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ON THE DERIVATIVE OF A POLYNOMIAL

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1. Introduction and statement of results

It is well known that if $p_n(z)=\sum_{\nu=0}^{n} c_\nu z^\nu$ is a polynomial of degree at most n, then (for references see [16])

$$(1) \qquad \max_{|z|=1} |p_n'(z)| \leq n \max_{|z|=1} |p_n(z)|,$$

where equality holds if and only if $p_n(z)$ is a constant multiple of z^n. If $p_n(z)\neq 0$ in $|z|<1$, then [11], [5], [2]

$$(2) \qquad \max_{|z|=1} |p_n'(z)| \leq \frac{n}{2} \max_{|z|=1} |p_n(z)|.$$

On the other hand, we have [18]

$$(3) \qquad \max_{|z|=1} |p_n'(z)| \geq \frac{n}{2} \max_{|z|=1} |p_n(z)|$$

if $p_n(z)$ is a polynomial of degree n having all its zeros in $|z|\leq 1$. Hence in (2) (as well as in (3)) equality holds for all polynomials $p_n(z)$ of degree n which have all their zeros on $|z|=1$.

Inequality (2) can be replaced [12], [9] by

$$(4) \qquad \max_{|z|=1} |p_n'(z)| \leq \frac{n}{1+K} \max_{|z|=1} |p_n(z)|$$

if $p_n(z)\neq 0$ in $|z|<K$, where $K>1$. Here, we have equality if

$$(5) \quad p_n(z)=c_0\left\{1+\binom{n}{1}\frac{1}{K} ze^{i\alpha}+\cdots+\binom{n}{\nu}\frac{1}{K^\nu}(ze^{i\alpha})^n+\cdots+\frac{1}{K^n}(ze^{i\alpha})^n\right\}.$$

Besides, it can be shown that if a polynomial $p_n(z)$ of degree n having all its zeros in $|z|\geq K>1$ is not of this form, then strict inequality holds in (4). In other words, there is equality in (4) for $p_n(z)=\sum_{\nu=0}^{n} c_\nu z^\nu \neq 0$ in $|z|<K$ $(K>1)$ if and only if $|c_1/c_0|=n/K$.

Now let us consider the following problem. Given that the polynomial

$$f_n(z)=\textstyle\sum_{\nu=1}^{n} a_\nu z^\nu$$

is univalent in $|z|<1$ how large can $(\max_{|z|=1}|f_n'(z)|)/\max_{|z|=1}|f_n(z)|$ be? We may apply (4) to the polynomial $p_{n-1}(z)=f_n(z)/z=:\sum_{\nu=0}^{n-1} c_\nu z^\nu$ which is of

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degree $n-1$ and does not vanish [10] in

$$|z|<2\sin \pi/n. \tag{6}$$

However, this cannot lead to a sharp estimate of $(\max_{|z|=1}|f_n'(z)|)/\max_{|z|=1}|f_n(z)|$ since [3], [4]

$$\left|\frac{c_1}{c_0}\right|=\left|\frac{a_2}{a_1}\right|\le\frac{2\sqrt{2}}{3} \quad \text{if} \quad n=3, \tag{7}$$

whereas, in general [6, p. 319]

$$\left|\frac{c_1}{c_0}\right|=\left|\frac{a_2}{a_1}\right|\le 2\cos\frac{\pi}{n+3}. \tag{7'}$$

We would do better if we knew the improvement that can be obtained in (4) when $|c_1/c_0|$ is given to be $\le cn/K$ where $0\le c\le 1$. Given that $p_n(z)=\sum_{\nu=0}^{n} c_\nu z^\nu\ne 0$ in $|z|<K$ $(K>1)$ it is indeed desirable to know the dependence of

$$\left(\max_{|z|=1}|p_n'(z)|\right)/\max_{|z|=1}|p_n(z)| \tag{8}$$

on the coefficients $c_0, c_1, \ldots, c_m$ $(1\le m\le n)$. It is clear that these coefficients are not quite arbitrary. For example, if

$$p_n(z)=\sum_{\nu=0}^{n} c_\nu z^\nu \ne 0 \quad \text{in} \quad |z|<K,$$

then $|c_1/c_0|\le n/K$,

$$(n-1)\left|\frac{2K^2}{n(n-1)}\frac{c_2}{c_0}-\frac{K^2}{n^2}\left(\frac{c_1}{c_0}\right)^2\right|\le 1-\frac{K^2}{n^2}\left|\frac{c_1}{c_0}\right|^2. \tag{9}$$

The latter relationship is not obvious but can be proved as follows.

