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POLYNOMIAL IDENTITIES OF NONASSOCIATIVE RINGS PART III: APPLICATIONS

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0. Introduction

In Parts I and II, a general theory was built which focused on certain classes of Ω-rings satisfying central polynomials. In this part we shall give applications to the classes of associative rings, alternative rings, alternative rings with involutions, and Jordan rings. {associative PI-rings}, {alternative PI-rings}, {alternative PI-rings with involution}, and many classes of Jordan PI-rings are Kaplansky and thus admit the structure theorems developed in [1, §§3 and 4]. These results were patterned after known results in associative ring theory and are therefore already known for associative rings, but many are new for alternative rings, and most are new for Jordan rings. In each section we will start by assuming Ω is an associative, commutative ring ϕ (possibly with an extra element denoting an involution); at the end of each section we consider the case where Ω is an arbitrary ring. The idea is to use easy, known facts and the results of [25] and [26] to prove that various classes of rings are Kaplansky, to find canonical identities and then to obtain properties of these classes by applying other results of [25, §§3 and 4] and [26, Theorem 4.10]; other results from [25, §1] and [25, §3] are also available. For the reader's convenience, we quote below the more major results of [26] and [25] which describe properties of Kaplansky classes. Recall that R denotes an Ω-ring with center Z; Nil( ), J( ), Jac( ), and BM( ) denote respectively the nil, Jacobson–Smiley, Jacobson–Brown ("generalized Jacobson"), and Brown–McCoy radicals. Also, two Ω-rings are *equivalent* if they satisfy the same identities.

A. Theorems about Ω-rings R belonging to a Kaplansky class (cf. [25, Def. 3.18])

THEOREM 0.1 [25, Theorem 3.3]. *If R is centrally admissible then every nonzero ideal of R hits Z.*

THEOREM 0.2 [25, Theorem 3.4]. *If R is strongly prime then R is centrally admissible (and thus absolutely prime).*

THEOREM 0.3 [25, Theorem 3.11(iii)]. *If R is primitive and semisimple then R is simple.*

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THEOREM 0.4 [25, Theorem 3.13]. *There is a* 1:1 *order-preserving correspondence between* {*identity-faithful absolutely prime ideals of* R} *and* {*identity-faithful prime ideals of* Z}, *given by* $\tilde{P} \to \tilde{P} \cap Z$. *This correspondence induces a* 1:1 *order-preserving correspondence between* {*identity-faithful maximal ideals of* R} *and* {*identity-faithful maximal ideals of* Z}.

THEOREM 0.5 [25, Proposition 3.22]. *If* $\mathrm{Nil}(R) = 0$ *and if* R *satisfies the ascending chain condition on central annihilators, then the* Ω*-ring of central quotients of* R *is a direct sum of simple PI-rings.*

THEOREM 0.6 [25, Theorem 3.17]. *If* $\mathrm{Nil}(R) = 0$ *then* $BM(R[\lambda]) = 0$.

THEOREM 0.7 [25, Proposition 3.10] and [26, Theorem 4.10]). *Under the conditions of* [26, *Theorem* 4.10], R *is semisimple iff* Z *is semisimple.*

THEOREM 0.8 [26, Theorem 3.17]. $\mathrm{Nil}(R) = BM(R)$ *if* R *is a universal PI-ring.*

B. Theorems about Azumaya type (cf. [25, Definition 4.11])

THEOREM 0.9 [25, Theorem 4.13]. *If* R *has Azumaya type* t *then the following three properties hold*:

(i) *For every maximal ideal* P *of* Z, *there exists* c *in* $Z - P$ *such that* $R_c \approx (Z_c)^t$ *as* Z_c*-modules*;

(ii) *There is a lattice isomorphism from* {*ideals of* R} *to* {*ideals of* Z}, *given by* $A \mapsto A \cap Z$, *with inverse* $A \mapsto AR$;

(iii) *If* P *is a maximal ideal of* Z, *then* R/PR *is a central simple* Z/P*-algebra of dimension* t.

THEOREM 0.10 [25, Theorem 4.15]. *If* R *is equivalent to a central simple* Ω*-ring with regular central polynomial and if* S *is a multiplicative subset of* Z *such that* $S \cap IR \neq 0$ (*where* I *is the "central kernel", the values of regular central polynomials*), *then* R_S *has Azumaya type.*

THEOREM 0.11 [25, Theorem 4.19]. *If* R *is mult-equivalent to a central simple* Ω*-ring with multilinear central polynomial, then* $R_{Z-\{0\}}$ *has Azumaya type.*

In order to use Theorem 0.7, we will want some knowledge about canonical central polynomials. Say a Kaplansky class $\mathscr{C}$ of Ω-rings has *property* F (with respect to a class of polynomials $\{g_n \mid n \in \mathbf{Z}^+\}$) if, for every semisimple Ω-ring R of $\mathscr{C}$, there is some k, such that g_k is R-central and g_t is an identity of R for all $t > k$.

Remark 0.12. Every Ω-ring in a Kaplansky class with property F is centrally admissible.

THEOREM 0.13. *If* $\mathscr{C}$ *is a Kaplansky class with property* F *then, for the same class of polynomials* $\{g_n \mid n \in \mathbf{Z}^+\}$, *for every strongly semiprime* Ω*-ring* R

of $\mathscr{C}$, there is some k such that g_k is R-central and g_t is an identity of R for all $t > k$. If every direct power of a member of $\mathscr{C}$ is in $\mathscr{C}$ then for every Ω-ring R of $\mathscr{C}$ there is some k such that a suitable power of g_t is an identity of R for all $t > k$.

Proof. If R is strongly semiprime then, by Theorem 0.6, $R[\lambda]$ is semisimple, so, for some k, g_t is an identity of R for all $t > k$. Take the smallest such k. Then g_k is not an identity of R, so g_k is not an identity of some strongly prime image $\bar{R}$ of R (cf. [25, Corollary 2.5]). But for each such $\bar{R}$, g_k is central for the ring of central quotients of $\bar{R}$, which is simple, by Theorem 0.2; thus, by [25, Corollary 2.1], g_k is $\bar{R}$-central. Hence we see g_k is central or an identity for each strongly prime image of R, so g_k is R-central.

The second assertion is an immediate application of [25, Theorem 3.23]. Q.E.D.

As we prove various classes of rings are Kaplansky satisfying property F, we shall not restate verbatim results 0.1 through 0.13, but these results are quite striking in the individual applications.

1. Associative algebras

The variety of associative *PI*-algebras was the model for our theory, so we do not expect many new results (although I believe [25, 3.13] and parts of [25, §1] and [26, §§1, 3] are new even for associative *PI*-algebras). Note that $J(R) = \mathrm{Jac}(R)$ when R is associative; hence, by [26, Theorem 2.3], one gets Amitsur's theorem, that $\mathrm{Nil}(R) = 0$ implies $\mathrm{Jac}(R[\lambda]) = 0$. We need the following additional well-known results in the category of associative algebras over a commutative, associative ring.

THEOREM A (Kaplansky's theorem). *Every (associative) primitive PI-algebra is simple, and is a finite dimensional algebra over its center.*

THEOREM B (Levitzki-Amitsur). *Every (associative) semiprime PI-algebra is strongly semiprime. (Thus "prime" means "strongly prime".)*

A central simple F-algebra of dimension n^2 over F is said to have *degree* n.

THEOREM C (Formanek [7]). *For each $n \in \mathbf{Z}^+$, there is a completely homogeneous polynomial $g_n(X_1, \ldots, X_{n+1})$, linear in $X_2, \ldots, X_{n+1}$, which is A-stable central for each (associative) central simple algebra A of degree n, and g_n is an identity for all central simple algebras of degree $< n$. By rearranging terms, Amitsur has shown that $g_n(X_1, X_2) \equiv g_n(X_1, X_2, \ldots, X_2)$ is also A-stable central for each associative central simple algebra of degree n.*

Theorem A shows that the variety of associative *PI*-algebras is Kaplansky and also, for every *PI*-algebra R, $BM(R) = \mathrm{Jac}(R)$.

Theorems A and C imply that {Associative *PI*-algebras} have property F, so, by Theorem B and Theorem 0.12, we see in particular that every associative semiprime *PI*-algebra satisfies a central polynomial. At this point we see that Theorems 0.1–0.13 are applicable, although Theorem 0.3 is subsumed by Kaplansky's theorem.

