THE SHAPE OF A CROSS-SECTION OF THE SOLUTION FUNNEL OF AN ORDINARY DIFFERENTIAL EQUATION

BY

JAMES T. ROGERS, JR.

Let $f\colon \mathbf{R}^{n+1} \to \mathbf{R}^n$ be a continuous map with support in the product of R and some compact set in $\mathbf{R}^n$. Let $y \in \mathbf{R}^n$, let $t \in \mathbf{R}$, and let $(t_0, y_0) = p \in \mathbf{R}^{n+1}$. A map $y\colon \mathbf{R} \to \mathbf{R}^n$ is called a solution of f through p provided that y is a solution of the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0. \tag{1}$$

The cross-section of the integral funnel at time t is the set

$$F_t(p) = \{(t, y(t)) \colon y \text{ is a solution of } f \text{ through } p\}.$$

The integral funnel or f-funnel through p is the set $F(p) = \bigcup_{t \in \mathbf{R}} F_t(p)$.

In the case that f is not Lipschitz continuous, the cross-section $F_t(p)$ may consist of more than one point, and it becomes an interesting problem to investigate the topological properties of this set. H. Kneser obtained the first results on this problem; he showed that under the above hypotheses, each cross-section $F_t(p)$ is a continuum, i.e., a compact, connected set.

In this note, we obtain necessary conditions for a continuum to be homeomorphic to a cross-section of a solution funnel. This yields examples of continua that can never be cross-sections of a solution funnel. Our tools are shape theory, continua theory, and Čech cohomology theory.

1. Results of Aronszajn and Pugh.

In this section, we recall two previous results on this problem that will be useful in our program, and we recast these theorems in the language of shape theory.

The overlying f-funnel $\tilde{F}(f, p)$ is the set of f-solutions through p in the function space $C^1(\mathbf{R}, \mathbf{R}^n)$. N. Aronszajn [1] considered overlying f-funnels in 1942, when he proved that each overlying f-funnel is the intersection of a decreasing sequence of compact absolute retracts. Thus each overlying f-funnel is a continuum. Since the evaluation function $y \to y(t)$ from $\tilde{F}(f, p)$ onto $F_t(p)$ is continuous, Aronszajn obtained another proof of Kneser's theorem that each cross-section is a continuum.

We assume that the reader is familiar with shape theory as expounded by Borsuk [2, 3]. Intuitively, shape theory is Čech homotopy theory. Continua shape-equivalent to a point are said to have trivial shape. D. Hyman [6] has proved that a continuum has trivial shape if and only if it is the intersection of a

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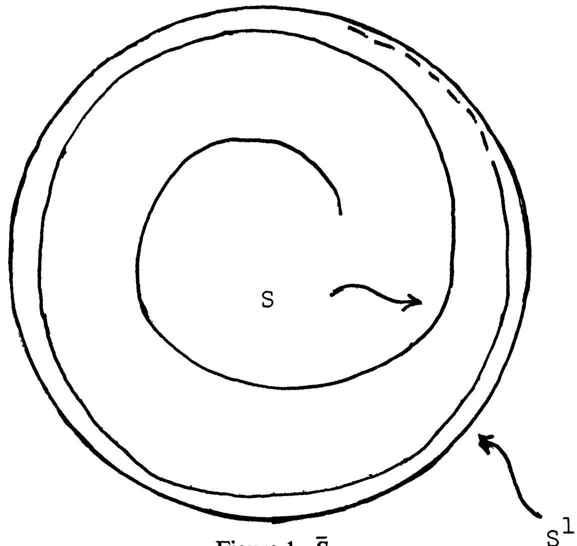

Figure 1. $\bar{S}$.

decreasing sequence of absolute retracts. We can therefore restate Aronszajn's results in the language of shape theory.

THEOREM 1. *Each overlying f-funnel is a continuum of trivial shape.*

A subset A of $\mathbf{R}^m$ is said to be a funnel-section if for some $n \geq m$, there exists a continuous map f: $\mathbf{R}^{n+1} \to \mathbf{R}^n$ as above and a point p in $\mathbf{R}^{n+1}$ such that $i(A) = F_t(p)$ for some t, where i: $\mathbf{R}^m \to \mathbf{R}^n$ is the standard injection.

C. C. Pugh [10, 11] has proved that there exists a plane continuum that is never a funnel-section. This plane continuum $\bar{S}$, the union of S^1 and an outward plane spiral S, is pictured in Figure 1. Pugh [10, 11] has also shown that any finite, connected polyhedron can be embedded in some $\mathbf{R}^m$ as a funnel-section. In particular, $S^1 \subset \mathbf{R}^2$ is a funnel-section. Since S^1 and $\bar{S}$ have the same shape, it follows that being a funnel-section is not a shape property.