The polynomial $p_n(Kz)\ne 0$ in $|z|<1$ and hence by a result of Dieudonné [7],

$$K\frac{p_n'(Kz)}{p_n(Kz)}=\frac{n}{z-\dfrac{1}{\varphi(z)}},$$

where $\varphi(z)$ is analytic and $|\varphi(z)|\le 1$ in $|z|<1$. Thus, if $\varphi(z)=\sum_{\nu=0}^{\infty}\gamma_\nu z^\nu$ then

$$K\frac{\sum_{\nu=1}^{\infty}\nu c_\nu (Kz)^{\nu-1}}{\sum_{\nu=0}^{\infty} c_\nu (Kz)^\nu}=\frac{n}{z-\dfrac{1}{\gamma_0}\dfrac{1}{1+\sum_{\nu=1}^{\infty}\dfrac{\gamma_\nu}{\gamma_0}z^\nu}}$$

or

$$K\frac{c_1}{c_0}+K^2\left\{2\frac{c_2}{c_0}-\left(\frac{c_1}{c_0}\right)^2\right\}z+\cdots=-n\gamma_0\left\{1+\left(\frac{\gamma_1}{\gamma_0}+\gamma_0\right)z+\cdots\right\}.$$

Comparing coefficients on the two sides, we get

$$K\frac{c_1}{c_0}=-n\gamma_0,$$

$$K^2\left\{2\frac{c_2}{c_0}-\left(\frac{c_1}{c_0}\right)^2\right\}=-n(\gamma_1+\gamma_0^2).$$

Since $|\varphi(z)|\leq 1$ in $|z|<1$, we have [13, p. 172, exercise # 9] $|\gamma_0|\leq 1$, $|\gamma_1|+|\gamma_0|^2\leq 1$, and therefore

$$\left|\frac{c_1}{c_0}\right|=\frac{n}{K}|\gamma_0|\leq\frac{n}{K},$$

$$(n-1)\left|\frac{2K^2}{n(n-1)}\frac{c_2}{c_0}-\frac{K^2}{n^2}\left(\frac{c_1}{c_0}\right)^2\right|=\left|\frac{K^2}{-n}\left\{2\frac{c_2}{c_0}-\left(\frac{c_1}{c_0}\right)^2\right\}-\frac{K^2}{n^2}\left(\frac{c_1}{c_0}\right)^2\right|$$

$$=|\gamma_1|\leq 1-|\gamma_0|^2\leq 1-\frac{K^2}{n^2}\left|\frac{c_1}{c_0}\right|^2.$$

Inequalities (10), (11) below give respectively, the dependence of (8) on $|c_1/c_0|$ and on c_0, c_1, c_2.

THEOREM 1. *If $p_n(z)=\sum_{\nu=0}^{n}c_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z|\geq K\geq 1$, then*

$$\max_{|z|=1}|p_n'(z)|\leq n\frac{n\,|c_0|+K^2\,|c_1|}{(1+K^2)n\,|c_0|+2K^2\,|c_1|}\max_{|z|=1}|p_n(z)|\,; \tag{10}$$

furthermore

(11)

$$\max_{|z|=1}|p_n'(z)|\leq\frac{n}{1+K}\frac{(1-|\lambda|)(1+K^2\,|\lambda|)+K(n-1)\,|\mu-\lambda^2|}{(1-|\lambda|)(1-K+K^2+K\,|\lambda|)+K(n-1)\,|\mu-\lambda^2|}\max_{|z|=1}|p_n(z)|,$$

where

$$\lambda=\frac{K}{n}\frac{c_1}{c_0},\qquad \mu=\frac{2K^2}{n(n-1)}\frac{c_2}{c_0}. \tag{12}$$

It follows from (9) that the quantity appearing on the right hand side of (11) is, in general, smaller than the one appearing on the right hand side of (10).

Equality in (10), (11). For even n, equality holds in (10) for

$$p_n(z)=c_0\frac{1}{K^n}(ze^{i\gamma}+Ke^{i\alpha})^{n/2}(ze^{i\gamma}+Ke^{-i\alpha})^{n/2}=c_0\left\{1+\frac{n}{K}(\cos\alpha)ze^{i\gamma}+\cdots\right\},$$

where γ and α are arbitrary real numbers. Whether n is even or odd, equality holds in (11) for

$$(13)\quad p_n(z)=c_0\frac{1}{K^n}(z+K)^{n_1}\left(z^2+2Kz\frac{na-n_1}{n-n_1}+K^2\right)^{(n-n_1)/2}$$
$$=c_0\left[1+a\frac{n}{K}z+\left\{1+(n-2)a^2-\frac{2n_1}{n-n_1}(1-a)^2\right\}\frac{n}{2K^2}z^2+\cdots\right],$$