Theorem 0.9 yields the following important result:

THEOREM 1.1 (Artin [6], Procesi [21]). *If R is associative and satisfies all identities of $M_n(\mathbf{Z})$, and if no simple homomorphic image of R has dimension $<n^2$ over its center, then R is an Azumaya algebra over its center, of rank n^2.*

Proof. Let $f(X_1, \ldots, X_{2n^2}) = \sum (\operatorname{sg} \pi) X_1 X_{\pi 2} X_3 X_{\pi 4} \cdots X_{2n^2-1} X_{\pi(2n^2)}$, summed over all permutations π of $(2, 4, 6, \ldots, 2n^2)$. It is an easy matter to check matrix units and show that $R_0^+(f) = R_0$ for any matrix algebra $R_0 = M_n(F)$, where F is an arbitrary field. Thus, setting

$$g' = g_n(X_1, \ldots, X_n, f(X_{n+1}, \ldots, X_{2n^2+n})),$$

and rearranging the indeterminates so that they start $X_{n+2}, X_{n+4}, \ldots$, we see that g' is n^2-normal and is central for every matrix algebra $M_n(F)$.

Now, every associative central simple algebra R_0 is equivalent to a matrix algebra over a field. (If the center is finite then R_0 is already a matrix algebra over a field, by Wedderburn's theorems; otherwise we "split" R_0 by the algebraic closure of its center and use [25, Remark 1.8].) Thus, it follows easily from the hypotheses that g' is R-central and $1 \in R(g')R$. Hence R has Azumaya type n^2, by [26, Definition 4.11], so we may apply Theorem 0.9. The conclusions yield classically that R is an Azumaya algebra. Q.E.D.

The theory of universal associative *PI*-algebras is very interesting, and we outline its main features, based on [26, §3]. (Of course, the results presented here are well known.) First, we recall [26, Theorem 3.17], which says that the Brown-McCoy radical of every universal, associative *PI*-algebra is nil. At this point, assume ϕ is an infinite integral domain and R is a ϕ-algebra. It is well known (cf. [2]) that for every n, there is a division ϕ-algebra D of dimension n^2 over its center. This gives rise to the universal algebra $\mathfrak{U}(D)$, which we call $\phi^{(n)}\{Y\}$, because, letting $\phi(\xi)$ be the field generated over ϕ by the commuting indeterminates $\{\xi_{ij}^{(k)} \mid 1 \le i, j \le n, 1 \le k < \infty\}$, we have $\phi^{(n)}\{Y\}$ canonically isomorphic to the ϕ-subalgebra of $M_n(\phi(\xi))$ generated by the "generic" matrices $Y_k = \sum_{i,j=1}^n \xi_{ij}^{(k)} e_{ij}$ (where e_{ij} are the canonical matric units), $1 \le k < \infty$. By [26, Corollary 3.26], $\phi^{(n)}\{Y\}$ is a domain equivalent to the *PI*-ring D, so $\phi^{(n)}\{Y\}$ has an ϕ-algebra of central quotients which we call $\phi^{(n)}(Y)$. Obviously $\phi^{(n)}(Y)$ is a division algebra equivalent to D, so $\phi^{(n)}(Y)$ also has dimension n^2 over its center. By [26, Corollary 3.26], $\phi^{(n)}\{Y\}$ satisfies every universal, sentence of an arbitrary simple ϕ-algebra having dimension n^2 over its center. Piecing together information about $\phi^{(n)}(Y)$ from various division algebras was one of the main techniques

of Amitsur in his paper [5], proving that $\phi^{(n)}(Y)$ is not a crossed product, for many n.

Let us make a brief digression concerning {associative *PI*-algebras with involution}. The starting point in this theory is the theorem of Amitsur [4], that if $(R, *)$ is a *PI*-algebra then R is a *PI*-algebra. (In fact, by [27, Theorem 8], if $(R, *)$ is a *PI*-algie then R is a *PI*-$\mathbf{Z}$-algebra. Thus, one has cosmetically the same theory as in §1, but the relevant objects are often different. This idea is exploited in [27].)

We briefly consider universal associative *PI*-algebras with involution, for example. We again use $M_n(\phi(\xi))$, where ϕ is an infinite integral domain, but we also consider the transpose involution (t), and let $\phi^{(n)}\{Y, Y^t\}$ be the ϕ-subalgebra of $M_n(\phi(\xi))$ generated by the generic matrices $Y_k = \sum_{i,j=1}^n \xi_{ij}^{(k)} e_{ij}$ and their transposes $Y_k^t = \sum_{i,j=1}^n \xi_{ji}^{(k)} e_{ij}$. $\phi^{(n)}\{Y, Y^t\}$ has an involution induced by (t), of the first kind (by [26, Example 3.24]); letting $(\phi^{(n)}(Y, Y^t), t)$ be the algebra of central quotients of $(\phi^{(n)}\{Y, Y^t\}, t)$, we see that $\phi^{(n)}(Y, Y^t)$ is central simple of dimension n^2. Moreover, there exists a ϕ-division algebra with involution of the first kind, having dimension n^2 over the center, iff n is a power of 2 (cf. [28, Proposition 28]); hence $\phi^{(n)}(Y, Y^t)$ is a division algebra iff n is a power of 2. One can do a similar analysis using the symplectic involution in place of (t), and it is fascinating to study these "generic" division algebras with involution, as well as the "involutory" analogue of the Artin-Procesi theorem, but we shall not digress that far here.

In case we consider algies, in place of algebras, one can state Theorems 2 and 6 in [27] as: If $(R, *)$ is an associative *PI*-algie over a ring Σ, then R is a *PI*-$\mathbf{Z}$-algebra.

2. Alternative algebras

Let *purely* (*resp absolutely*) *alternative* mean purely (resp. absolutely) nonassociative alternative, cf. [25, §5]. There is an extensive structure theory of purely alternative algebras, due largely to Albert, Bruck, Kleinfeld, and Slater. Much of this theory now follows as a special case of the above theory, applied to alternative algebras; also, new information is obtained about the lattice of prime ideals. First recall by [25, Remark 1.2] that the identities $[X_1, X_1, X_2]$ and $[X_2, X_1, X_1]$, which define alternative algebras, are stable identities of every alternative algebra, hence, by [25, Proposition 1.3], $R[\lambda]$ is alternative if R is alternative. Thus, every class of alternative algebras satisfies criteria (ii) and (iii) of [25, Definition 3.16] of central class. Another immediate consequence of the alternative identities is:

LEMMA 2.1. *If A is an ideal of an alternative algebra R, then*

$$\{r \in R \mid rA = 0\} \quad \textit{and} \quad \{r \in R \mid Ar = 0\}$$

are ideals of R.

Thus, if R is prime and $rA = 0$, then either $r = 0$ or $A = 0$. We shall now prove that every variety of alternative PI-algebras is a Kaplansky class. We shall need the easily verified Moufang identities

$$((X_1X_2)X_1)X_3 - X_1(X_2(X_1X_3)) \quad \text{and} \quad (X_1(X_2X_3))X_1 - (X_1X_2)(X_3X_1)$$

cf. [12, p. 16]), as well as the following five basic results:

THEOREM D (Artin). *Any alternative algebra generated over its center by two elements is associative.*

THEOREM E (Kleinfeld [13] and Bruck). $[[X_1, X_2]^4, X_3, X_4]$ *is an identity of every alternative algebra.*

THEOREM F (Kleinfeld [14, proof of Lemma 3.5]). *For every prime, nonassociative, alternative algebra* R, $Z(R) = N(R)$. (*Hence* $[X_1, X_2]^4$ *is either central or an identity of* R.)

THEOREM G (Kleinfeld [13, Theorem 4.1]). *If* $[X_1, X_2]^4$ *is an identity of an alternative algebra* A, *not necessarily with* 1, *then* $\mathrm{Nil}(A) = \{$*nilpotent elements of* $A\}$.

THEOREM H (Kleinfeld [14]). *Suppose a nonassociative, alternative algebra* R *has a maximal left ideal which contains no nonzero two-sided ideals. Then* R *is a Cayley-Dickson algebra of octonians (abbreviated CD algebra). In other words* R *has the following form: The additive structure of* R *is* $Q \oplus Q$, Q *an arbitrary (associative) quaternion algebra with involution* $q \to \bar{q}$ *such that* $q \in Z(Q)$ *iff* $\bar{q} = q$; *multiplication on* R *is given by*

$$(q_1, q_2)(q_3, q_4) = (q_1q_3 + v\overline{q_4}q_2, q_4q_1 + q_2\overline{q_3}),$$

v *a suitable fixed nonzero element of* $Z(Q)$.

