$$

(4) *there exists $z_1 \in \mathbf{C}$ such that for all $M \in \Gamma$,*

$$v(z, M) \sim \left(\frac{z - z_1}{z - M^{-1}z_1}\right)^*;$$

(5) *there exists $z_1 \in \mathbf{C}$ such that for all $M \in \Gamma$,*

$$v(z, M) \sim \left(\frac{z - M^{-1}z_1}{z - z_1}\right)^*.$$

Proof. Without loss of generality, Γ is a countable group. In Lemmas 1 through 5 below, it is assumed that for every L in Γ, $v(z, L)$ is restricted to types (1), (6), and (7) of Theorem 2. After Lemma 5 is the general proof of Theorem 3 except for the case that Γ consists solely of M such that $M^2 = \pm I$, which Lemma 6 then handles.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

belong to Γ throughout.

LEMMA 1. *If*

$$v(z, M) \sim \left(\frac{z - z_1}{z - M^{-1}z_1}\right) \neq 1 \quad \textit{and} \quad v(z, N) \sim \left(\frac{z - z_2}{z - N^{-1}z_2}\right) \neq 1,$$

then either $z_1 = z_2$ or both $M^2 = \pm I$ and $N^2 = \pm I$.

Proof.

$$\begin{aligned} v(z, MN) &= v(z, N)v(Nz, M) \\ &= \left(\frac{z - z_2}{z - N^{-1}z_2}\right)\left(\frac{Nz - z_1}{Nz - M^{-1}z_1}\right) \\ &\sim \left(\frac{z - z_2}{z - N^{-1}z_2}\right)\left(\frac{z - N^{-1}z_1}{z - N^{-1}M^{-1}z_1}\right)^* \end{aligned}$$

by procedure (I). But $MN \in \Gamma$, so either $z_2 = N^{-1}M^{-1}z_1$ or $N^{-1}z_2 = N^{-1}z_1$. Thus if $z_1 \neq z_2$, then $z_2 = N^{-1}M^{-1}z_1$. By procedure (II),

$$v(z, M^2) \sim \left(\frac{z - z_1}{z - M^{-2}z_1}\right),$$

so unless $M^2 = \pm I$, we may repeat the above argument with M^2 in place of M to conclude that if $z_1 \neq z_2$, then $z_2 = N^{-1}M^{-2}z_1$. Thus if both $M^2 \neq \pm I$ and $z_1 \neq z_2$, then MNz_2 equals both z_1 and $M^{-1}z_1$, so that $z_1 = M^{-1}z_1$ which violates the supposition that $v(z, M)$ is not proportional to one. Therefore $z_1 \neq z_2$ implies $M^2 = \pm I$. An interchange of M with N yields: $z_2 \neq z_1$ implies $N^2 = \pm I$.

LEMMA 2. *If*

$$v(z, M) \sim \left(\frac{z - M^{-1}z_1}{z - z_1}\right) \neq 1 \quad \textit{and} \quad v(z, N) \sim \left(\frac{z - N^{-1}z_2}{z - z_2}\right) \neq 1,$$

then either $z_1 = z_2$ *or both* $M^2 = \pm I$ *and* $N^2 = \pm I$.

Proof. The reciprocal of a factor of automorphy is a factor of automorphy, so Lemma 1 may be applied to $1/v$.