and in fact for $p_n(ze^{i\gamma})$ for all real γ, if n_1 is an integer such that $n/3\le n_1\le n$, $n-n_1$ is even, and

$$\frac{3n_1-n}{n+n_1}\le a\le 1.$$

The hypotheses $n_1\ge n/3$ and

$$a\ge\frac{3n_1-n}{n+n_1}\ge\frac{3n_1-n}{2n}$$

make sure that $\max_{|z|=1}|p_n(z)|$ and $\max_{|z|=1}|p_n'(z)|$ are both attained at $z=1$. Hence

$$\frac{\max\limits_{|z|=1}|p_n'(z)|}{\max\limits_{|z|=1}|p_n(z)|}=\frac{n_1\left(1+2K\frac{na-n_1}{n-n_1}+K^2\right)+(n-n_1)(1+K)\left(1+K\frac{na-n_1}{n-n_1}\right)}{(1+K)\left(1+2K\frac{na-n_1}{n-n_1}+K^2\right)}.$$

This is easily seen to be equal to

$$\frac{n}{1+K}\,\frac{(1-|\lambda|)(1+K^2|\lambda|)+K(n-1)\,|\mu-\lambda^2|}{(1-|\lambda|)(1-K+K^2+K|\lambda|)+K(n-1)\,|\mu-\lambda^2|}$$

since for our polynomial $\lambda=a$ and

$$\mu-\lambda^2=\frac{1}{n-1}\left\{1-a^2-2(1-a)^2\frac{n_1}{n-n_1}\right\}$$

which is non-negative because we supposed that

$$a\ge\frac{3n_1-n}{n+n_1}\ge\frac{3n_1-n}{2n}.$$

It may be mentioned that not all polynomials of degree n (having all their zeros in $|z|\ge K>1$) for which equality holds in (11) are covered by $p_n(ze^{i\gamma})$, where $p_n(z)$ has the form (13).

Coming back to our problem about univalent polynomials, if $f_3(z)=a_1z+a_2z^2+a_3z^3$ is univalent in $|z|<1$, then we may apply (10) to the quadratic

$$p_2(z)=a_1+a_2z+a_3z^2$$

which does not vanish in $|z|<\sqrt{3}$ and where $|a_2|\le(2\sqrt{2}/3)|a_1|$, to conclude that

$$\max_{|z|=1}|p_2'(z)|\le\frac{1+\sqrt{2}}{2+\sqrt{2}}\max_{|z|=1}|p_2(z)|\,.$$

Since $f_3(z)=zp_2(z)$, we have

$$\max_{|z|=1}|p_2(z)|=\max_{|z|=1}|f_3(z)|\,,$$

$$\max_{|z|=1}|f_3'(z)|\le\max_{|z|=1}(|p_2(z)|+|p_2'(z)|)\le\frac{3+2\sqrt{2}}{2+\sqrt{2}}\max_{|z|=1}|p_2(z)|\,,$$

and the following corollary holds.

COROLLARY 1. *If* $f_3(z)=a_1z+a_2z^2+a_3z^3$ *is univalent in* $|z|<1$, *then*

$$\max_{|z|=1}|f_3'(z)|\le\frac{3+2\sqrt{2}}{2+\sqrt{2}}\max_{|z|=1}|f_3(z)|\,. \tag{14}$$

The estimate is sharp and equality holds for

$$f_3(z)=a_1\left(z+\frac{2\sqrt{2}}{3}z^2+\tfrac{1}{3}z^3\right).$$

If $p_n(z)$ is a polynomial of degree n, then $q_n(z):=z^n\overline{p_n(1/\bar{z})}$ is a polynomial of degree at most n and $|p_n(z)|=|q_n(z)|$ on $|z|=1$. Besides,

$$\max_{|z|=1}|p_n'(z)|\ge n\max_{|z|=1}|p_n(z)|-\max_{|z|=1}|q_n'(z)|\,. \tag{15}$$

If $p_n(z)$ has all its zeros in $|z|\le k\le 1$, then $q_n(z)$ has all its zeros in $|z|\ge 1/k\ge 1$. Hence we may apply Theorem 1 to $q_n(z)$ and deduce from (15) the following