Pugh [11] has proved that any subcontinuum of $\mathbf{R}^m$ whose complement is diffeomorphic to the complement of a point is a funnel-section (in fact, if $m \neq 4$, then "diffeomorphic" may be replaced by "homeomorphic"). T. A. Chapman [5] has proved that any finite-dimensional continuum of trivial shape can be embedded in some $\mathbf{R}^m$ such that its complement is homeomorphic to the complement of a point. We can therefore restate Pugh's theorem in the language of shape theory.

THEOREM 2. *Each finite-dimensional continuum of trivial shape can be a funnel-section.*

2. Necessary conditions for a continuum to be a funnel-section.

A circle is a continuum homeomorphic to S^1. A continuum X is said to be circle-like if it is homeomorphic to the inverse limit of an inverse sequence of circles with all bonding maps being surjective. If, in addition, each bonding map is a nontrivial covering map, then X is called a solenoid. The first Čech cohomology group of a continuum X with integral coefficients is denoted $H^1(X)$.

A nonzero element g of an abelian group G is said to have infinite height if there exists a sequence $P = \{p_1, p_2, \ldots\}$ of integers such that (1) each $p_i > 1$, and (2) for each positive integer n, there exists an element h of G such that $(p_1 p_2 \cdots p_n)h = g$.

THEOREM 3. *If X is a continuum and if $H^1(X)$ contains an element of infinite height, then X can never be a funnel section.*

Proof. The main theorem of [13] states that a continuum X can be mapped onto a solenoid if and only if $H^1(X)$ contains an element of infinite height. Thus, there is a solenoid S and a map $g: X \to S$ of X onto S. If X were homeomorphic to a cross section $F_t(p)$, for some $f: R^{n+1} \to R^n$, then there would exist a continuous map of the overlying f-funnel $\tilde{F}(f, p)$ onto X. Hence there would exist a continuous map of the overlying f-funnel $\tilde{F}(f, p)$ onto the solenoid S. This would imply that $H^1(\tilde{F}(f, p))$ contains an element of infinite height; but such an implication contradicts the fact that $\tilde{F}(f, p)$ has trivial shape and thus trivial Čech cohomology. Thus X cannot be a funnel section.

In particular, no solenoid can be a funnel section; thus there is a collection of cardinality c of topologically distinct, compact, connected abelian groups such that no one of them is ever a funnel section.

More can be said by using some results of shape theory. Q is the Hilbert cube. First we recall a definition. A pointed continuum $(X, x_0) \subset (Q, x_0)$ is said to be 1-movable if for every neighborhood U of X, there exists a neighborhood V of X with the property that every loop in V based at x_0 can be deformed within U into any neighborhood of X, the basepoint remaining fixed at x_0 throughout the deformation. Roughly, X is 1-movable means that fundamental groups of X-neighborhoods are nicely related. If (X, x_0) is 1-movable, and if $x_1 \in X$, then (X, x_1) is 1-movable. Thus we may speak of pointed 1-movable continua.

THEOREM 4. *If the continuum X can be a funnel section, then X is pointed 1-movable.*

Proof. Let X be homeomorphic to a cross section $F_t(p)$, for some $f: R^{n+1} \to R^n$. Let $\tilde{F}(f, p)$ be the overlying f-funnel. Since $\tilde{F}(f, p)$ has the shape of a point, by Theorem 1, it follows that $\tilde{F}(f, p)$ is pointed 1-movable. Pointed 1-movability is a continuous invariant [7], [8]; hence $F_t(p)$ is pointed 1-movable because the evaluation map ev: $\tilde{F}(f, p) \to F_t(p)$ is a continuous surjection. Hence X is pointed 1-movable.

THEOREM 5. *If the curve (i.e., one-dimensional continuum) X is a funnel section, then X is shape-equivalent to a point or a countable bouquet of circles.*

Proof. For curves, movability and pointed 1-movability are equivalent concepts. A. Trybulec [14] has shown that every movable curve has the shape of a plane continuum. K. Borsuk [2] has shown that every plane continuum is shape equivalent to a countable bouquet of circles (regard a point as an empty bouquet of circles). Thus the theorem is proved.

Since the Čech cohomology functor factors through the shape category, we have the following corollary:

COROLLARY 6. *If the curve X is a funnel section, then $H^1(X) \cong \oplus_n Z$, where Z is the set of integers and $n = 0, 1, 2, \ldots$ or ∞.*

The converse to this corollary fails in all cases.