Remark 2.2. *CD* algebras are not associative and thus satisfy the central polynomial $[X_1, X_2]^4$. (In fact, $[X_1, X_2]^2$ is central, cf. Theorem 2.8 below.)

THEOREM 2.3. (1) *If* R *is a strongly prime, alternative PI-algebra, then* R *is absolutely prime.*

(2) *The variety of alternative PI-algebras is Kaplansky.*

Proof. (1) We are done unless R is nonassociative. Let $A \neq 0$ be an ideal of R. We need to prove $A \cap Z \neq 0$. For each a in A, r in R, $[a, r]^4 \in Z$, so we are done unless $[a, r]^4 = 0$, for all a in A, all r in R.

Choose r arbitrarily and let R' be the Z-subalgebra of R generated by Z, A, and r. R' has an associative, commutative Z-subalgebra $R'' = Z[r]$; any element of R' has the form $a + x$, for suitable a in A and x in R''. Since $[a_1 + x_1, a_2 + x_2] \in A$ for all a_i in A, all x_i in R'', Theorem F yields $[a_1 + x_1, a_2 + x_2]^4 \in A \cap Z$, so we are done unless $[X_1, X_2]^4$ is an identity of R'.

By Theorem G, we then have $\mathrm{Nil}(R') = \{$nilpotent elements of $R'\}$, so ar, ra, and $a + a'$ are nilpotent for all nilpotent elements a, a' of A. Since r was

taken arbitrarily in R, we conclude that {nilpotent elements of A} is a nil ideal of R, which must thus be 0. Hence A has no nilpotent elements.

On the other hand, since $[X_1, X_2]^4$ is an identity of R', $[r, a]$ must be nilpotent for each element a of A, so $[r, a]=0$. In other words, $[R, A]=0$.

The rest of the proof is standard, but we argue it out for completeness. Section 1B example $(i\beta)$ in [25] yields the identity

$$3[X_1, X_2, X_3]+X_1[X_2, X_3]+[X_1, X_3]X_2-[X_1X_2, X_3]$$

for every alternative ring. We apply this identity to R. Specializing X_3 to A shows $3[R, R, A]=0$; by left-right symmetry, $[3A, R, R]=0$. Next, specializing X_1 to A (and X_2, X_3 to R) shows $A[R, R]=0$; by Lemma 2.1, $[R, R]=0$. Using the Moufang identities and Artin's theorem we get, by an easy computation, $[r_1^3, r_2, r_3]=0$ for all r_i in R, implying $r_1^3 \in N(R)=Z$. In particular, $a^3 \in Z \cap A$ for all a in A. Since $\mathrm{Nil}(R)=0$, $a^3 \neq 0$ for some a in A, so we conclude that $Z \cap A \neq 0$.

(2) First we shall prove that {alternative *PI*-rings} is central, as defined in [25, Definition 3.16.] As remarked earlier, it suffices to verify condition (i), that every strongly prime alternative *PI*-algebra R has an R-stable central polynomial. This is clear unless R is nonassociative. Then, by part (1) above, Theorem H, and [25, Corollary 2.1], R is equivalent to a *CD* algebra, which then, by Remark 2.2, satisfies the central polynomial $[X_1, X_2]^4$, so $[X_1, X_2]^4$ is R-central. But $R[\lambda]$ is also strongly prime and nonassociative, so $[X_1X_2]^4$ is $R[\lambda]$-central; thus $[X_1, X_2]^4$ is stable R-central, and we have proved condition (i), and know that {alternative *PI*-algebras} is a central class. Then, by Theorem 3.17 ((3)$\Rightarrow$(1)), {alternative *PI*-algebras} is Kaplansky. Q.E.D.

At this point, we are ready to apply immediately most of the results given in §0, and obtain much information about prime and semiprime alternative *PI*-algebras; the main point is Theorem F, which says that every prime, nonassociative, alternative algebra is a *PI*-algebra! Slater [30], [31] obtained structure theorems about semiprime purely alternative algebras; these can be seen to be special cases of the theorems in §0 via the following results.

Remark 2.4. Every strongly semiprime purely alternative algebra is *PI*. (Proof. Special case of [25, Remark 5.5].)

Remark 2.5. (cf. Slater [29]). If R is prime, nonassociative and alternative then R is strongly prime *unless* $3R=0$ and R has a nonzero locally nilpotent ideal. (Proof. By Kleinfeld [15], R is strongly prime unless $\mathrm{Ann}\{3\} \equiv \{r \in R \mid 3r=0\} \neq 0$. Since $(3R)\,\mathrm{Ann}\{3\}=0$, R is strongly prime unless $3R=0$. Even so, by a result of Shirshov given in [17], R must have a nonzero locally nilpotent ideal.)

Having seen how well the *PI*-theory works in presenting known results, we shall show that {alternative *PI*-algebras} is as well behaved as we could

wish, enabling us to obtain new results for alternative algebras. In view of Theorem H, we should first investigate {*CD*-algebras} more closely. An easy verification shows, in the notation of Theorem H, that a *CD*-algebra R is a domain if, for all q in Q, $v \neq q\bar{q}$. It is then a simple matter to construct a "generic" *CD* division algebra; let v be an indeterminate over ϕ, and let Q be the algebra of central quotients $(\phi(v))^{(2)}(Y, Y^t)$ of $(\phi[v])^{(2)}\{Y, Y^t\}$ (as defined above in §1).

On the other hand, notation as in Theorem H, if $v = 1$ and $Q \approx M_2(F)$, the ring of 2×2 matrices over a field F, then R is called a *split CD-algebra.* It is easy to see (cf. [29, p. 32]), for any *CD*-algebra R whose center Z has algebraic closure Z', that $R \otimes_Z Z'$ is split. Let $CD(F)$ denote the split *CD*-algebra with center F.

Suppose ϕ is an infinite field and R is a *CD*-algebra whose center Z has algebraic closure Z'. R is equivalent to $R \otimes_Z Z' \approx CD(Z') \approx CD(\phi) \otimes_\phi Z'$, which is equivalent to $CD(\phi)$. Hence $U(R) = U(CD(\phi))$ for every *CD*-algebra R. But we have seen that there exist *CD* division algebras (over ϕ); hence, by [26, Corollary 3.26], $U(R)$ is a purely nonassociative domain. So we see that there is precisely one universal algebra for all *CD*-algebras over ϕ, and this is a purely nonassociative domain which we call $\phi^{CD}\{Y\}$. Using central quotients, we see that $\phi^{CD}\{Y\}$ is equivalent to every prime, nonassociative, alternative ϕ-algebra, so, by [25, Remark 5.5], we have:

PROPOSITION 2.6. *$\phi^{CD}\{Y\}$ is equivalent to every strongly semiprime, purely alternative ϕ-algebra.*

Of course, the algebra of central quotients of $\phi^{CD}\{Y\}$ is *CD*, and is in fact the "generic" *CD* division algebra described above; details are left to the interested reader. Since every *CD*-algebra contains an (associative) quaternion subalgebra, we know that all identities of $\phi^{CD}\{Y\}$ are identities of $M_2(\phi)$. As a partial converse, we have the following result:

THEOREM 2.7. *If ϕ is an infinite field then $\phi^{CD}\{Y\}$ satisfies every 2-identity (i.e. every identity in 2 indeterminates) of $M_2(\phi)$.*

Proof. Let $f(X_1, X_2)$ be a 2-identity of $M_2(\phi)$. Theorem D says that, for any elements x_1, x_2 in $\phi^{CD}\{Y\}$, the subalgebra A of $\phi^{CD}\{Y\}$ generated by x_1 and x_2 is associative. Moreover, A is a domain in which $[X_1, X_2]^4$ is either an identity or central. Hence, the ring of central quotients of A is central simple of degree ≤ 2, so A satisfies all the 2-identities of $M_2(\phi)$, implying $f(x_1, x_2) = 0$. Therefore $f(X_1, X_2)$ is an identity of $\phi^{CD}\{Y\}$. Q.E.D.