LEMMA 3. *If*

$$v(z, M) \sim \left(\frac{z - z_1}{z - M^{-1}z_1}\right) \neq 1 \quad \textit{and} \quad v(z, N) \sim \left(\frac{z - N^{-1}z_2}{z - z_2}\right) \neq 1.$$

then either $M^2 = \pm I$ *or* $N^2 = \pm I$. *If* $M^2 = \pm I$, *it is immediate that*

$$v(z, M) \sim \left(\frac{z - M^{-1}w_1}{z - w_1}\right) \neq 1 \quad \textit{where } w_1 = M^{-1}z_1,$$

so that Lemma 2 *is applicable. If* $N^2 = \pm I$, *then*

$$v(z, N) \sim \left(\frac{z - w_2}{z - N^{-1}w_2}\right) \neq 1 \quad \textit{where } w_2 = N^{-1}z_2,$$

so that Lemma 1 *is applicable.*

Proof. Suppose neither $M^2 = \pm I$ nor $N^2 = \pm I$.
By the consistency condition,

$$v(z, MN) = v(z, N)v(Nz, M) \sim \left(\frac{z - N^{-1}z_2}{z - z_2}\right)\left(\frac{z - N^{-1}z_1}{z - N^{-1}M^{-1}z_1}\right)^*$$

so either $z_2 = N^{-1}z_1$ or $z_2 = M^{-1}z_1$. We may use procedure (II) to justify repetition of this process with N replaced by N^2. So either $z_2 = N^{-2}z_1$ or $z_2 = M^{-1}z_1$. If $z_2 \neq M^{-1}z_1$, then $z_2 = N^{-1}z_1$ and $z_2 = N^{-2}z_1$, which implies $N^{-1}z_2 = z_2$, a violation of the hypothesis $v(z, N) \nsim 1$. So $z_2 = M^{-1}z_1$.

By procedure (II), we are justified in the repetition of both steps in the above argument with M replaced by M^2. Thus also $z_2 = M^{-2}z_1$, so that $z_1 = M^{-1}z_1$, again a violation of hypotheses. The original assumption was thus false.

LEMMA 4. *If*

$$v(z, M) \sim \left(\frac{z - z_1}{z - M^{-1}z_1}\right) \neq 1 \quad \textit{and} \quad v(z, N) \sim 1,$$

then either (i) N *fixes* $w_1 = M^{-1}z_1$ *and* $M^2 = \pm I$, *or* (ii) $Nz_1 = z_1$. *In the former case*

$$v(z, M) \sim \left(\frac{z - M^{-1}w_1}{z - w_1}\right) \neq 1 \quad \textit{and} \quad v(z, N) \sim \left(\frac{z - N^{-1}w_1}{z - w_1}\right) = 1.$$

In the latter case

$$v(z, M) \sim \left(\frac{z - z_1}{z - M^{-1}z_1}\right) \neq 1 \quad \textit{and} \quad v(z, N) \sim \left(\frac{z - z_1}{z - N^{-1}z_1}\right) = 1.$$

Proof. Since $v(z, NM)$ is assumed to be of type (1), (6), or (7) of Theorem 2, we know it is proportional to either

$$\left(\frac{z - z_2}{z - (NM)^{-1}z_2}\right) \quad \text{or} \quad \left(\frac{z - (NM)^{-1}z_2}{z - z_2}\right)$$

for some z_2. One of these must match

$$v(z, NM) = v(z, M)v(Mz, N) \sim \left(\frac{z - z_1}{z - M^{-1}z_1}\right) \cdot 1.$$

If the first matches, $z_1 = z_2$ and $M^{-1}z_1 = M^{-1}N^{-1}z_2$ so $z_1 = N^{-1}z_2$. Thus $z_1 = N^{-1}z_1$, the second of the allowed conclusions. If the second matches, $z_1 = M^{-1}N^{-1}z_2$ and $z_2 = M^{-1}z_1$, so $z_1 = M^{-1}N^{-1}M^{-1}z_1$. Lemma 3 may be applied to the pair M and NM to conclude either $M^2 = \pm I$ or $(NM)^2 = \pm I$. If $M^2 = \pm I$, then $M^{-1}z_1 = Mz_1 = N^{-1}M^{-1}z_1$ so that N leaves w_1 fixed where $w_1 = M^{-1}z_1$, the first of the allowed conclusions. If $(NM)^2 = \pm I$, then $(M^{-1}N^{-1})^2 = \pm I$, so that

$$M^{-1}z_1 = (M^{-1}N^{-1})^2M^{-1}z_1 = M^{-1}N^{-1}(M^{-1}N^{-1}M^{-1}z_1) = M^{-1}N^{-1}z_1$$

and thus $z_1 = Nz_1$, again the second conclusion.

