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COMMUTING SUBNORMAL OPERATORS

BY

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If M and N are commuting normal operators on a Hilbert space $\mathscr{K}$, if $\mathscr{H}$ is a subspace of $\mathscr{K}$ which is invariant for M and N, and if S and T are the restrictions to $\mathscr{H}$ of M and N respectively, then S and T are commuting subnormal operators with commuting normal extensions. A recent example of Lubin shows that commuting subnormal operators need not have commuting normal extensions [6]. However, commuting subnormal operators S and T have commuting normal extensions if either S or T is normal [2, Theorem 8], if either S or T is cyclic [9, Theorem 3], or if either S or T is an isometry [8, Theorem 1].

In this paper it is shown that two commuting subnormal operators S and T have commuting normal extensions if the spectrum of T is finitely connected and the spectrum of the minimal normal extension of T is contained in the boundary of the spectrum of T. This generalizes a result of Slocinski who proves the theorem under the additional hypotheses that T is pure and that the spectrum of T does not divide the plane [8, Theorem 5]. In addition, an example is presented of two commuting subnormal operators without commuting normal extensions. This example is different from the aforementioned example of Lubin and perhaps more elementary.

The main theorem is proved in Section 1, the example is presented in Section 2, and two problems are stated in Section 3. In this paper, all Hilbert spaces are complex and separable, all subspaces are closed, and all operators are bounded.

1. A theorem on the existence of commuting normal extensions

The main theorem is a consequence of seven known theorems and two elementary facts. These nine results are recorded as Lemmas 1 through 9 below and the main theorem follows. Let X be a compact subset of the complex plane, let χ be the function $\chi(z) = z$, let $\int \oplus \mathscr{H}_x \, d\mu(x)$ be a direct integral over X, and let M_χ on $\int \oplus \mathscr{H}_x \, d\mu(x)$ be the operator defined by the equation $M_\chi(f) = \chi f$. An operator A on $\int \oplus \mathscr{H}_x \, d\mu(x)$ is said to be decomposable if for each x in X there is an operator A_x on $\mathscr{H}_x$ such that the function $x \to \|A_x\|$ is bounded and Borel measurable on X and

$$A(f)(x) = A_x(f(x)) \quad d\mu\text{-a.e.}$$

for all f in $\int \oplus \mathscr{H}_x \, d\mu(x)$. The operator A is denoted $\int \oplus A_x \, d\mu(x)$. The first two lemmas are proved in Dixmier [3, p. 208 and p. 164] and the third lemma is due to Bastian [1, Theorem 4.4].

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LEMMA 1. *A normal operator with spectrum X is unitarily equivalent to M_χ on a direct integral space $\int \oplus \mathscr{H}_x \, d\mu(x)$ over X.*

LEMMA 2. *An operator on $\int \oplus \mathscr{H}_x \, d\mu(x)$ commutes with M_χ if and only if it is decomposable.*

LEMMA 3. *A decomposable operator $A = \int \oplus A_x \, d\mu(x)$ on $\int \oplus \mathscr{H}_x \, d\mu(x)$ is subnormal if and only if A_x is subnormal $d\mu$-a.e.*

Let $C(X)$ be the Banach space of continuous complex functions on X and let $C_+(X)$ be the cone of nonnegative functions in $C(X)$. For a subset F of $C_+(X)$, the positive linear span of F is the set of linear combinations of the form $\lambda_1 f_1 + \cdots + \lambda_n f_n$ with $\lambda_k \geq 0$ and f_k in F. Let $R(X)$ be the closure in $C(X)$ of the rational functions with poles off X, let ∂X denote the boundary of X, and let $R(X) \mid \partial X$ be the space of functions $\{\phi \mid \partial X \colon \phi \text{ in } R(X)\}$. The following two lemmas are due to Mlak [7].

LEMMA 4. *If X is a finitely connected compact subset of the plane, then the positive linear span of the set*

$$\{|\phi|^2 \colon \phi \text{ in } R(X) \mid \partial X\}$$

is dense in $C_+(\partial X)$.

LEMMA 5. *If T is a subnormal operator with minimal normal extension N, if the spectrum of T is finitely connected, and if the spectrum of N is contained in the boundary of the spectrum of T, then every operator in the commutant of T is the restriction of an operator in the commutant of N.*

Lemma 6 below is due to Halmos [4, Theorem 3] and Bram [2, Theorem 1] and Lemma 7 is due to Bram [2, Theorem 8].

LEMMA 6. *If S is a subnormal operator on a Hilbert space $\mathscr{H}$, then*

$$\sum \langle S^j(f_k), S^k(f_j)\rangle \geq 0$$

for every finite sequence $f_1, \ldots, f_n$ of vectors in $\mathscr{H}$. Conversely, if $\mathscr{W}$ is a dense subset of $\mathscr{H}$ and if the inequality above holds for every finite sequence $f_1, \ldots, f_n$ in $\mathscr{W}$, then S is subnormal.