Note that $\phi^{CD}\{Y\}$, being nonassociative, cannot satisfy $[X_1, X_2, X_3]$, a 3-identity of $M_2(\phi)$. On the other hand, each Formanek polynomial $g_n(X_1, X_2)$ is an identity of $M_2(\phi)$, for all $n > 2$. Thus $g_n(X_1, X_2)$ is an

identity of $\phi^{CD}\{Y\}$, for all $n>2$. It is easy to check that $g_2=[X_1, X_2]^2$ is central for $CD(\phi)$, and thus for $\phi^{CD}\{Y\}$. Now multilinearize g_2 to the polynomial

$$g_2'=[X_1, X_2][X_3, X_4]+[X_1, X_4][X_3, X_2]+[X_3, X_2][X_1, X_4]$$
$$+[X_3, X_4][X_1, X_2].$$

Then g_2' is either $\phi^{CD}\{Y\}$-central or is an identity of $\phi^{CD}\{Y\}$. On the other hand, it is well known (cf. [7]) that g_2' is *not* an identity of any quaternion algebra, and thus is not an identity of any CD-algebra. Thus g_2' is $CD(\phi)$-central, and thus $\phi^{CD}\{Y\}$-central.

Now let ϕ be an arbitrary integral domain. Any CD-algebra obviously can be embedded into a split CD-algebra with infinite center (via splitting by the algebraic closure of ϕ). Thus we have proved:

THEOREM 2.8. *Every CD-algebra satisfies the identities $g_n(X_1, X_2)$, $n>2$, and the central polynomials g_2 and g_2'.*

Also note that every CD-algebra has dimension 8 over its center, so every 9-normal polynomial is an identity. We are now ready to prove that the variety of alternative PI-algebras satisfies property F.

THEOREM 2.9. (1) *If A is a strongly semiprime, purely alternative algebra, then $[X_1, X_2]^2$ and g_2' are A-central, $g_n(X_1, X_2)$ is an identity of A for all $n>2$, and every 9-normal polynomial is an identity of A.*

(2) *The variety of alternative PI-algebras satisfies property* F *with respect to* $\{|g_n \mid n\geq 1\}$.

Proof. (1) Since A is a subdirect product of strongly prime, nonassociative algebras, it suffices to verify the assertion when A is prime. But then A is absolutely prime, by Theorem 2.3(1), so A is equivalent to its algebra of central quotients, a CD-algebra, and, by theorem 2.8, we are done.

(2) We want to show that for every semisimple alternative PI-algebra A, there is some number k such that g_k is A-central and g_n is an identity of A for all $n>k$. By [25, Proposition 5.4], and [26, Corollary 2.6], we have A as the subdirect product of semisimple PI-algebras A_1 and A_2, with A_1 associative and A_2 purely alternative. Then we are done by putting together Theorem C and Theorem 2.9(1). Q.E.D.

By Theorem 2.9, we have the whole theory of §0 above to draw from in the study of alternative PI-algebras; more specifically, purely alternative algebras satisfy the multilinear central polynomial g_2' and are thus amenable to Theorem 0.4. Next, we see how we can use our version of the Artin-Procesi theorem to study alternative algebras. The object is to prove an alternative algebra has "Azumaya type" and then to apply Theorem 0.9. First note, by [25, Theorem 4.5] and Theorem 2.9, that every CD-algebra satisfies an 8-normal, central polynomial $g<g_2'$. Now suppose R is alternative, and g_2' is R-central, and, using notation following [25, Lemma 4.9], let

$G' = \{g \in \mathcal{G}_8 \mid g < g_2'\}$. If R is absolutely nonassociative then $R^+(G')$ is not contained in any maximal ideal of R, so, by [25, Proposition 4.12], R has Azumaya type 8, yielding:

THEOREM 2.10. *If R is absolutely nonassociative, alternative, and if g_2' is R-central, then R has Azumaya type 8.*

COROLLARY 2.11. *If R is absolutely nonassociative, alternative, and if* $\mathrm{Nil}(R) = 0$, *then R has Azumaya type 8.*

Proof. By Theorem 2.9(1), g_2' is R-central, so apply Theorem 2.10. Q.E.D.

Thus, in view of Theorem 0.9, the structure of absolutely nonassociative, alternative algebras is very closely connected to the center. We close this section with a nostalgic result:

THEOREM 2.12. *Every alternative PI-algebra satisfies a power of a suitable standard identity. Every strongly semiprime alternative PI-algebra satisfying an identity of degree m also satisfies the standard identity S_t for all* $t \geq \max(9, 2[m/2])$.

Proof. In view of [25, Theorem 1.9], we need only verify the second assertion. But then a strongly semiprime, alternative PI-algebra is a subdirect product of an associative PI-algebra (which satisfies S_t for all $t \geq 2[m/2]$) and a purely alternative algebra which, by Theorem 2.9(1), satisfies S_t for all $t > 8$. Q.E.D.

3. Alternative algebras with involution

Since the theory of alternative PI-algebras follows closely the theory of associative PI-algebras, we should expect the theory of alternative PI-algebras with involution to match the theory of associative PI-algebras with involution. Indeed, the theories are analogous, seen through the following result:

THEOREM 3.1. *If $(R, *)$ is alternative and satisfies a strong identity f of degree d, then R is a PI-algebra, satisfying the identity $g_{d+1}(X_1, X_2)^t$ for suitable t.*

Proof. We rely heavily on a theorem of Herstein [8]–Martindale [18]–Amitsur [4]: If $(R, *)$ is associative and satisfies a strong identity of degree d, then R satisfies some power of the standard polynomial S_{2d}; in particular, if R is semiprime then R satisfies S_{2d}, so g_{d+1} is an identity of R. Now, to prove Theorem 3.1, first assume R is strongly semiprime. By [25, Remark 6.11], $(R, *)$ is the subdirect product of an associative semiprime PI-algebra with involution $(R_1, *)$ and a strongly semiprime, purely alternative algebra with involution $(R_2, *)$; f is a strong identity of each. By Amitsur's theorem,

g_{d+1} is an identity of R_1. Moreover, by Theorem 27, g_m, $m>2$, are identities of R_2. If $d=1$ then $R_2=0$, so it follows in all cases that g_{d+1} is an identity of R.

Let $\mathscr{C}$ be the class of alternative *PI*-algebras with involution for which f is a strong identity. $\mathscr{C}$ is closed under direct powers, so we conclude the proof of Theorem 3.1 by using [25, Theorem 1.9]. Q.E.D.

Thus far, all of our results on alternative algebras would apply without restriction on Ω, as will be seen shortly. If we want special results which rely on ϕ being associative and commutative, we can use the flavor of Amitsur [5] as follows: Given a generalized monomial h of an element f of $\phi\{X\}$, note that the canonical image $\bar{h}$ of h in the free, associative ϕ-algebra has the form $\alpha\bar{X}_{i_1}\cdots\bar{X}_{i_m}$ for a suitable α in ϕ. Call α the *coefficient* of h, and say f is *R-correct* if, given r in R, one can find a generalized monomial with some coefficient α such that $\alpha r\neq 0$. Note that every R-strong identity of a ϕ-algebra with involution $(R, *)$ is R-correct. Amitsur's approach to [5], coupled with his results in [4], then yields:

THEOREM 3.2. *If $(R, *)$ is an alternative algebra satisfying an R-correct identity f of degree d, then R is a PI-algebra satisfying the identity $g_{d+1}(X_1, X_2)^t$ for suitable t.*

Proof. The details in the associative case are given in [28]. The extension to the alternative case is the same as in Theorem 3.1.

For completeness, we show that the *PI*-algie theory is also the same as the *PI*-algebra theory for {alternative rings}. Q.E.D.

THEOREM 3.3. *Any alternative PI-algie R satisfies the identity $g_d(X_1, X_2)^t$ for suitable d and t. If $(R, *)$ is an alternative PI-algie with involution, then R is a PI-algie.*

Proof. Using [25, Theorem 1.9], we may assume $\mathrm{Nil}(R)=0$. But every strongly semiprime, purely alternative ring satisfies the central polynomial g_2 and the identities g_d, $d>2$, so we may apply [25, Remark 6.11] to conclude the proof (in the same way as in Theorems 2.9 and 3.1, respectively). Q.E.D.

4. Jordan algebras

For simplicity, assume $\frac{1}{2}\in\phi$, and define the *variety of Jordan algebras* as

$$\mathscr{V}\{[X_1, X_2], (X_1^2X_2)X_1 - X_1^2(X_2X_1)\}.$$

Since $\frac{1}{2}\in\phi$ the latter identity is easily seen to be "stable", in the sense of [25, §1], so, in particular, every central extension of a Jordan algebra is Jordan. Let $\mathscr{J}$ denote an arbitrary Jordan ϕ-algebra. In view of commutativity, we have the equality of $BM(\mathscr{J})$ and $\mathrm{Jac}(\mathscr{J})$. McCrimmon [16] has

characterized $J(\mathcal{J})$, but I believe it is an open question whether $J(\mathcal{J})$ equals $\mathrm{Jac}(\mathcal{J})$ in general. Write FJ^t for the free (i.e. universal) Jordan algebra on $X_1, \ldots, X_t$.