COROLLARY 2. *If* $p_n(z)=\sum_{\nu=0}^{n}c_\nu z^\nu$ *is a polynomial of degree n having all its zeros in* $|z|\le k\le 1$, *then*

$$\max_{|z|=1}|p_n'(z)|\ge n\,\frac{n\,|c_n|+|c_{n-1}|}{(1+k^2)n\,|c_n|2\,|c_{n-1}|}\max_{|z|=1}|p_n(z)|; \tag{16}$$

furthermore

(17)

$$\max_{|z|=1}|p_n'(z)|\ge\frac{n(1-|\omega|)(1+k^2\,|\omega|)+(n-1)k/\Omega-\omega^2|}{1+k(1-|\omega|)(1-k+k^2+k\,|\omega|)+(n-1)k\,|\Omega-\omega^2|}\max_{|z|=1}|p_n(z)|,$$

where

$$\omega=\frac{1}{nk}\frac{c_{n-1}}{c_n},\qquad \Omega=\frac{2}{n(n-1)k^2}\frac{c_{n-2}}{c_n}\,. \tag{18}$$

Like (10), (11), the inequalities (16), (17) cannot, in general, be improved. Corollary 2 improves upon the estimate [12]

$$\max_{|z|=1} |p_n'(z)| \geq \frac{n}{1+k} \max_{|z|=1} |p_n(z)|, \tag{19}$$

which also happens to be sharp.

An application of (16). If we wish to study the location of the zeros of a polynomial $p_n(z) = \sum_{v=0}^{n} c_v z^v$ in terms of its norm (cf. [15])

$$\|p_n\| = \max_{|z|=1} |p_n(z)|$$

we need some further information about the polynomial in order to get non-trivial conclusions. Thus, to mention a simple example, the polynomial $p_n(z)$ has at least one simple zero in the disk

$$|z| \leq \left(\frac{\|p_n\|}{|c_n|} - 1\right)^{1/n},$$

and this result is best possible as long as the coefficient c_n is the only additional information given. In view of the well known Gauss–Lucas theorem we may apply inequality (16) repeatedly to get the following analogous result involving $|c_0|$, $|c_1|$ instead of $|c_n|$.

COROLLARY 3. *The polynomial* $p_n(z) = \sum_{\nu=0}^{n} c_\nu z^\nu$ *has at least one zero in*

$$|z| \leq \max\left\{1, \left(\left(1 + \frac{1}{n}\left|\frac{c_1}{c_0}\right|\right)^{(n-2)/(n-1)} \left(\frac{\|p_n\|}{|c_0|}\right)^{1/(n-1)} - \frac{2}{n}\left|\frac{c_1}{c_0}\right| - 1\right)^{-1/2}\right\}, \tag{20}$$

where $\|p_n\| = \max_{|z|=1} |p_n(z)|$.

As a supplement to (19) it was shown by Govil [8] that if $K \geq 1$, then for a polynomial $p_n(z)$ of degree n having all its zeros in $|z| \leq K$,

$$\max_{|z|=1} |p_n'(z)| \geq \frac{n}{1+K^n} \max_{|z|=1} |p_n(z)|. \tag{21}$$

Here we shall give a simpler proof of the latter inequality. In fact, it is not much harder to prove the following more general result.

THEOREM 2. *Let $f(z)$ be an entire function of order* 1 *type τ having all its zeros in* $\operatorname{Im} z \geq \eta$, *where* $\eta \leq 0$. *If*

$$h_f(\pi/2) := \limsup_{r \to \infty} \frac{\log |f(re^{i\pi/2})|}{r} \leq 0$$

and $\sup_{-\infty<x<\infty} |f(x)| = M < \infty$, *then*

$$\sup_{-\infty<x<\infty} |f'(x)| \geq \frac{\tau}{1+e^{-\tau\eta}} M. \tag{22}$$

The result is sharp and equality in (22) *holds for the function* $e^{i\tau z} - e^{-\tau\eta}$.

Inequality (21) can be obtained by applying Theorem 2 to the function $p_n(e^{iz})$.

Finally, we prove:

THEOREM 3. *If $f(z)$ is an entire function of exponential type τ such that $h_f(\pi/2) \leq \sigma \leq \tau$, then at an arbitrary point x_0 of the real axis which is not a simple zero of $f(z)$, we have*

$$\left| \frac{d}{dx} |f(x)| \right|_{x=x_0} \leq \frac{\tau+\sigma}{2} \sqrt{M^2 - |f(x_0)|^2}, \tag{23}$$

where $M = \sup_{-\infty<x<\infty} |f(x)|$. The example $f(z) = e^{i\tau z} - e^{-i\sigma z}$ shows that the result is best possible.

