ILLINOIS JOURNAL OF MATHEMATICS
Volume 21, Number 4, December 1977

ON THE FACTORIZATION OF ENTIRE FUNCTIONS

BY

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1. Introduction

Following [1], an entire function $F(z) = f(g(z))$ is said to have meromorphic functions $f(z)$ and $g(z)$ as left and right factors respectively. $F(z)$ is prime if every factorization of the above form implies that one of the factors $f(z)$ or $g(z)$ is linear. For example, obviously, all polynomials of prime degree are prime. The first interesting nontrivial prime function is $e^z + z$, which was mentioned by Rosenbloom in [13] without proof. This was proved later by F. Gross [5]. As a further study, I. N. Baker and F. Gross [1] proved that every function of the form $e^z + p(z)$ (where $p(z)$ is a nonconstant polynomial) is prime. For generalizations of this and some other interesting classes of prime entire functions we refer the reader to [3], [4], [5], [6], [10], [12], [14], etc.

When factors are restricted to entire functions, it is said to be factorization in entire sense. Furthermore, F is said to be left-prime (right-prime) if every factorization $F = f(g)$ in the entire sense implies f is linear whenever g is transcendental (g is linear whenever f is transcendental). Recently several interesting results on the factorization in the above-mentioned senses were obtained. Among them the following one is very useful which is due to Ozawa [12].

THEOREM. *Let $F(z)$ be an entire function of finite order whose derivative $F'(z)$ has infinitely many zeros. Assume that the number of common roots of $F(z) = c$ and $F'(z) = 0$ is finite for every constant c. Then $F(z)$ is left-prime in the entire sense.*

In this note we shall exhibit a new class of prime entire functions, which, in particular, contains the function $e^z + z$ as a special case. We are able to prove that every entire function of the form $F(z) = aze^{miz} + p(e^{iz}, e^{-iz})$ is prime, where m is an integer, $p(u, v)$ is a polynomial in u and v, and a is a nonzero constant. We shall study the possible forms of the right factors of entire functions of the form $F(z) = ze^{H_1(z)} + H_2(z)$, where $H_1(z)$ and $H_2(z)$ are periodic entire functions with the same period σ. Finally, we obtain all the right factors and left factors of entire functions of the form $F(z) = z + e^{H(z)}$, where $H(z)$ is a periodic entire function.

Received June 24, 1976.

2. Prime entire functions

THEOREM 1. *Let a be a nonzero constant, m a nonzero integer, and $p(u, v)$ a polynomial in u and v. Then every entire function of the form*

$$F(z) = aze^{miz} + p(e^{iz}, e^{-iz})$$

is prime.

Proof. Let $F(z) = aze^{miz} + p(e^{iz}, e^{-iz})$. Consider the simultaneous equations

$$F(z) = c \text{ for constant } c, \quad F'(z) = 0.$$

In order to apply Ozawa's theorem we must prove that $F'(z) = 0$ has infinitely many zeros. This can be proved easily by applying the well-known impossibility of Borel's identity (see e.g., [2]). Now if the above system of equations has only finitely many solutions, then F is left-prime by the Ozawa theorem. For any common root z of the above equations we have

$$aze^{miz} + p(e^{iz}, e^{-iz}) = c,$$

$$amize^{miz} + ae^{miz} + ip_u(e^{iz}, e^{-iz})e^{iz} - ip_v(e^{iz}, e^{-iz})e^{-iz} = 0.$$

Hence

$$cmi - mip(e^{iz}, e^{-iz}) + ae^{miz} + ip_u(e^{iz}, e^{-iz}) - ip_v(e^{iz}, e^{-iz}) = 0.$$

This has the form $Q(e^{iz}) = 0$ for some polynomial $Q(x) = A \prod_{j=1}^{s} (x - \alpha_s)$. It follows that if $e^{iz_j} = \alpha_j$, $j = 1, \ldots, s$, are all the roots of $Q(x) = 0$, then $\bigcup_{i=1}^{s} \{z_i + 2k\pi\}$ for $k = \pm 1, \pm 2, \ldots$, are the only other possible roots of $Q(e^{iz}) = 0$. However, it is easy to see that for any integer $k \neq 0$ and any j,

$$a(z_j + 2k\pi)e^{miz_j} + p(e^{iz_j}, e^{-iz_j}) = c + 2ka\pi e^{miz_j} \neq c.$$

Hence $z_j + 2k\pi$ is not a common root for any j and integer $k \neq 0$. Therefore the system of equations can have at most s common roots $z_1, z_2, \ldots, z_j$.