LEMMA 5. *If*

$$v(z, M) \sim \left(\frac{z - M^{-1}z_1}{z - z_1}\right) \neq 1 \quad \textit{and} \quad v(z, N) \sim 1,$$

then either

(i) $$v(z, M) \sim \left(\frac{z - w_1}{z - M^{-1}w_1}\right) \neq 1 \textit{ and } v(z, N) \sim \left(\frac{z - w_1}{z - N^{-1}w_1}\right) = 1$$

where $w_1 = M^{-1}z_1$,

or

(ii) $v(z, M) \sim \left(\dfrac{z - M^{-1}z_1}{z - z_1}\right) \neq 1$ *and* $v(z, N) \sim \left(\dfrac{z - N^{-1}z_1}{z - z_1}\right) = 1.$

Proof. The method of proof of Lemma 2 is applied to Lemma 4.

Proof of Theorem 3. Pick a fixed

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in Γ such that $M^2 \neq \pm I$ and $v(z, M) \nsim 1$ (not proportional to 1). This may not be possible, which case is handled by Lemma 6. Let

$$N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

belong to Γ. Reference to procedure (I), before Theorem 3, shows T can be chosen so that for every $\hat{L}$ in $\hat{\Gamma}$, $\hat{v}(\zeta, \hat{L})$ is one of the three types (1), (6), or (7) of Theorem 2. An application of Lemmas 1 through 5 yields a complex ζ_1 such that either

$$\hat{v}(\zeta, \hat{M}) \sim \frac{\zeta - \zeta_1}{\zeta - \hat{M}^{-1}\zeta_1} \quad \text{and} \quad \hat{v}(\zeta, \hat{N}) \sim \frac{\zeta - \zeta_1}{\zeta - \hat{N}^{-1}\zeta_1},$$

or

$$\hat{v}(\zeta, \hat{M}) \sim \frac{\zeta - \hat{M}^{-1}\zeta_1}{\zeta - \zeta_1} \quad \text{and} \quad \hat{v}(\zeta, \hat{N}) \sim \frac{\zeta - \hat{N}^{-1}\zeta_1}{\zeta - \zeta_1}.$$

If $v(z, M)$ is of type (3), (4), or (6), the former must hold so

$$\begin{aligned} v(z, M) &= \hat{v}(\zeta, \hat{M}) \\ &= \hat{v}(T^{-1}z, T^{-1}MT) \\ &\sim \left(\frac{T^{-1}z - \zeta_1}{T^{-1}z - T^{-1}M^{-1}T\zeta_1}\right) \\ &\sim \left(\frac{z - T\zeta_1}{z - M^{-1}T\zeta_1}\right)^* \end{aligned}$$

and likewise

$$v(z, N) \sim \left(\frac{z - T\zeta_1}{z - N^{-1}T\zeta_1}\right)^*.$$

The basic point is that N is arbitrary in Γ. If $T\zeta_1 = \infty$, this becomes the third of the five possibilities in the statement of Theorem 3: $v(z, N) \sim (\gamma z + \delta)^{-1}$ for all N. If $T\zeta_1 \neq \infty$, we set $z_1 = T\zeta_1$, and have the fourth possibility. Similarly, if $v(z, M)$ is of type (2), (5), or (7) of Theorem 2, then the second or fifth possibilities of Theorem 3 are fulfilled. So the theorem is proved unless $M^2 = \pm I$ for all M such that $v(z, M) \nsim 1$.