THEOREM 7. *Let n be a nonnegative integer or let $n = \infty$. Then there is a curve X such that $H^1(X) \cong \oplus_n Z$ and X is not a funnel section.*

Proof. Pugh's example (described above) is easily modified to cover the cases $n \geq 1$ by adding suitably-chosen concentric circles and spirals. As an example of an acyclic curve that is never a funnel section, we refer to the Case-Chamberlain example [4]. This curve is described as an inverse limit of figure-eights; the bonding maps are so conceived, however, that the curve has trivial Čech cohomology. Sibe Mardešić and Jack Segal [9] have shown that the Case-Chamberlain curve is not movable. Therefore, it is not a funnel section.

By the Alexander duality theorem, no continuum such that $H^1(X)$ contains an element of infinite height can be embedded in $\mathbf{R}^2$. Thus no solenoid embeds in $\mathbf{R}^2$. Since solenoids are circle-like continua (while Pugh's example is not), we might ask whether each circle-like continuum in the plane is a funnel-section. Circle-like continua X in the plane satisfy one of two conditions: $H^1(X) \cong 0$ or $H^1(X) \cong Z$. In the former case, X is embedded in $\mathbf{R}^2$ such that its complement in $\mathbf{R}^2$ is diffeomorphic to the complement of the origin. Hence by results of Pugh, X can be a funnel-section. In the latter case, results of Pugh state that S^1 can be a funnel-section. We find, on the other hand, the following result.

THEOREM 8. *There exist circle-like continua in the plane that are never funnel-sections.*

Proof. Let the circle-like continuum X be a funnel-section. Then X is the continuous image of a continuum with trivial shape, namely some overlying f-funnel $\tilde{F}(f, p)$. J. Krasinkiewicz [7] has proved that such a continuum X is also a continuous image of an arc-like continuum. Hence to find a circle-like plane continuum that can never be a funnel section, it suffices to find one that is not a continuous image of an arc-like continuum. Such continua have been studied previously [12]; in particular, the pseudo-circle is such a continuum [12].

We close with the following example.

THEOREM 9. *There exists an arcwise-connected curve in* $\mathbf{R}^3$ *that is never a funnel-section.*

Proof. The example T is a modification of the plane continuum $\bar{S}$. Recall that $\bar{S} = S^1 \cup S$, where

$$S = \{(r, \theta) : r = 1 - \theta^{-1}, 2\pi \leq \theta < \infty\}.$$

$\bar{S}$ is pictured in Figure 1. Consider $\bar{S}$ as a subcontinuum of $\mathbf{R}^3$. Let A be an arc in $\mathbf{R}^3$ such that

(1) one endpoint of A is the endpoint of S,
(2) the other endpoint of A is $(1, 0, 0)$,
(3) all other points of A have third coordinate $z > 0$,
(4) A does not intersect the z-axis.

Finally, define $T = \bar{S} \cup A$.

Suppose that $T = F_s(p)$ for a suitable $f: \mathbf{R}^{n+1} \to \mathbf{R}^n$. Let

$$\pi: \mathbf{R}^3 \to \mathbf{R}^3 - (z\text{-axis})$$

be the universal covering space of $\mathbf{R}^3 - (z\text{-axis})$. Proceeding as in [11, Theorem 3.1], we find that the evaluation map,

$$\text{ev}: \tilde{F}(f, p) \to \mathbf{R}^3 - (z\text{-axis}),$$

which maps the overlying f-funnel into $\mathbf{R}^3 - (z\text{-axis})$, lifts to $\mathbf{R}^3$. Hence we have the commutative diagram:

$$\begin{tikzcd} & \mathbf{R}^3 \\ \tilde{F}(f, p) \arrow[ur, "\tilde{e}"] \arrow[r, "\text{ev}"] & \mathbf{R}^3 - (z\text{-axis}). \arrow[u, "\pi"'] \end{tikzcd}$$

Restricting π to $\pi^{-1}(T)$, we have the commutative diagram:

$$\begin{tikzcd} & \pi^{-1}(T) \arrow[d, "\pi"] \\ \tilde{F}(f, p) \arrow[ur, "\tilde{e}"] \arrow[r, "\text{ev}"] & T \end{tikzcd}.$$

See Figure 2 for a picture of a homeomorph of $\pi^{-1}(T)$. The arcwise-connected set $\pi^{-1}(T)$ consists of 3 types of lines:

(1) the x-axis, which covers S^1,
(2) half-lines $\{H_n\}$ that converge asymptotically to the x-axis and cover S,
(3) arcs $\{A_n\}$ that cover A and that join the endpoint of H_n to the point $(n, 0, 0)$.