LEMMA 7. *If S is subnormal, T is normal, and S commutes with T, then S and T have commuting normal extensions.*

LEMMA 8. *If S on $\mathscr{H}$ is subnormal with minimal normal extension N and if X is the spectrum of S, then $\mathscr{H}$ is invariant for $\phi(N)$ for all ϕ in $R(X)$.*

Proof. By standard approximation arguments, it is sufficient to prove the theorem for $\phi(z) = 1/(z - \lambda)$ with λ not in X. In this case, each f in $\mathscr{H}$ can be written $f = (S - \lambda)(g)$ with g in $\mathscr{H}$. Thus, $\phi(N)(f) = \phi(N)(N - \lambda)(g) = g$ which proves the lemma.

An invariant subspace $\mathscr{M}$ for M_χ on $\int \oplus \mathscr{H}_x \, d\mu(x)$ is said to be full if the smallest subspace containing $\mathscr{M}$ which reduces M_χ is the entire space $\int \oplus \mathscr{H}_x \, d\mu(x)$.

LEMMA 9. *Let $\int \oplus \mathscr{H}_x \, d\mu(x)$ be a direct integral over X, let $\mathscr{M}$ be a full invariant subspace for M_χ, let $\mathscr{W}$ be a countable dense subset of $\mathscr{M}$, and for x in X let $\mathscr{W}_x$ be the closed linear span of $\{f(x): f$ in $\mathscr{W}\}$. Then $\mathscr{W}_x = \mathscr{H}_x$ $d\mu$-almost-everywhere.*

Proof. By [3, Proposition 9, p. 150], the space $\mathscr{N}$ of vectors f in $\int \oplus \mathscr{H}_x \, d\mu(x)$ such that $f(x)$ is in $\mathscr{W}_x$ $d\mu$-a.e. is a reducing subspace for M_χ. If f is in $\mathscr{M}$ and if $f_n \to f$ with f_n in $\mathscr{W}$, then there is a subsequence of the sequence $\{f_n\}$ converging to f $d\mu$-a.e. It follows that $\mathscr{M}$ is contained in $\mathscr{N}$. Since $\mathscr{M}$ is a full invariant subspace for M_χ, the space $\mathscr{N}$ is $\int \oplus \mathscr{H}_x \, d\mu(x)$. It follows from the definition of the direct integral that $\mathscr{N}$ contains a countable set $\mathscr{D}$ such that the linear span of $\{f(x): f$ in $\mathscr{D}\}$ is dense in $\mathscr{H}_x$ for each x [3, p. 141]. Consequently, $\mathscr{H}_x = \mathscr{H}_x$ $d\mu$-almost-everywhere.

THEOREM. *If S and T are commuting subnormal operators, if the spectrum of T is finitely connected, and if the spectrum of the minimal normal extension of T is contained in the boundary of the spectrum of T, then S and T have commuting normal extensions.*

Proof. Suppose that S and T act on the Hilbert space $\mathscr{H}$ and let N on $\mathscr{K}$ be the minimal normal extension of T. By Lemma 5, there is an operator $\hat{S}$ on $\mathscr{K}$ such that $\hat{S}$ commutes with N and the restriction of $\hat{S}$ to $\mathscr{H}$ is S. It is sufficient by Lemma 7 to prove that $\hat{S}$ is subnormal.

Let X denote the spectrum of N. By Lemma 1, there is a unitary operator U from $\mathscr{K}$ onto a direct integral space $\int \oplus \mathscr{H}_x \, d\mu(x)$ over X such that $UNU^* = M_\chi$. By Lemma 2, there is a decomposable operator $A = \int \oplus A_x \, d\mu(x)$ such that $U\hat{S}U^* = A$. Let $\mathscr{M} = U(\mathscr{H})$. Since N is the minimal normal extension of S, the space $\mathscr{M}$ is a full invariant subspace for M_χ. Observe also that $\mathscr{M}$ is invariant for A and that $U|\mathscr{M}$ establishes a unitary equivalence between S and $A|\mathscr{M}$. Hence, the operator $A|\mathscr{M}$ is subnormal. To prove the theorem it must be shown that A is subnormal.