LEMMA 6. *If* $M^2 = \pm I$ *for every* M *in* Γ *such that* $v(z, M) \nsim 1$, *then Theorem* 3 *remains valid.*

Proof. Let T be as above. $M^2 = \pm I$ if and only if $\hat{M}^2 = \pm I$. $v(z, M) \sim 1$ if and only if $\hat{v}(\zeta, \hat{M}) \sim 1$. Suppose $\hat{v}(\zeta, \hat{M}) \nsim 1$, and $\hat{v}(\zeta, \hat{N}) \nsim 1$. Since $\hat{M}^2 = \pm I$, $\hat{v}(\zeta, \hat{M})$ can be expressed as both

$$\left(\frac{\zeta - \zeta_1}{\zeta - \hat{M}^{-1}\zeta_1}\right) \quad \text{and} \quad \left(\frac{\zeta - M^{-1}w_1}{\zeta - w_1}\right)$$

where $w_1 = \hat{M}\zeta_1$. Likewise since $\hat{N}^2 = \pm I$, $\hat{v}(\zeta, \hat{N})$ can be expressed as both

$$\left(\frac{\zeta - \zeta_2}{\zeta - \hat{N}^{-1}\zeta_2}\right) \quad \text{and} \quad \left(\frac{\zeta - \hat{N}^{-1}w_2}{\zeta - w_2}\right)$$

where $w_2 = \hat{N}\zeta_2$. Now

$$\hat{v}(\zeta, \hat{M}\hat{N}) = \hat{v}(\zeta, \hat{N})\hat{v}(\hat{N}\zeta, \hat{M}) \sim \left(\frac{\zeta - \zeta_2}{\zeta - \hat{N}^{-1}\zeta_2}\right)\left(\frac{\zeta - \hat{N}^{-1}\zeta_1}{\zeta - \hat{N}^{-1}\hat{M}^{-1}\zeta_1}\right)^*.$$

So either $\zeta_1 = \zeta_2$ or $\zeta_2 = \hat{N}^{-1}\hat{M}^{-1}\zeta_1$. The last can be written $\hat{N}^{-1}\zeta_2 = \hat{M}^{-1}\zeta_1$. That is, either the numerators or the denominators must match.

Let $\{\hat{Q}_i\}$ be the set of matrices in $\hat{\Gamma}$ such that $\hat{v}(\zeta, \hat{Q}_i) \nsim 1$. Thus

$$\hat{Q}_i^2 = \pm I \quad \text{and} \quad \hat{v}(\zeta, \hat{Q}_i) \sim \left(\frac{\zeta - \zeta_i}{\zeta - \hat{Q}_i^{-1}\zeta_i}\right).$$

Either $\zeta_i = \zeta_1$ for all i, or $\hat{Q}_i^{-1}\zeta_i = \hat{Q}_i^{-1}\zeta_1$ for all i. If this failed to be true, we could take one matrix with a deviant numerator in its factor of automorphy $\hat{v}$ (and therefore a nondeviant denominator), and another matrix with the deviancy reversed. These two would then have $\hat{v}$ with neither numerators nor denominators matched, which is contrary to the result immediately above.

If $\zeta_i = \zeta_1$ for all i, then the already familiar method for translating results about $\hat{v}(\zeta, \hat{M})$ to results about $v(z, M)$, previous to Lemma 6, yields two possibilities. If $T\zeta_1 = \infty$, then for all

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in Γ with $v(z, M) \nsim 1$, necessarily $v(z, M) \sim (cz + d)^{-1}$. If $T\zeta_1 = \infty$, then there is a complex z_1, namely $z_1 = T\zeta_1$, so that for all M in Γ with $v(z, M) \nsim 1$, necessarily

$$v(z, M) \sim \left(\frac{z - z_1}{z - M^{-1}z_1}\right)^*.$$

Similarly, if $\hat{Q}_i^{-1}\zeta_i = \hat{Q}_1^{-1}\zeta_1$ for all i, then since $\hat{Q}_i^2 = \pm I$, each $\hat{v}(\zeta, \hat{Q}_i) \nsim 1$ can also be written

$$\hat{v}(\zeta, \hat{Q}_i) \sim \left(\frac{\zeta - \hat{Q}_i^{-1}w_i}{\zeta - w_i}\right)$$

where $w_i = Q_i^{-1}\zeta_i = \hat{Q}_1^{-1}\zeta_1$. All denominators are identical, and either for all