Let A be a (not necessarily associative) ϕ-algebra, and define $a \cdot b = \frac{1}{2}(ab + ba)$, for elements a, b of A. Define A^+ to be the algebra obtained by substituting the operation $(\ \cdot\ )$ for multiplication in A. Of course, if A is already commutative then A^+ is the same as A, a fact which will be useful in evaluating polynomials.

$\mathcal{J}$ is a *special* Jordan algebra if, for some associative algebra A, $\mathcal{J}$ is a Jordan subalgebra of A^+. Note, by [12, Theorem 3, p. 15], if A is alternative then A^+ is special. Let $\phi^{(t)}(\bar{X})$ denote the universal associative algebra. The *free special Jordan algebra* with t *free generators* $FSJ^{(t)}$ is defined as the subalgebra of $(\phi^{(t)}\{\bar{X}\})^+$ generated by $\bar{X}_1, \ldots, \bar{X}_t$. There is a canonical map $\psi_t: FJ^{(t)} \to FSJ^{(t)}$, sending respective generator to respective generator.

THEOREM I (Macdonald's Theorem [12, p. 40]). *Suppose* Y_1, Y_2, Y_3 *generate* $FJ^{(3)}$, *and* $f(Y_1, Y_2, Y_3) \in \ker \psi_3, f$ *linear in the third indeterminate. Then* $f(Y_1, Y_2, Y_3) = 0$.

An immediate consequence of Theorem I is the following: Suppose $\mathcal{J}$ is a Jordan algebra in a variety of Jordan algebras and $f(X_1, X_2, X_3)$ is a polynomial, linear in X_3, which is an identity of every special Jordan algebra in this variety. Then f is an identity of $\mathcal{J}$. In particular, $FJ^{(2)} = FSJ^{(2)}$.

THEOREM J (Cohn [12, p. 9]). *If* $t \leq 3$, *then* $FSJ^{(t)}$ *is the subspace of symmetric elements of* $\phi^{(t)}(\bar{X})$, *under the involution given by* $\bar{X}_i^* = \bar{X}_i$ (*i.e.* $(\bar{X}_{k_1} \cdots \bar{X}_{k_j})^* = \bar{X}_{k_j} \cdots \bar{X}_{k_1}$).

It has been observed by Amitsur that the Formanek polynomial g_n is symmetric under the involution given in Theorem J, so we can find a polynomial $\tilde{g}_n$ in $FSJ^{(2)} = FJ^{(2)}$, obtained from g_n by using $(\ \cdot\ )$ instead of the old multiplication.

Using the definition of absolutely primitive idempotents from [12, p. 197], say $\mathcal{J}$ is *reduced* if 1 is a finite sum of absolutely primitive idempotents. Reduced simple Jordan algebras are classified in [12, p. 203]; they are important because all simple finite-dimensional Jordan algebras over algebraically closed fields are reduced. I have been unable to obtain sweeping results for Jordan *PI*-algebras in general; instead, a structure theory will be developed here which includes all reduced simple Jordan algebras. (Another direction, taken in a work to appear in J. Algebra, is the study of Jordan algebras satisfying normal identities.)

Our strategy here is to use results of Racine [22] and Jacobson (concerning central polynomials of simple, reduced Jordan algebras) to find Kaplansky classes of Jordan *PI*-algebras. First, if $\mathcal{J}$ is commutative and associative, then obviously we can apply the associative *PI*-theory.

Next, suppose $\mathcal{J}$ is "generically algebraic of degree 2", defined in terms of a quadratic form as follows: Given a subring C of $Z(\mathcal{J})$ and a quadratic form $Q: \mathcal{J} \to C$ with $Q(1) = 1$ and bilinearization

$$Q(a, b) = Q(a+b) - Q(a) - Q(b),$$

multiplication in $\mathcal{J}$ is given by $ab = \frac{1}{2}(T(a)b + T(b)a - Q(a, b))$, where $T(a) = Q(a, 1)$. Let

$$\text{Rad } Q = \{a \in R \mid Q(a, r) = 0 \text{ for all } r \text{ in } R\};$$

clearly $(\text{Rad } Q)^2 = 0$. $\mathcal{J}/\text{Rad } Q$ has a quadratic form induced by Q. We shall write $(\mathcal{J}, Q, C)$ to denote the Jordan algebra $\mathcal{J}$ defined by the quadratic form $Q: \mathcal{J} \to C$.

Note that, without loss of generality, we could take $C = Z(\mathcal{J})$, but I prefer to isolate C. It has been customary to work in the category of algebras generically algebraic over a given field, but here we want to be able to change C in order to pass to central extensions. We now need the following computation, due to Jacobson.

THEOREM K (Racine [22, Theorem 1]). *If $\mathcal{J}$ is generically algebraic (of degree 2), then $\tilde{g}_2$ is either $\mathcal{J}$-central or an identity of $\mathcal{J}$. Moreover, if $Z(\mathcal{J})$ is a field and $\tilde{g}_2$ is an identity of $\mathcal{J}$, then $\mathcal{J}$ is commutative and associative.*

LEMMA 4.1. *If $(\mathcal{J}, Q, C)$ is a generically algebraic Jordan algebra, and H is a commutative, associative C-algebra, then $(J \otimes_C H, Q, H)$ is a generically algebraic Jordan algebra, via the map*

$$Q(\sum a_{i1} \otimes h_{i1}, \sum a_{j2} \otimes h_{j2}) = \sum Q(a_{i1}, a_{j2}) \otimes h_{i1} h_{j2}.$$

Proof. Standard application of the categorical properties of tensor product. Q.E.D.

Note that every associative Jordan algebra $\mathcal{J}$ is commutative and associative, and thus has the "trivial" quadratic form $Q(x) = x^2$ for all x in $\mathcal{J}$; also g_2 is an identity of $\mathcal{J}$.

THEOREM 4.2. *Let $\mathscr{C}$ be the class of generically algebraic Jordan algebras of degree 2. Suppose $(\mathcal{J}, Q, C) \in \mathscr{C}$ and P is a prime ideal of $\mathcal{J}$, such that $\mathcal{J}/P$ is not associative. Then $Q(\mathcal{J}, P) \subseteq P$, $Q(P) \subseteq P$, and Q induces a quadratic form on $\mathcal{J}/P$.*

Proof. Let $\bar{\cdot}$ denote the canonical image in $\bar{\mathcal{J}} = \mathcal{J}/P$. By hypothesis, we have elements a_1, a_2, a_3 in $\mathcal{J}$, such that $[\bar{a}_1, \bar{a}_2, \bar{a}_3] \neq 0$. Suppose $b \in P$. Then

$$T(b)a_1 - Q(a_1, b) = 2a_1 b - T(a_1)b \in P,$$

so $\overline{T(b)}\bar{a}_1 \in \overline{Z(\mathcal{J})}$, implying $\overline{T(b)}[\bar{a}_1, \bar{a}_2, \bar{a}_3] = 0$. Since $\bar{\mathcal{J}}$ is prime (and thus torsion-free over its center), we have $\overline{T(b)} = 0$, i.e. $T(b) \in P$. Thus, for any b

in P and x in $\mathcal{J}$, we have

$$Q(b, x) = -2bx + T(b)x + T(x)b \in P,$$

implying $Q(P, \mathcal{J}) \subseteq P$. In particular, $Q(P) \subseteq P$. Now define $\bar{Q}$ on $\bar{\mathcal{J}}$ by $Q(\bar{x}) = \overline{Q(x)}$. For any element b of P,

$$\overline{Q(x+b)} = \overline{Q(x)} + \overline{Q(b)} + \overline{Q(x, b)} = \overline{Q(x)},$$

so $\bar{Q}$ is well defined, and is obviously a quadratic form. Q.E.D.

PROPOSITION 4.3. *If $\mathcal{J}$ is a prime, generically algebraic Jordan algebra of degree 2, then $\mathcal{J}$ is absolutely prime.*

Proof. Let $\mathcal{J}'$ be the algebra of central quotients of $\mathcal{J}$. Clearly $Z(\mathcal{J}')$ is a field, and the quadratic form Q of $\mathcal{J}$ extends canonically to $\mathcal{J}'$, by Lemma 4.1. Moreover, $(\operatorname{Rad} Q)^2 = 0$, so $\operatorname{Rad} Q = 0$ (since $\mathcal{J}'$ is prime), and it is well known and straightforward to conclude that $\mathcal{J}'$ is simple. Q.E.D.