Let $\mathscr{W}$ be a countable dense subset of $\mathscr{M}$, let $\mathscr{P}$ be the set of vectors $\lambda_1 g_1 + \cdots + \lambda_m g_m$ with each g_k in $\mathscr{W}$ and each λ_k of the form $a + ib$ with a and b rational, let $f_1, \ldots, f_n$ be in $\mathscr{P}$, and let ϕ be in $R(X)$. By Lemma 8, the vectors $\phi f_1, \ldots, \phi f_n$ are in $\mathscr{M}$. Thus, applying Lemma 6 to the subnormal operator $A|\mathscr{M}$,

$$\begin{aligned} 0 &\le \sum \langle A^j(\phi f_k), A^k(\phi f_j)\rangle \\ &= \int |\phi|^2(x) \sum \langle A_x^j(f_k(x)), A_x^k(f_j(x))\rangle \, d\mu(x). \end{aligned}$$

It follows from Lemma 4 that there is a Borel subset $G(f_1, \ldots, f_n)$ of X such that $\mu(X \backslash G(f_1, \ldots, f_n)) = 0$ and

$$\sum \langle A_x^j(f_k(x)), A_x^k(f_j(x)) \rangle \geq 0$$

for all x in $G(f_1, \ldots, f_n)$. Thus, the set

$$G = \bigcap \{G(f_1, \ldots, f_n) \colon n \geq 0, f_1, \ldots, f_n \text{ in } \mathscr{P}\}$$

is a Borel set, $\mu(X \backslash G) = 0$, and

$$\sum \langle A_x^j(f_k(x)), A_x^k(f_j(x)) \rangle \geq 0$$

for all $f_1, \ldots, f_n$ in $\mathscr{P}$ and all x in G. It follows from Lemma 9 that there is a Borel subset G' of X such that $\mu(X \backslash G') = 0$ and, for x in G', the set $\{f(x) \colon f \text{ in } \mathscr{P}\}$ is dense in $\mathscr{H}_x$. Thus, by Lemma 6, the operator A_x is subnormal for each x in $G \cap G'$. Hence, by Lemma 3, the operator A is subnormal and this completes the proof of the theorem.

2. An example

Let Z_+ denote the nonnegative integers and let $\mathscr{G}$ be a Hilbert space with an orthonormal basis $\{e_k \colon k \text{ in } Z_+\}$. For n in Z_+, define the operator A_n on $\mathscr{G}$ by the equations

$$A_n(e_0) = 2^{-n} e_1, \qquad A_n(e_k) = e_{k+1} \text{ for } k \geq 1,$$

and define the operator D on $\mathscr{G}$ by the equations

$$D(e_0) = 2e_0, \qquad D(e_k) = e_k \text{ for } k \geq 1.$$

Let $\mathscr{H} = \mathscr{G} \oplus \mathscr{G} \oplus \mathscr{G} \oplus \cdots$ and define S and T on $\mathscr{H}$ by setting

$$S = \begin{bmatrix} 0 & & & & 0 \\ D & 0 & & & \\ & D & 0 & & \\ & & D & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & & \cdot \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} A_0 & & & & 0 \\ & A_1 & & & \\ & & A_2 & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \cdot \end{bmatrix}.$$

The following facts about S and T are straightforward. (1) The operator A_k is unitarily equivalent to multiplication by z on

$$H^2(m + (2^{2n} - 1)\delta)$$

where m is normalized linear Lebesgue measure on the unit circle and δ is a unit point mass at the origin (see [5, Problem 156]). Thus, each A_k is subnormal and therefore T is subnormal. (2) The operator S is quasinormal (S commutes with S^*S) and therefore S is subnormal [5, Problem 154]. (3) For all n in Z_+, $DA_n = A_{n+1}D$ and from this it follows that S and T commute. (4) If f in $\mathscr{H}$ is the vector $(e_1, e_0, 0, 0, 0, \ldots)$ and if $C = S + T$, then

$$\langle (C^*C - CC^*)f, f \rangle = -11/4.$$

Thus, $S + T$ is not hyponormal, hence not subnormal [5, p. 103], and so S and T do not have commuting normal extensions.

3. Two problems

The example of Lubin [6] and the example in this paper of commuting subnormal operators S and T without commuting normal extensions both involve operators of infinite multi-cyclicity. At the other extreme, if either S or T is cyclic, then S and T have commuting normal extensions [9, Theorem 3]. This raises the following question.

Problem 1. If S and T are commuting subnormal operators and if either S or T is n-multi-cyclic for some finite n, must S and T have commuting normal extensions?

The following proposition is easily proved; one direction is an immediate consequence of Lemma 7 and the other direction is an application of the Fuglede theorem.

PROPOSITION. *Two commuting subnormal operators S and T have commuting normal extensions if and only if there is a subnormal extension of S which commutes with the minimal normal extension of T.*

This proposition raises the following question.

Problem 2. If S and T are commuting subnormal operators and if S extends to an operator which commutes with the minimal normal extension of T, must this extension be subnormal?

If the question posed in Problem 2 has an affirmative answer, then two commuting subnormal operators S and T have commuting normal extensions whenever the commutant of S or T lifts to the commutant of its minimal normal extension. The theorem in this paper and every other known positive result in this direction would follow as corollaries.

Added in proof. The question posed in Problem 2 has been answered in the negative by A. Lubin.

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