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in Γ with $v(z, M) \nsim 1$, necessarily $v(z, M) \sim (cz + d)$, or there is a complex z_1 so that for all M in Γ with $v(z, M) \nsim 1$, necessarily

$$v(z, M) \sim \left(\frac{z - M^{-1}z_1}{z - z_1}\right)^*.$$

The following four statements complete the proof of Lemma 6.

(a) If for all

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in Γ such that $v(z, M) \nsim 1$ we have $v(z, M) \sim (cz + d)^{-1}$, and if

$$N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

is in Γ and $v(z, N) \sim 1$, then $\gamma = 0$, so that $v(z, N) \sim (\gamma z + \delta)^{-1}$, and the third possibility in Theorem 3 follows.

(b) If for all M in Γ such that $v(z, M) \nsim 1$ we have

$$v(z, M) \sim \left(\frac{z - z_1}{z - M^{-1}z_1}\right)^*,$$

and if N belongs to Γ and $v(z, N) \sim 1$, then $N^{-1}z_1 = z_1$ so that

$$v(z, N) \sim \left(\frac{z - z_1}{z - N^{-1}z_1}\right),$$

and the fourth possibility in Theorem 3 follows.

(c) If in (a), $(cz + d)$ replaces $(cz + d)^{-1}$, then still $\gamma = 0$ and $(\gamma z + \delta)$ replaces $(\gamma z + \delta)^{-1}$. The second possibility of Theorem 3 follows.

(d) If in (b),

$$\left(\frac{z - M^{-1}z_1}{z - z_1}\right)^* \quad \text{replaces} \quad \left(\frac{z - z_1}{z - M^{-1}z_1}\right)^*,$$

then still $z_1 = N^{-1}z_1$ and

$$\left(\frac{z - N^{-1}z_1}{z - z_1}\right) \quad \text{replaces} \quad \left(\frac{z - z_1}{z - N^{-1}z_1}\right).$$

The fifth possibility for Theorem 3 follows.

For the proof of these, T is chosen as above to produce only $\hat{v}(\zeta, \hat{L})$ of type (1), (6), or (7) of Theorem 2. In (a) or (b), $\hat{\Gamma}$ is restricted to the case that all numerators are the same for the $\hat{v}(\zeta, \hat{Q}_i)$ above. Thus

$$\hat{v}(\zeta, \hat{M}) \sim \left(\frac{\zeta - \zeta_1}{\zeta - \hat{M}^{-1}\zeta_1}\right)$$

so

$$\begin{aligned}\hat{v}(\zeta, \hat{M}\hat{N}) &= \hat{v}(\zeta, \hat{N})\hat{v}(\hat{N}\zeta, \hat{M}) \\ &\sim 1 \cdot \left(\frac{\hat{N}\zeta - \zeta_1}{\hat{N}\zeta - \hat{M}^{-1}\zeta_1}\right) \\ &\sim \left(\frac{\zeta - \hat{N}^{-1}\zeta_1}{\zeta - \hat{N}^{-1}\hat{M}^{-1}\zeta_1}\right) \\ &\nsim 1.\end{aligned}$$

But $\hat{M}\hat{N}$ is in $\hat{\Gamma}$, so

$$\hat{v}(\zeta, \hat{M}\hat{N}) \sim \left(\frac{\zeta - \zeta_1}{\zeta - (\hat{M}\hat{N})^{-1}\zeta_1}\right).$$