THEOREM 4.4. *Let $\mathscr{C} = \{$Homomorphic images of generically algebraic Jordan algebras of degree 2$\}$.*

(1) *Every prime member of $\mathscr{C}$ is generically algebraic of degree 2.*

(2) *If $\mathcal{J} \in \mathscr{C}$ and is prime, then either $\mathcal{J}$ is associative or $\tilde{g}_2$ is $\mathcal{J}$-central.*

(3) *$\mathscr{C}$ is Kaplansky and satisfies property F with respect to $\{X_1, \tilde{g}_2\}$.*

Proof. (1) Immediate from Theorem 4.2.

(2) Suppose $\mathcal{J}$ is nonassociative. By (1), $\mathcal{J}$ has a quadratic form. Let $\mathcal{J}'$ be the algebra of central quotients of $\mathcal{J}$. $Z(\mathcal{J}')$ is a field; by Lemma 4.1, $\mathcal{J}'$ is generically algebraic of degree 2, so, by Theorem K, $\tilde{g}_2$ is $\mathcal{J}'$-central. But $\mathcal{J}$ is equivalent to $\mathcal{J}'$, so $\tilde{g}_2$ is $\mathcal{J}$-central.

(3) First we verify that $\mathscr{C}$ is a central class, checking the conditions of [25, Definition 3.16]. We have (i) by assertions (1), (2) above; (ii) is immediate; (iii) follows easily from Lemma 4.1. Next, we show $\mathscr{C}$ is Kaplansky by verifying that every strongly prime algebra $\mathcal{J}$ of $\mathscr{C}$ is absolutely prime (cf. [25, Theorem 3.17(3) and Definition 3.18]); this is immediate by part (1) above and Proposition 4.3. Finally, it is clear from (2) that $\mathscr{C}$ satisfies property F. Q.E.D.

At this point one can already apply Theorems 0.1 through 0.13, but we want to push these results even further. First we remark without proof that Levitzki's theorem, that the lower radical of an associative ring is the intersection of the prime ideals, goes over to nonassociative rings, so Proposition 4.3 and Theorem 4.4(1) easily yield:

PROPOSITION 4.5. *Every semiprime member of $\mathscr{C}$ (defined in Theorem 4.4), is strongly semiprime.*

PROPOSITION 4.6. *If $\mathcal{J} \in \mathscr{C}$ and $\mathcal{J}$ is semiprime, then $\mathcal{J}$ is the subdirect product of a semiprime associative, commutative algebra and a purely nonassociative, semiprime, generically algebraic algebra of degree 2.*

Proof. Using the decomposition of [25, Proposition 5.4], we need only show that every purely nonassociative, semiprime member of $\mathfrak{C}$ is generically algebraic of degree 2. So suppose $R \in \mathfrak{C}$ is purely nonassociative and semiprime, and $R = \mathcal{J}/B$ where $\mathcal{J}$ is generically algebraic of degree 2, with the quadratic form Q. By Proposition 4.5, R is strongly semiprime, so by [25, Remark 5.5], $B = \{\text{strongly prime ideals } P \text{ of } \mathcal{J} \mid B \subseteq P \text{ and } \mathcal{J}/P \text{ is nonassociative}\}$. But for each such P, we have $Q(\mathcal{J}, B) \subseteq Q(\mathcal{J}, P) \subseteq P$, by Theorem 4.2, so we conclude that $Q(\mathcal{J}, B) \subseteq B$, implying Q induces a quadratic form $\bar{Q}$ in R (by $\bar{Q}(x+B) = Q(x)+B$). Q.E.D.

The rest of our study of Jordan algebras is motivated as follows: Suppose we are given a simple Jordan algebra $\mathcal{J}$ whose center Z (a field) has algebraic closure Z'. Call $\mathcal{J}$ *quasi-reduced* if $\mathcal{J} \otimes_Z Z'$ is reduced. Note, by [12, Theorem 4, p. 197], that any simple finite-dimensional Jordan algebra is quasi-reduced and thus satisfies all the identities of the appropriate reduced algebra. Our objective will be to find varietally-defined Kaplansky classes of Jordan algebras whose simple members are quasi-reduced.

First, we need a definition of Jordan *PI*-algebra which requires identities other than the usual Jordan identities. To achieve this end, define inductively a ϕ-linear map $\psi: \phi\{X\} \to \phi\{X\}$ by $\psi(1) = 1$, $\psi(X_i) = X_i$, and for monomials h_1 and h_2,

$$\psi(h_1 h_2) = \tfrac{1}{2}(\psi(h_1)\psi(h_2) + \psi(h_2)\psi(h_1)).$$

An element f of $\phi\{X\}$ will be called *Jordan-correct* if, given x in $\mathcal{J}$, we can find a generalized monomial h of ψf, with coefficient α such that $\alpha x \neq 0$; $\mathcal{J}$ is *PI-Jordan* if $\mathcal{J}$ has a Jordan-correct identity. It is easy to see that every multilinearization of a Jordan-correct identity is Jordan-correct. Hence, every *PI*-Jordan algebra has a multilinear Jordan-correct identity. (Note that all generically algebraic Jordan algebras of degree 2 are *PI*-Jordan, by Theorem 4.4(2) and [25, Theorem 1.9].)

We shall now look at more Kaplansky classes of *PI*-Jordan algebras. Let $(R, *)$ be an algebra with involution and let $\mathcal{S}(R, *)$ be the algebra of symmetric elements of R, under the new multiplication

$$s_1 \cdot s_2 = \tfrac{1}{2}(s_1 s_2 + s_2 s_1)$$

(cf. [1, Theorem 6.11]). Suppose that $\mathcal{S}(R, *)$ is a Jordan algebra $\mathcal{J}$. If $\mathcal{J}$ satisfies a Jordan-correct identity $f(X_1, \ldots, X_m)$, then

$$\psi f(X_1 + X_1^*, \ldots, X_m + X_m^*)$$

is an identity of $(R, *)$. Moreover, given r in R, one can find a generalized monomial of ψf with coefficient α such that $\alpha(r + r^*) = \alpha r + (\alpha r)^* \neq 0$, so $\alpha r \neq 0$. This shows that ψf is R-correct. Hence, if $\mathcal{J}$ is a *PI*-Jordan algebra then $(R, *)$ is a *PI*-algebra with involution.

The most interesting case of the above situation occurs when R has the

form $M_n(A)$, the algebra of $n \times n$ matrices with entries in an algebra A; i.e. we let $\{e_{ij} \mid 1 \leq i, j \leq n\}$ be a set of (associative) matrix units, and define $(\sum a_{ij}e_{ij})(\sum a_{ku}e_{ku}) = \sum \delta_{jk}a_{ij}a_{ku}e_{iu}$.

Every multilinear generalized monomial which is an identity of A is an identity of $M_n(A)$, which is "why" $M_n(A)$ is associative if A is associative. If A has an involution $*: a \to \bar{a}$, then $M_n(A)$ has an involution $(*)$ given by $(\sum a_{ij}e_{ij})^* = \sum \bar{a}_{ji}e_{ij}$. It is easy to see that any ideal of $M_n(A)$ has the form $M_n(B)$, B an ideal of A; hence any ideal of $(M_n(A), *)$ has the form $(M_n(B), *)$, $(B, *)$ an ideal of $(A, *)$.

It has been shown that, given n, there is a variety $\mathcal{V}_n$ of algebras with involution such that $\mathcal{S}(M_n(A), *)$ is a Jordan algebra if and only if $(A, *) \in \mathcal{V}_n$ (cf. [12, Theorem 1, p. 127; ex. 4, p. 132]). For example, $\mathcal{V}_n = \{\text{associative algebras}\}$ for $n \geq 4$; $\mathcal{V}_3 = \{\text{alternative algebras with involution satisfying } [X_1 + X_1^*, X_2, X_3]\}$, i.e. those alternative algebras with involution whose symmetric elements are nuclear. The Jordan algebras $\mathcal{S}(M_n(A), *)$ have most interest when A is alternative (in view of [12, Theorem 8, p. 203]), so let us examine $\mathcal{V}_3$ a little more closely. If $(A, *)$ is simple and if $(A, *) \in \mathcal{V}_3$, then either (i) A is associative, (ii) A is Cayley-Dickson, or (iii) A is the direct sum of a CD-algebra and its opposite, with $(*)$ the exchange involution. However, since the symmetric elements are in $N(A)$, we may rule out (iii). Moreover, in (ii), $N(A) = Z(A)$, a field, so all symmetric elements are invertible; hence, by [12, Theorem 8, p. 170] $(*)$ may be assumed to be the *standard involution*, defined as $(q_1, q_2)^* = (\bar{q}_1, -q_2)$ in the notation of Theorem H. (Note that the involution $q \mapsto \bar{q}$ in Q must be symplectic in order that symmetric elements be nuclear.)