Thus we have proved that F is left-prime in the entire sense. Therefore every nontrivial factorization $F = f(g)$ implies that f is transcendental entire and g is a nonlinear polynomial. However, in this case, for any z, $g(z + 2\pi) - g(z) = c \neq 0$ by $F(z + 2\pi) - F(z) = 2\pi ae^{miz}$. Hence g can only be a linear function, giving a contradiction. Finally we note that F is not a periodic function, hence, by a result of Gross [4], F is prime. This also completes the proof of the theorem.

3. Results about right factors

According to [1], an entire function $F(z)$ is said to be periodic mod h (where h is an entire function) with period σ iff $F(z + \sigma) - F(z) = h(z)$. When h is restricted to be a function of order less than 1, the following result was obtained.

LEMMA 1 [1]. *Every right factor $g(z)$ of an entire function F (where F is periodic* mod *h with period σ) is of the form $g(z) = H_1(z) + h_1(z)e^{H_2(z)+az}$, where $H_i(z)$, $i = 1, 2$, are periodic entire functions with the same period σ, a is a constant, and $h_1(z)$ is an entire function of order less than or equal to* 1. *If h is a polynomial then h_1 is also.*

Remark. We would like to point out here that when h is a constant, then from the proof of Lemma 1 we will find that $h_1(z)$ must be linear.

For an entire function F which is periodic mod h (where h is a function of order greater than or equal to 1) we have the following.

THEOREM 2. *Let $H_1(z)$ and $H_2(z)$ be two periodic entire functions with the same period σ. Then every right factor $g(z)$ of the entire function $ze^{H_1(z)} + H_2(z)$ is of the form*

(a) $g(z) = \omega_1(z)e^{2k_1\pi iz} + \omega_2(z)e^{(\log c)z}$

or

(b) $g(z) = \omega_1(z)e^{2k_1\pi iz} + z\omega_2(z)e^{2k_1\pi iz}$

where $\omega_1(z)$ and $\omega_2(z)$ are two periodic entire functions with the same period σ, k_1 and k_2 are two integers, and c is a constant unequal to 1.

Proof. Suppose that

$$F(z) = f(g(z)) \tag{1}$$

where f, g are entire functions with $g(z)$ being nonlinear. Then we have

$$\begin{aligned} F(z+\sigma) - F(z) &= f(g(z+\sigma)) - f(g(z)) \\ &= (z+\sigma)e^{H_1(z)} + H_2(z) - [ze^{H_1(z)} + H_2(z)] \\ &= \sigma e^{H_1(z)}. \end{aligned} \tag{2}$$

Similarly

$$f(g(z+2\sigma)) - f(g(z)) = 2\sigma e^{H_1(z)}. \tag{3}$$

It follows from this and (2) that

$$g(z+\sigma) - g(z) = e^{\alpha(z)} \tag{4}$$

and

$$g(z+2\sigma) - g(z) = e^{\beta(z)} \tag{5}$$

where $\alpha(z)$ and $\beta(z)$ are two entire functions. Subtracting (4) from (5) and comparing the result with (4) in which z is being replaced by $z + \sigma$, we obtain

$$e^{\beta(z)} - e^{\alpha(z)} = e^{\alpha(z+\sigma)}, \tag{6}$$

$$e^{\beta(z)-\alpha(z+\sigma)} - e^{\alpha(z)-\alpha(z+\sigma)} = 1. \tag{7}$$

This is impossible (by applying little Picard theorem to the function $e^{\beta(z)-\alpha(z+\sigma)}$)

