THE FIXED-POINT PROPERTY FOR ALMOST CHAINABLE HOMOGENEOUS CONTINUA

BY

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A continuum M is *almost chainable* if for each positive number ε, there exists an ε-cover $\mathscr{D}$ of M and a chain $\mathscr{C} = \{L_i: 1 \le i \le n\}$ of elements of $\mathscr{D}$ such that no L_i $(i > 1)$ intersects an element of $\mathscr{D} - \mathscr{C}$ and every point of M is within ε of some element of $\mathscr{C}$. In [3], C. E. Burgess proved that every k-junctioned tree-like homogeneous continuum is almost chainable. Burgess [3] also showed that a homogeneous continuum is almost chainable if and only if all of its proper subcontinua are pseudo-arcs. In this paper we prove that every continuous function of an almost chainable homogeneous continuum into itself has a fixed point. Hence we have another proof of L. Fearnley and J. T. Rogers' theorem [5], [15] that the pseudo-circle is not homogeneous.

A *continuum* is a nondegenerate compact connected metric space. A finite collection $\{L_i: 1 \le i \le n\}$ of open sets is a *chain* provided that $L_i \cap L_j \neq \emptyset$ if and only if $|i - j| \le 1$. If $n > 2$ and L_1 also intersects L_n, the collection is called a *circular chain*. If each L_i has diameter less than ε, the chain (circular chain) is called an *ε-chain* (*ε-circular chain*). A continuum M is *chainable* (*circularly chainable*) if for each $\varepsilon > 0$, there exists an ε-chain (ε-circular chain) that covers M. In [2], R. H. Bing characterized the pseudo-arc by showing that it is homeomorphic to every hereditarily indecomposable chainable continuum. The pseudo-arc is also circularly chainable.

A space S has the *fixed-point property* if for each continuous function f of S into S, there exists a point x of S such that $f(x) = x$. O. H. Hamilton [10] proved that every chainable continuum has the fixed-point property.

A finite coherent collection $\mathscr{T}$ of open sets is a *tree chain* if no subcollection of $\mathscr{T}$ is a circular chain. If the diameter of each element of a tree chain is less than ε, then it is called an *ε-tree chain*. A continuum M is *tree-like* if for each $\varepsilon > 0$, there exists an ε-tree chain that covers M.

A *junction link* of a tree chain is an element that intersects at least three other elements of the tree chain. A tree-like continuum is said to be *k-junctioned* if k is the least integer such that, for each $\varepsilon > 0$, the continuum can be covered by an ε-tree chain with k junction links. In [11], W. T. Ingram constructed an atriodic 1-junctioned tree-like continuum that is not chainable.

A space is *homogeneous* if for each pair x, y of its points, there exists a homeomorphism of the space onto itself that takes x to y. The pseudo-arc is the only tree-like continuum known to be homogeneous [1], [13].

Henceforth, M is a homogeneous continuum with metric ρ. Let x be a point

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of M, and let ε be a given positive number. In [9, Lemma 4], a theorem of E. G. Effros [4, Theorem 2.1] is used to prove that x belongs to an open subset W of M with the following property:

The ε-push property. For each pair y, z of points of W, there exists a homeomorphism h of M onto M such that $h(y) = z$ and $\rho(v, h(v)) < \varepsilon$ for every point v of M.

THEOREM. *If M is an almost chainable homogeneous continuum, then M has the fixed-point property.*

Proof. Suppose there exists a continuous function f of M into itself that moves every point of M. Define ε to be a positive number such that $\rho(v, f(v)) > 2\varepsilon$ for every point v of M. Let $\mathscr{W}$ be a cover of M consisting of finitely many open sets with the ε-push property.

For each positive integer j, there exists a j^{-1}-cover $\mathscr{D}_j$ of M and a chain $\mathscr{C}_j = \{L_{i,j}\colon 1 \le i \le n_j\}$ of elements of $\mathscr{D}_j$ such that no element of $\mathscr{D}_j - \mathscr{C}_j$ intersects $\bigcup \{L_{i,j}\colon 2 \le i \le n_j\}$ and every point of M is within j^{-1} of some element of $\mathscr{C}_j$.

For each j, let p_j be a point of $L_{n_j,j}$, let K_j be the p_j-component of $M - L_{1,j}$, and let x_j be a point of K_j that belongs to the boundary of $L_{1,j}$.

Let j be an integer such that $j^{-1} < \varepsilon$ and K_j intersects each element of $\mathscr{W}$. Define W to be an element of $\mathscr{W}$ that contains $f(x_j)$. Let h be a homeomorphism of M onto M such that $hf(x_j)$ belongs to $W \cap K_j$ and $\rho(v, h(v)) < \varepsilon$ for every point v of M. Note that $\rho(v, hf(v)) > \varepsilon$ for every point v of M.

For each integer k $(2 \le k \le n_j)$, define

$$A_k = \{p \in K_j \cap L_{k,j}\colon hf(p) \in \bigcup \{L_{i,j}\colon k \le i \le n_j\}\}$$

and

$$B_k = \{p \in K_j \cap L_{k,j}\colon hf(p) \in M - \bigcup \{L_{i,j}\colon k \le i \le n_j\}\}.$$

Note that x_j and p_j belong to A_2 and B_{n_j} respectively. Since $\mathscr{C}_j$ is an ε-chain,

$$A = \bigcup \{A_k\colon 2 \le k \le n_j\} \quad \text{and} \quad B = \bigcup \{B_k\colon 2 \le k \le n_j\}$$

are nonempty disjoint closed sets. Since $A \cup B$ is the connected set K_j, this is impossible. This contradiction completes the proof. ■

The fixed-point theorem in [8] is generalized by the following:

COROLLARY 1. *Every k-junctioned tree-like homogeneous continuum has the fixed-point property.*

Proof. Every k-junctioned tree-like homogeneous continuum is almost chainable [3, Theorem 13], [12, Theorem 1]. ■

Let X be a hereditarily indecomposable circularly chainable continuum that is not chainable. Fearnley [6] proved that if X is a plane continuum, it is topologically equivalent to Bing's pseudo-circle [2, Example 2]. Burgess [3, Theorem 4] proved that X is almost chainable. If X is planar, then Theorem 6.1 of [6] can be used to define a fixed-point free homeomorphism of X onto itself. If X is not planar, the existence of such a mapping follows from the argument in [7, Theorem 3.2 (Proof of sufficiency)] (also see [14]). Hence our theorem implies that X is not homogeneous. Thus we have the following result of Fearnley [5] and Rogers [15]:

COROLLARY 2. *The pseudo-arc is the only homogeneous, hereditarily indecomposable, circularly chainable continuum.*

F. Burton Jones pointed out that our principal theorem, which was originally proved for homeomorphisms, holds for all continuous functions. The author is also indebted to Clifford Arnquist, Sandra Barkdull, Richard Cleveland, and Gerald Ungar for helpful conversations about homogeneous spaces.

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