PROPOSITION 4.7. *Suppose* $(A, *)$ *is simple. Then* $\mathcal{S}(M_n(A), *)$ *is simple under any one of the three following additional assumptions*: (*i*) $n \geq 3$; (*ii*) $\mathcal{S}(A, *)$ *is simple*; (*iii*) $(A, *) \in \mathcal{V}_3$.

Proof. The first assertion is standard and easy, having the proof of [12, p. 129], and the second assertion is even easier, proved using the same ideas. For the final assertion, we have observed above that A is either associative, or Cayley-Dickson with $(*)$ the standard involution. In the latter case $\mathcal{S}(A, *)$ is a field, so $\mathcal{S}(M_n(A), *)$ is already seen to be simple. On the other hand, if A is associative then $M_n(A)$ is associative so $\mathcal{S}(M_n(A), *)$ is simple by [8], Theorem 2.6 or Theorem 1.1]. Q.E.D.

Say a Jordan algebra $\mathcal{J}$ has *type* n if $\mathcal{J} = \mathcal{S}(M_n(A), *)$ for some $(A, *)$ in $\mathcal{V}_3$. In other words, either $\mathcal{J} = \mathcal{S}(R, *)$ for some associative R (which, as noted above, must be the case if $n \geq 4$) or $\mathcal{J} = S(M_n(A), *)$ where $n \leq 3$ and A is alternative with the symmetric elements of A in the nucleus. Hence, in the definition of "type n" we may always assume $n \leq 3$.

LEMMA 4.8. *If* A *is an alternative PI-algebra then* $\mathrm{Nil}(M_n(A)) = M_n(\mathrm{Nil}(A))$.

Proof. Clearly $\mathrm{Nil}(M_n(A)) = M_n(B)$ for some nil ideal B of A, so we must merely prove $\mathrm{Nil}\, A \subseteq B$, or, equivalently, that $\mathrm{Nil}\,(A/B) = 0$. Let $B_1 = \mathrm{Nil}(A/B)$. Then, by Shirshov's theorem (cf. [17]), B_1 is locally nilpotent. But then $M_n(B_1)$ is nil, so

$$M_n(B_1) \subseteq \mathrm{Nil}(M_n(A/B)) = \mathrm{Nil}(M_n(A)/M_n(B)) = 0,$$

implying $B_1 = 0$, as desired. Q.E.D.

In particular, A is strongly semiprime iff $M_n(A)$ is strongly semiprime, a useful fact in the subsequent results.

THEOREM 4.9. *Let $\mathscr{C}_n$ be the class of PI-Jordan algebras of type n, for n fixed, under the categorical definition of ideal (i.e. ideals are kernels of surjections $\mathscr{J}_1 \to \mathscr{J}_2$, where $\mathscr{J}_1, \mathscr{J}_2 \in \mathscr{C}_n$). The following three conditions are equivalent:*

(1) *$\mathscr{J}$ is simple (resp. strongly prime, strongly semiprime) in $\mathscr{C}_n$.*

(2) *$\mathscr{J} = \mathscr{S}(M_n(A_1), *)$ for suitable simple (resp. strongly prime, strongly semiprime) $(A_1, *)$ in $\mathscr{V}_n$.*

(3) *$\mathscr{J}$ is simple (resp. absolutely prime, strongly semiprime) as a ϕ-algebra (i.e. with respect to the usual definition of ideal).*

Proof. Since there are at least as many algebra-ideals as ideals with respect to the class $\mathscr{C}_n$, surely (3) implies (1), so we prove (1)$\Rightarrow$(2)$\Rightarrow$(3). Write $\mathscr{J} = \mathscr{S}(M_n(A), *)$.

(1)$\Rightarrow$(2). Choose an ideal $(P, *)$ of $(A, *)$, maximal with respect to $\mathscr{J} \cap M_n(P) = 0$, and let $(A_1, *) = (A/P, *)$. $\mathscr{J} \approx \mathscr{S}(M_n(A_1), *)$, and clearly, in view of Lemma 4.8, $(A_1, *)$ is simple (resp. strongly prime, strongly semiprime) if $\mathscr{J}$ is.

(2)$\Rightarrow$(3). If $(A_1, *)$ is simple then $\mathscr{J}$ is a simple algebra, by Proposition 4.7. If $(A_1, *)$ is strongly prime then, by passing to the algebra of central quotients, we see that $\mathscr{J}$ is absolutely prime. Finally, if $\mathrm{Nil}(A_1) = 0$, write $(A_1, *)$ as a subdirect product of strongly prime $(A_\gamma, *)$; then $\mathscr{J}$ is a subdirect product of the absolutely prime $\mathscr{S}(M_n(A_\gamma), *)$, so clearly $\mathscr{J}$ is strongly semiprime. Q.E.D.

Theorem 4.9 shows in particular that our category $\mathscr{C}_n$ is quite natural, and we examine the property (2) more carefully.

PROPOSITION 4.10. *If $\mathscr{J} = \mathscr{S}(M_n(A), *)$ is PI-Jordan and $\mathrm{Nil}(A, *) = 0$, then $Z(M_n(A), *) = Z(\mathscr{J})$.*

Proof. First we note that obviously $Z(M_n(A), *) \subseteq Z(\mathscr{J})$. To prove the opposite direction, first we recall that $(A, *)$, and thus A, is *PI*. If $(A, *)$ is simple then we know (by Theorem 4.9) that $\mathscr{J}$ is simple; moreover, since $(A, *)$ is finite dimensional over the field $Z(A, *)$, $\mathscr{J}$ is also finite dimensional, and we may conclude that $Z(\mathscr{J}) = Z(M_n(A), *)$ by inspecting the classification of finite-dimensional Jordan algebras over a field. If $(A, *)$ is strongly prime then, $(A, *)$ is absolutely prime, so, letting $S = Z(A, *) - \{0\}$,

we have $(A_S, *)$ simple, so $\mathcal{J}_S = \mathcal{S}(M_n(A_S), *)$ (by [25, Theorem 6.11(ii)]) and thus $Z(\mathcal{J}_S) = Z(M_n(A_S), *)$. Thus

$$Z(\mathcal{J}) = (M_n(A), *) \cap Z(M_n(A_S), *) \subseteq Z(M_n(A), *).$$

Finally, in the general situation that $\mathrm{Nil}(A) = 0$, we write $(A, *)$ as a subdirect product of strongly prime $(A_\gamma, *)$, and let $\mathcal{J}_\gamma = \mathcal{S}(M_n(A_\gamma), *)$. By [25, Theorem 6.11(iii)]), $\mathcal{J}_\gamma$ is a homomorphic image of $\mathcal{J}$, and $\mathcal{J}$ is obviously a subdirect product of the $\mathcal{J}_\gamma$. But we saw above that

$$Z(\mathcal{J}_\gamma) \subseteq Z(M_n(A_\gamma), *),$$

for each γ, so we conclude $Z(\mathcal{J}) \subseteq Z(M_n(A), *)$. Q.E.D.

THEOREM 4.11. *For any n, {PI-Jordan algebras of type n} is Kaplansky, having property* F *with respect to* $\{\tilde{g}_t(X_1, X_2) \mid t \geq 1\}$. *In fact, if* $\mathrm{Nil}(\mathcal{J}) = 0$, *and if* $\mathcal{J}$ *has type n and satisfies a Jordan-correct identity of degree d, then there exists* $t \leq d$ *such that* $\tilde{g}_t$ *is* $\mathcal{J}$*-central and* $\tilde{g}_k$ *is an identity of* $\mathcal{J}$, *for all* $k > t$.