Thus $\hat{N}^{-1}\zeta_1 = \zeta_1$. If $T\zeta_1 = \infty$, we are in case (a), and $\zeta_1 = \hat{N}\zeta_1 = T^{-1}NT\zeta_1$ becomes $\infty = T\zeta_1 = NT\zeta_1 = N\infty$. Thus $\gamma = 0$ and (a) is proved. If $T\zeta_1 \neq \infty$, we are in case (b), and since $z_1 = T\zeta_1$, the equation $\zeta_1 = \hat{N}^{-1}\zeta_1$ implies

$$T^{-1}z_1 = (T^{-1}NT)^{-1}T^{-1}z_1 = T^{-1}N^{-1}z_1$$

so that $z_1 = N^{-1}z_1$ and (b) is proved.

If we reciprocate (a) and (b), we arrive at (c) and (d) respectively.

THEOREM 4. *If $f(Mz) = v(z, M)f(z)$ where for every M in Γ, $v(z, M)$ is a linear fractional transformation in z with coefficients depending on*

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then one of the following holds.

(1) *$f(Mz) \sim f(z)$ for every M in Γ;*

(2) *$f(Mz) \sim (cz + d)f(Mz)$ for every M in Γ;*

(3) *$f(Mz) \sim (cz + d)^{-1}f(Mz)$ for every M in Γ;*

(4) *there is a complex number z_1 such that if $g(z) = f(z)(z - z_1)$, then*

$$g(Mz) \sim (cz + d)^{-1}g(z) \quad \textit{for every } M \textit{ in } \Gamma;$$

(5) *there is a complex number z_1 such that if $g(z) = f(z)/(z - z_1)$, then*

$$g(Mz) \sim (cz + d)g(z) \quad \textit{for every } M \textit{ in } \Gamma.$$

Proof. The first three possibilities follow from the first three possibilities of Theorem 3.

Suppose in Theorem 3, the fourth holds. If $g(z) = f(z)(z - z_1)$, then

$$\begin{aligned} g(Mz) &= f(Mz)(Mz - z_1) \\ &\sim \left(\frac{z - z_1}{z - M^{-1}z_1}\right)^* f(z)\left(\frac{az + b}{cz + d} - z_1\right) \\ &\sim (cz + d)^{-1}f(z)\left(\frac{z - z_1}{z - M^{-1}z_1}\right)^* (z - M^{-1}z_1)^* \\ &= (cz + d)^{-1}f(z)(z - z_1) \\ &= (cz + d)^{-1}g(z). \end{aligned}$$

This is (4) above.

Similarly (5) follows from the fifth possibility in Theorem 5.

Suppose $\Gamma \subset SL(2, R)$ is discontinuous. A natural question is what kind of $f(z)$ corresponds with a given linear fractional factor of automorphy on Γ. In the first two cases of Theorem 4, this has already been answered by the information in the last paragraph of the section on polynomial factors of automorphy. In the third, fourth, and fifth cases, we apply this information to $1/f(z)$, $1/g(z)$, and $g(z)$ respectively.

A result in [7] states that if $r = \pm 1$ or 0, and if for $\Gamma \subset SL(2, R)$, $h(z)$ is an unrestricted automorphic form of degree r, meromorphic on the whole complex plane, then either Γ is finite, or Γ consists solely of translations, or $h(z)$ has one of the few following forms:

(a) $h(z) = A(z - z_1)^r$; z_1 real, Γ consists of parabolic and hyperbolic matrices, both with $Mz_1 = z_1$.

(b) $h(z) = A(z - z_1)^r$; z_1 nonreal, Γ consists of elliptic matrices such that $Mz_1 = z_1$.

(c) $h(z) = A(z - z_1)^s$; z_1 real, s an integer, Γ consists of hyperbolic matrices such that $Mz_1 = z_1$ and $M\infty = \infty$.