unless

$$\beta(z) - \alpha(z + \sigma) = c_1 \tag{8}$$

and

$$\alpha(z) - \alpha(z + \sigma) = c_2 \tag{9}$$

for some constants c_1 and c_2. Then

$$\frac{g(z + 2\sigma) - g(z)}{g(z + 2\sigma) - g(z + \sigma)} = e^{\beta(z) - \alpha(z + \sigma)} = e^{c_1} \tag{10}$$

and

$$\frac{g(z + \sigma) - g(z)}{g(z + 2\sigma) - g(z + \sigma)} = e^{\alpha(z) - \alpha(z + \sigma)} = e^{c_2}. \tag{11}$$

It follows that

$$g(z + \sigma) - g(z) = c(g(z + 2\sigma) - g(z)) \tag{12}$$

for some nonzero constant c. Hence

$$cg(z + 2\sigma) - g(z + \sigma) + (1 - c)g(z) = 0. \tag{13}$$

This is a homogeneous linear difference equation of second order. The characteristic equation of this equation is

$$c\rho^2 - \rho + (1 - c) = 0. \tag{14}$$

The roots of it are $\rho = 1$ and $\rho = (1 - c)/c$. If $c \neq 1/2$, then the two roots of equation (14) are distinct. Otherwise, equation (14) has a double root at $\rho = 1$. Thus, according to [8, Chap. XII], the complete solutions of equation (13) can be expressed as

$$g(z) = \omega_1(z)e^{2k_1\pi iz} + z\omega_2(z)e^{2k_1\pi iz} \tag{15}$$

or

$$g(z) = \omega_1(z)e^{2k_1\pi iz} + \omega_2(z)e^{(\log c)z} \tag{16}$$

depending on whether $c = 1/2$ or not, where $\omega_1(z)$ and $\omega_2(z)$ are two periodic entire functions with the same period σ. This also completes the proof of Theorem 2.

Remark. By putting $a = -1$, $m = 0$, and $p(e^{iz}, e^{-iz}) = \sin z$, we obtain a result which was proved earlier in [14].

4. Right factors and left factors of functions of the form $z + e^{H(z)}$

THEOREM 3. *Let $H(z)$ be a nonconstant periodic entire function with period 1. Then every right factor $g(z)$ and left factor $f(z)$ of the function $z + e^{H(z)}$ are of the* *forms* $g(z) = H_1(z) + l_1(z)$ *and* $f(z) = G_1(z) + l_2(z)$ *respectively, where* $H_1(z)$ *is a periodic entire function with period* 1, *and* $G_1(z)$ *is also a periodic entire function,* $l_i(z)$ $i = 1, 2$, *are linear functions.*

We first prove a lemma.

LEMMA 2. *Let* $H(z)$ *be a nonconstant entire function with period* 1. *Then every right factor* $l(z)$ *of the function* $z + e^{H(z)}$ *can be expressed as*

$$l(z) = K_1(z) + q(z)e^{K_2(z)},$$

where K_i, $i = 1, 2$ *are periodic entire functions with the same period* 1, *and* $q(z)$ *is linear.*

Proof. Let

$$F(z) = z + e^{H(z)} \tag{17}$$

and

$$F(z) = f(g(z)) \tag{18}$$

where f and g are entire. Then, clearly, the function F is periodic mod 1 with period 1. Therefore, according to Lemma 1, we conclude that every right factor $g(z)$ of $F(z)$ has the form

$$g(z) = H_1(z) + q_1(z)e^{H_2(z)+az} \tag{19}$$

where H_i, $i = 1, 2$, are periodic entire functions with period 1, $q_1(z)$ is linear, and a is a constant. Now, by substituting z by $z + e^{2\pi iz}$ into (19) and from (17), we have

$$\begin{aligned} F(z + e^{2\pi iz}) &= f(g(z + e^{2\pi iz})) \\ &= z + e^{2\pi iz} + \exp H(z + e^{2\pi iz}). \end{aligned} \tag{20}$$