Proof. Let $\mathcal{C}_n = \{PI\text{-Jordan algebras of type } n\}$. First we verify that $\mathcal{C}_n$ is central, as per [25, Definition 3.16]. Condition (iii) is immediate from [25, Theorem 6.11(ii)]. To check that condition (i) holds, we shall now verify the stronger assertion given above, that if $\mathcal{J} \in \mathcal{C}_n$, $\mathrm{Nil}(\mathcal{J}) = 0$ and $\mathcal{J}$ satisfies a Jordan-correct identity f of degree d, then there exists $t \leq d$ such that $\tilde{g}_t$ is $\mathcal{J}$-central and $\tilde{g}_k$ is an identity of $\mathcal{J}$, for all $k > t$; in view of the multilinearization technique, we assume that f is multilinear. We then write f as $f(X_1, \ldots, X_d)$. Writing $\mathcal{J} = \mathcal{S}(M_n(A), *)$, where $\mathrm{Nil}(A) = 0$ (cf. Theorem 4.9(2)), we see that f is an identity of $\mathcal{J}[\lambda] = \mathcal{S}(M_n(A[\lambda]), *)$ and since $(A[\lambda], *)$ is semisimple, we can use the decomposition into simple alternative algebras with involution, coupled with Theorem 6.11(iii), to reduce the assertion to the case when $(A, *)$ is simple. Let us assume that the coefficient of $X_1 X_2 \cdots X_d$ in f is $\alpha \neq 0$, and write $\{e_{ij} \mid 1 \leq i, j \leq n\}$, a set of matric units for $M_n(A)$. If $n > d$, substituting $e_{i,i+1}$ for X_i, $1 \leq i \leq d$, would yield

$$0 = e_{11} f(e_{12} + e_{21}, e_{23} + e_{32}, \ldots, e_{d,d+1} + e_{d+1,d}) = \alpha e_{1,d+1},$$

a contradiction, so we have $n \leq d$.

At this point, we shall try to maneuver into a position to apply the results of Racine [22]. First assume A is nonassociative. In the discussion before Proposition 4.7, we saw that A is a CD-algebra and $(*)$ is the "standard" involution; moreover, by [12, Theorem 1, p. 127], $n \leq 3$. If $n = 1$ then $\mathcal{J} = Z(A)^+$ and the assertion is immediate; for $n = 3$, our assertion is essentially [22, Theorem 3]. For $n = 2$, we need to prove that $g_2 = [X_1 + X_1^*, X_2 + X_2^*]$ is $(M_2(A), *)$-central, and all $g_k(X_1 + X_1^*, X_2 + X_2^*)$ are identities, for $k > 2$. By a well-known Zariski topology density argument (similar to the one used in the proof of [22, Theorem 3]), if we want to prove, for all symmetric x_1, x_2 in $M_2(A)$, that $g_2(x_1, x_2)$ is scalar and $g_k(x_1, x_2) = 0$ for $k > 2$, we may assume that x_1 is diagonal; in other words,

$$x_1 = \alpha e_{11} + \beta e_{22} \quad \text{and} \quad x_2 = \alpha_1 e_{11} + \alpha_2 e_{22} + a e_{12} + a^* e_{21},$$

for suitable α, β, α_1, α_2 in $Z(A)$ and a in A. Let $H=(Z(A))[a]$, a commutative, associative ring. Since $a^*=\mathrm{tr}(a)-a\in H$, we have x_1, $x_2\in M_2(H)$, so our assertion is a consequence of Formanek's theorem (for associative algebras).

Thus we are left with the case that $(A, *)$ is simple associative. Let $C=Z(A, *)$, a field with algebraic closure C'. Then

$$\mathcal{J}\otimes_C C'\approx\mathcal{S}(M_n(A)\otimes_C C', *)\approx\mathcal{S}(M_t(A'), *),$$

for suitable t, with either $A'=C'$ or $(A', *)=(C'\otimes C', \circ)$. Now our assertion reduces essentially to [22, Theorem 2].

So we have finally verified (i) of [25, Definition 3.16]. Next, we check (ii). If $\mathcal{J}=\mathcal{S}(M_n(A), *)\in\mathscr{C}_n$, then let $A'=A/\mathrm{Nil}(A')$, and consider $\mathcal{J}'=\mathcal{S}(M_n(A'), *)$. We know, from Theorem 4.9, together with [25, Theorem 6.11(iii)], that $\mathcal{J}'$ is the subdirect product of strongly prime images $\mathcal{S}(M_n(A'), *)$ of $\mathcal{J}'$; moreover, this implies $\mathrm{Nil}(\mathcal{J}')=0$. In addition, by Lemma 4.8, the canonical map $A\to A'$ induces a homomorphism $\mathcal{J}\to\mathcal{J}'$, whose kernel is nil, so we conclude that $\mathcal{J}/\mathrm{Nil}(\mathcal{J})\approx\mathcal{J}'\in\mathscr{C}_n$, and is a subdirect product of strongly prime members of $\mathscr{C}_n$. This verifies condition (ii), and we have proved that $\mathscr{C}_n$ is central *and* satisfies property F.

Finally, Theorem 4.9 shows that any strongly prime member of $\mathscr{C}_n$ is absolutely prime, so $\mathscr{C}_n$ is Kaplansky. Q.E.D.

Remark 4.12. The classes $\mathscr{C}_n$ are "sufficient", in the sense of [25, Definition 4.8], as is verified trivially.

Example 4.13. For any alternative algebra A, we have $A^+\approx\mathcal{S}(A\oplus A^{\mathrm{op}}, \circ)$ by the map $x\to(x, x)$, so A^+ has type 1. Consequently, all finite-dimensional central simple Jordan algebras either are generically algebraic of degree 2 or have some type n (cf. [29, p. 101]).

Example 4.14. For $m\leq 3$, the free special Jordan algebra $FSJ(m)$ has type 1, since $FSJ(m)=\mathcal{S}(\phi^{(m)}\{\bar{X}\}, r)$, where r denotes the reversal involution. (This is [12, Corollary, p. 9].) Since $FSJ(2)=FJ(2)$, we see that $FJ(2)$ has type 1.

LEMMA 4.15. *Any special Jordan algebra generated by* 3 *or fewer elements has type* 1. *In particular, any subalgebra which is generated by* 2 *elements, of a Jordan algebra, has type* 1.

Proof. Follows easily from [12, Lemma 1, p. 10].

This gives us some canonical identities:

THEOREM 4.16. *If $\mathcal{J}$ satisfies a Jordan-correct identity f of degree d then, for each $k>d$, there exists u such that $\tilde{g}_k(X_1, X_2)^u$ is an identity of $\mathcal{J}$.*

Proof. For a, b in $\mathcal{J}$, let $\mathcal{J}_{a,b}$ be the Jordan ϕ-algebra generated by a and b, and let $\mathcal{J}'$ be the complete direct product of all $\mathcal{J}_{a,b}$ (for all pairs (a, b)). By Lemma 4.14, each $\mathcal{J}_{a,b}$ has type 1, implying $\mathcal{J}'$ has type 1, and f is a

Jordan-correct identity of $\mathcal{J}'$. The theorem then follows easily from Theorem 4.11 and [25, Theorem 1.9]. Q.E.D.

Our definition of type n is an alternative analogue of reflexive so it may be worthwhile to explore the alternative analogue of the universal envelope of a Jordan algebra.

One could try to study the class of algebraic *PI*-Jordan algebras, in the hope of proving that it was Kaplansky. The proof of [12, Theorem 4 p. 197] shows that if $\mathcal{J}$ is algebraic over an algebraically closed field, and if 1 is a finite sum of primitive idempotents, then $\mathcal{J}$ is reduced; this leads to:

Conjecture 4.16. If $\mathcal{J}$ is a simple *PI*-Jordan algebra, algebraic over an algebraically closed field, then 1 is a sum of orthogonal, primitive idempotents.

Given Conjecture 4.16, one would see immediately that {algebraic *PI*-Jordan algebras} is Kaplansky, satisfying property F. Algebraic Jordan algebras without nilpotent elements are studied extensively by Loustau (Comm. Alg., vol. 4 (1976), pp. 1045–1070).

Discussion of Results. It seems quite remarkable that the theory of polynomial identities, which has produced many beautiful theorems for associative rings, can be placed in a setting beyond the reach of many associative ring-theoretic methods. For example, modules (and in particular one-sided ideals) are not significant in this paper, and it is tempting to believe that the smoothest general treatment of *PI*-rings would not involve modules.

One very important class of nonassociative algebras which has been conspicuously absent in this paper is the class of Lie algebras. Since the center of a Lie algebra is a nilpotent ideal, strongly semiprime Lie algebras have trivial centers and therefore have no central polynomials. There are ways to try to overcome this difficulty; one method is to pass to representations in associative algebras and study Lie algebras with associative representations satisfying polynomial identities. In particular, Lie algebras of the form A^-, A associative, can be handled in this manner. Still, this approach involves a blatant mixing of categories.

The techniques of this study obviously work for any variety of algebras for which simple algebras are associative or *CD*. Recently, such a property has been shown to hold for a large number of varieties.

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