Since $\tilde{F}(f, p)$ is compact, its image $\tilde{e}(F(f, p))$ is bounded in $\mathbf{R}^3$. Let m be a positive integer such that any point (x, y, z) in $\tilde{e}(F(f, p))$ lies in the set $B = \{(x, y, z): -m < x < m\}$. Since $\tilde{F}(f, p)$ is connected, its image $\tilde{e}(F(f, p))$ lies

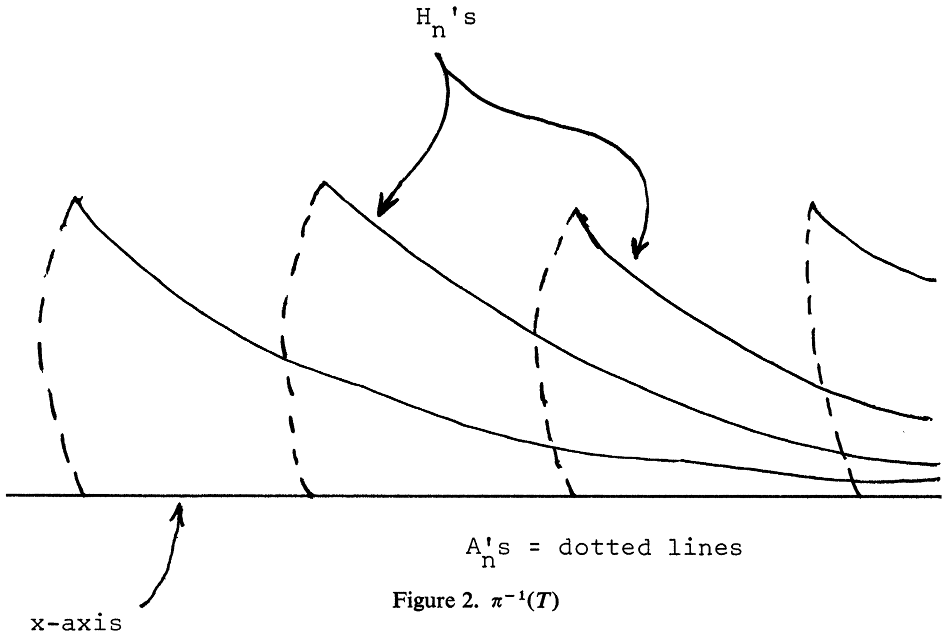

Figure 2. $\pi^{-1}(T)$

in one component of $B \cap \pi^{-1}(T)$. But no component of $B \cap \pi^{-1}(T)$ projects onto T. Therefore T is not a funnel-section.

Still unresolved is the question of whether every finite-dimensional Peano continuum can be a funnel-section or even if every finite-dimensional, compact, connected ANR can be a funnel-section.

Bibliography

1. N. Aronszajn, *Le correspondant topologique de l'unicité*, Ann. of Math., vol. 43 (1942), pp. 730–738.
2. K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math., vol. 62 (1968), pp. 223–254.
3. ———, *On movable compacta*, Fund. Math., vol. 66 (1969), pp. 137–146.
4. J. H. Case and R. E. Chamberlain, *Characterizations of tree-like continua*, Pacific J. Math., vol. 10 (1966), pp. 73–84.
5. T. A. Chapman, *Shapes of finite-dimensional compacta*, Fund. Math., vol. 76 (1972), pp. 261–276.
6. D. Hyman, *On decreasing sequences of compact absolute retracts*, Fund. Math., vol. 64 (1969), pp. 91–97.
7. J. Krasinkiewicz, *Continuous images of continua and 1-movability* (preprint).
8. D. R. McMillan, Jr., "One-dimensional shape properties and three-manifolds" in *Studies in topology*, Academic Press, New York, 1975, pp. 367–382.
9. S. Mardesić and J. Segal, *Movable compacta and ANR-systems*, Bull. Pol. Acad. Sci., vol. 18 (1970), pp. 649–654.
10. C. Pugh, *Cross-sections of solution funnels*, Bull. Amer. Math. Soc., vol. 70 (1964), pp. 580–583.

11. ———, *Cross-sections of solution funnels*, J. Differential Equations, vol. 19 (1975), pp. 270–295.
12. J. T. Rogers, Jr., *Mapping the pseudo-arc onto circle-like, self-entwined continua*, Mich. J. Math., vol. 17 (1970), pp. 91–96.
13. ———, *A cohomological characterization of pre-images of nonplanar, circle-like continua*, Proc. Amer. Math. Soc., vol. 49 (1975), pp. 232–236.
14. A. Trybulec, *On shapes of movable spaces*, Bull. Pol. Acad. Sci., vol. 21 (1973), pp. 727–733.

Tulane University
New Orleans, Louisiana