Clearly, the new function $F(z + e^{2\pi iz})$ also satisfies the assumptions of Lemma 1. Accordingly, we have

$$g(z + e^{2\pi iz}) = H_3(z) + q_2(z)e^{H_4(z)+bz} \tag{21}$$

where H_3, H_4 are periodic entire functions with the same period 1, q_2 is linear (see the remark following Lemma 1), and b is a constant. From this and equation (18) we obtain

$$H_1(z + e^{2\pi iz}) + q_1(z + e^{2\pi iz})e^{H_2(z + e^{2\pi iz})+az} \equiv H_3(z) + q_2(z)e^{H_4(z)+bz} \tag{22}$$

and so

$$H_1(z + e^{2\pi iz}) - H_3(z) \equiv q_2(z)e^{H_4(z)+bz} - q_1(z + e^{2\pi iz})e^{H_2(z + e^{2\pi iz})+az}. \tag{23}$$

We note the left hand side of this identity is a periodic function. Thus, by substituting z by $z+1$ into the above equation we have

$$(24)\quad q_2(z+1)e^{H_4(z)+b(z+1)} - q_1(z+1+e^{2\pi iz})e^{H_2(z+e^{2\pi iz})+a(z+1)} \equiv q_2(z)e^{H_4(z)+bz} - q_1(z+e^{2\pi iz})e^{H_2(z+e^{2\pi iz})+az}$$

(Here we have made use of the fact that both H_2 and H_4 are periodic with period 1.) Then

$$(25)\quad [e^b q_2(z+1) - q_2(z)]e^{H_4(z)+bz} \equiv [e^a q_1(z+1+e^{2\pi iz}) - q_1(z+e^{2\pi iz})]e^{H_2(z+e^{2\pi iz})+az}$$

From this and by the linearity of q_1 and q_2, one can conclude easily that

$$(26)\qquad e^b = e^a = 1.$$

Therefore e^{az} has a period 1. Lemma 2 is thus proved from this and (19).

The following two lemmas will also be needed in proving Theorem 3.

LEMMA 3 [7, p. 54]. *If $f(z)$ and $g(z)$ are transcendental entire then*

$$T(r, f(g))/T(r, g) \to \infty \quad \text{as } r \to \infty,$$

where $T(r, f)$ is the Nevanlinna characteristic function for f.

LEMMA 4 [9]. *Let $a_0(z), a_1(z), \ldots, a_n(z)$ and $g_1(z), \ldots, g_n(z)$ be entire functions. Suppose that*

$$T(r, a_j(z)) = o\left(\sum_{i=1}^{n} T(r, e^{g_i})\right) \quad (j = 0, 1, 2, \ldots, n).$$

If the identity $\sum_{i=1}^{n} a_i(z)e^{g_i(z)} = a_0(z)$ holds, then there is an identity

$$\sum_{i=1}^{n} c_i a_i(z)e^{g_i(z)} = 0,$$

where the c_i $(i = 1, 2, \ldots, n)$ are constants that are not all zero.

Proof of Theorem 3. Let $F(z) \equiv z + e^{H(z)} = f(g)(z)$ for some entire functions f and g. We assume that g is not linear. By Lemma 2, we have

$$(27)\qquad g(z) = H_1(z) + q(z)e^{H_2(z)},$$

where H_1, H_2 are periodic functions with the same period 1, and $q(z)$ is linear. We note that if f and g are entire, then $f(g)$ has infinitely many fix-points iff $g(f)$ has infinitely many fix-points (see, e.g., [6, p. 214]). From this and the factorization of F above, we have

$$(28)\qquad g(f(z)) = z + e^{\alpha(z)},$$

where α is an entire function. Substituting $g(z)$ for z in the above equation we obtain

$$(29)\qquad g(f(g(z))) = g(z) + e^{\alpha(g(z))}.$$

On the other hand, by virtue of (27), we have

$$g(f(g(z))) = H_1(f(g(z))) + q(f(g(z)))e^{H_2(f(g(z)))}. \tag{30}$$

It follows that

$$g(z) + e^{\alpha(g(z))} = H_1(f(g(z))) + q(f(g(z)))e^{H_2(f(g(z)))}. \tag{31}$$