(d) $h(z) = A(z - z_1)^a/(z - z_2)^b$; a, b positive integers, $r = a - b$, z_1 and z_2 real, and Γ consists of hyperbolic matrices such that $Mz_1 = z_1$ and $Mz_2 = z_2$.

(e) $h(z) = A(z - z_1)^a/(z - z_2)^b$; a, b positive integers, $r = a - b$, $z_1 = z_2$ not real, and Γ consists of elliptic matrices such that $Mz_1 = z_1$ and $Mz_2 = z_2$.

(f) $h(z) = A$; if $r = 0$, Γ is arbitrary; if $r \neq 0$, Γ consists of parabolics and hyperbolics, both with $M\infty = \infty$.

Thus if a factor of automorphy on $\Gamma \subset SL(2, R)$ is always linear fractional for an $f(z)$ which is meromorphic on the whole plane, then unless Γ is finite or contains only translations, either $f(z)$, $f(z)(z - z_1)$, or $f(z)/(z - z_1)$ is one of the types (a) through (f). Further, if Γ is discontinuous, then except for the trivial case ($h(z) = A$, $r = 0$, Γ arbitrary), the above combines nicely with Theorems 2F and 2H of Chapter I in [10] to show that as a transformation group, Γ must be cyclic.

REFERENCES

1. P. APPELL, *Sur les fonctions périodiques de deux variables*, J. Math. Pures Appl., vol. 7 (1891), pp. 157–219.
2. S. BOCHNER, *Algebraic and linear dependence of automorphic functions in several variables*, J. Indian Math. Soc., vol. 6 (1952), pp. 1–6.

2a. ———, "Functions in one complex variable as viewed from the theory of functions in several variables," in *Functions of a complex variable*, ed. W. Kaplan, U. of Mich. Press, Ann Arbor, 1955.

3. G. FROBENIUS, *Über due Grundlagen der Theorie der Jacobischen Funktionen*, J. Reine Angew, Math., vol. 97 (1884), pp. 16–48.
4. R. GUNNING, *General factors of automorphy*, Proc. Nat. Acad. Sci., vol. 44 (1955), pp. 496–498.
5. ———, The structure of factors of automorphy, Amer. J. Math., vol. 78 (1956), pp. 357–382.
6. ———, Factors of automorphy and other formal cohomology groups for Lie groups, Ann. Math., vol. 69 (2) (1959), pp. 314–326, correction, p. 734.
7. D. JAMES, *Functions automorphic on large domains*, Trans. Amer. Math., Soc. vol. 181 (1973), pp. 385–400.
8. F. KLEIN AND R. FRICKE, *Theorie der Elliptischen Modulfunktionen*, B. G. Teubner Verlaggesellschaft, Stüttgart, 1890.
9. M. KNOPP, *Polynomial automorphic forms and nondiscontinuous groups*, Trans. Amer. Math. Soc. (2), vol. 123 (1966), pp. 506–520.
10. J. LEHNER, *A short course in automorphic functions*, Holt, Rinehart and Winston, N.Y. 1966.
11. H. PETERSSON, Theorie der automorphen Formen beliebiger reeler Dimension und ihre Darstellung durch eine neue Art Poincaréscher Reihen, Math. Annal., vol. 103 (1930), pp. 369–436.
12. ———, Zur analytischen Theorie der Grenzkreisgruppen, Teil I, Math. Ann., vol. 115 (1937), pp. 23–67.
13. ———, Über die Transformations Faktoren der relativen Invarianten linearer Substitutionsgruppen, Monatsh. Math., vol. 53 (1949), pp. 17–41.
14. H. POINCARÉ, Memoire sur les Fonctions Fuchsiennes, Acta Math., vol. 1 (1882), pp. 193–294.
15. C. L. SIEGEL, *Topics in Complex Function Theory*, vol. III, Wiley Interscience, N.Y. 1966.

WAYNE STATE UNIVERSITY
DETROIT, MICHIGAN