Then, by substituting $z + 1$ for z in the equation, we have

$$g(z+1) + e^{\alpha(g(z+1))} = H_1(f(g(z))) + q(f(g(z)) + 1)e^{H_2(f(g(z)))}. \tag{32}$$

(Here we have made use of the fact that H_1 and H_2 are periodic with period 1.) By subtracting (32) from (31) we obtain

$$e^{\alpha(g(z))} - e^{\alpha(g(z+1)))} + g(z) - g(z+1) = -Ae^{H_2(f(g(z)))} \tag{33}$$

where A is the constant such that $q(z) \equiv Az + B$. Further, from (27) we have

$$g(z) - g(z+1) = -Ae^{H_2(z)}. \tag{34}$$

So (33) becomes

$$e^{\alpha(g(z))} - e^{\alpha(g(z+1)))} = A(e^{H_2(z)} - e^{H_2(f(g(z)))}), \tag{35}$$

from which we have

$$1 - e^{\alpha(g(z+1)) - \alpha(g(z))} = A(e^{H_2(z)} - e^{H_2(f(g(z)))})e^{-\alpha(g(z))}. \tag{36}$$

We proceed to apply Lemma 4 to identity (36) to show that H_2 must be a constant by dividing it into two cases separately.

Case (i). All the exponents $\alpha(g(z+1))) - \alpha(g(z))$, $H_2(z) - \alpha(g(z))$, and $H_2(f(g(z))) - \alpha(g(z))$ are constants. Then $H_2(f(g(z))) - H_2(z) = c$ for some constant c. This is impossible by virtue of Lemma 3 unless $H_2(z)$ is a constant.

Case (ii). At least one of the exponents is a nonconstant function. Then, according to Lemma 4, there exist constants c_1, c_2, and c_3 that are not all zero such that

$$c_1 e^{\alpha(g(z+1)) - \alpha(g(z))} + c_2 e^{H_2(z) - \alpha(g(z))} + c_3 e^{H_2(f(g(z))) - \alpha(g(z))} \equiv 0. \tag{37}$$

Hence

$$c_1 e^{\alpha(g(z+1))} + c_2 e^{H_2(z)} + c_3 e^{H_2(f(g(z)))} \equiv 0 \tag{38}$$

and, by the periodicity of $H_2(z)$,

$$c_1 e^{\alpha(g(z))} + c_2 e^{H_2(z)} + c_3^{H_2(f(g(z)))} \equiv 0. \tag{39}$$

If $c_1 = 0$, then from (38) we can deduce that $c_2 = c_3 = 0$ by applying Lemma 3 unless $H_2 \equiv$ constant. If $c_1 \neq 0$, then by subtracting (39) from (38), we have

$$c_1(e^{\alpha(g(z+1))} - e^{\alpha(g(z))}) \equiv 0. \tag{40}$$

Hence

$$e^{\alpha(g(z+1))} \equiv e^{\alpha(g(z))} \tag{41}$$

and thus identity (36) yields

$$e^{H_2(z)} - e^{H_2(f(g(z)))} \equiv 0. \tag{42}$$

This is impossible again by Lemma 3 unless $H_2(z) \equiv$ constant. Thus, from the above analysis we find that it is necessary that H_2 be a constant. This completes the proof for the right factors of F. Now we turn to the left factors of F. We have just shown that H_2 must be a constant. One can deduce easily from this and identities (34) and (41) that

$$\alpha(z + B) - \alpha(z) = 2\pi k i, \tag{43}$$

where B is a nonzero constant and k is an integer. It follows from this and (28) that $g(f(z))$ is an entire function periodic mod B with periodic B. As in the proof for the right factors case, we conclude that $f(z) = G_1(z) + l_2(z)$ where $G_1(z)$ is a periodic entire function and $l_2(z)$ is linear. This also completes the proof of Theorem 3.

We conclude the paper with the following conjecture: Let H be a periodic entire function, then $z + e^{H(z)}$ is prime.

Acknowledgment. The author would like to express his thanks to the referee for his suggestion of the present proof of Theorem 1.

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