Remark. Notice that

$$\left| \frac{d}{dx} |f(x)| \right| \leq |f'(x)|.$$

However, the function $f(z) = e^{i\tau z} - \varepsilon e^{-i\sigma z}$ with sufficiently small $\varepsilon > 0$ shows that under the assumptions of Theorem 3 nothing better than Bernstein's inequality $|f'(x)| \leq \tau M$ can be obtained as an upper bound for $|f'(x)|$.

COROLLARY 4. *If $p_n(z)$ is a polynomial of degree at most n, then at an arbitrary point $e^{i\theta_0}$ on the unit circle which is not a simple zero of $p_n(z)$, we have*

$$\left| \frac{d}{d\theta} |p_n(e^{i\theta})| \right|_{\theta=\theta_0} \leq \frac{n}{2} \sqrt{M^2 - |p_n(e^{i\theta_0})|^2} \quad \text{where} \quad M = \max_{|z|=1} |p_n(z)|.$$

2. A lemma

For the proof of inequality (11) we shall need the following:

LEMMA. *If $f(z)$ is analytic and $|f(z)| \leq 1$ in $|z| < 1$, then*

$$|f(z)| \leq \frac{(1-|a|)\,|z|^2 + |bz| + |a|\,(1-|a|)}{|a|\,(1-|a|)\,|z|^2 + |bz| + (1-|a|)} \quad (|z| < 1), \tag{24}$$

where $a = f(0)$, $b = f'(0)$. The example

$$f(z) = \left(a + \frac{b}{1+a} z - z^2\right) \Big/ \left(1 - \frac{b}{1+a} z - az^2\right)$$

shows that the estimate is sharp.

Proof. Let us assume that $|f(0)| < 1$ since otherwise the result is obvious. Choose γ such that $f(0)e^{i\gamma} = |f(0)|$ and consider the function

$$\varphi(z) = \frac{1}{z} \frac{e^{i\gamma} f(z) - |f(0)|}{|f(0)|\, e^{i\gamma} f(z) - 1}$$

which is analytic in $|z|<1$ and satisfies $|\varphi(z)|\le 1$ there. Further

$$\varphi(0)=\frac{f'(0)e^{i\gamma}}{|f(0)|^2-1}.$$

Hence by a well known inequality [13, p. 167], which is proved by applying Schwarz's Lemma to the function

$$\Phi(z):=\frac{\varphi(z)-\varphi(0)}{\overline{\varphi(0)}\varphi(z)-1},$$

we have

$$|\varphi(z)|\le\frac{|z|+\dfrac{|f'(0)|}{1-|f(0)|^2}}{\dfrac{|f'(0)|}{1-|f(0)|^2}|z|+1}=:\Lambda(z)\quad(|z|<1),\tag{25}$$

i.e.

$$\left|\frac{e^{i\gamma}f(z)-|f(0)|}{|f(0)|\,e^{i\gamma}f(z)-1}\right|\le|z|\Lambda(z)\quad(|z|<1).$$

From this an upper bound for $|f(z)|$ can be deduced in precisely the same way as (25) is obtained from the inequality $|\Phi(z)|\le|z|$. Indeed, $e^{i\gamma}f(z)$ lies in the disk D which is the image of $\{w\colon |w|\le|z|\,\Lambda(z)\}$ by the Möbius transformation

$$\zeta=\frac{w-|f(0)|}{|f(0)|w-1},$$

this leads to the desired result.

3. Proofs of the theorems

Proof of Theorem 1. Since $p_n(z)\ne 0$ in $|z|<K$, we have [17, p. 33]

$$np_n(z)+(\zeta-z)p_n'(z)\ne 0\quad\text{for}\quad|\zeta|<K,\ |z|<K,$$

i.e.

$$np_n(z)-zp_n'(z)\ne-\zeta p_n'(z)\quad\text{for}\quad|\zeta|<K,\ |z|<K.$$

Consequently,

$$\left|\frac{p_n'(z)}{np_n(z)-zp_n'(z)}\right|\le\frac{1}{K}\quad\text{for}\quad|z|\le K.$$

Hence if

$$f(z)=\frac{Kp_n'(Kz)}{np_n(Kz)-Kzp_n'(Kz)},$$

then $|f(z)|\le 1$ for $|z|<1$, $f(0)=Kc_1/nc_0$, so that [13, p. 167]

$$|f(z)|\le \frac{|z|+\dfrac{K}{n}\left|\dfrac{c_1}{c_0}\right|}{\dfrac{K}{n}\left|\dfrac{c_1}{c_0}\right||z|+1} \quad (|z|<1).$$

Thus in particular

$$|p_n'(z)|\le \frac{1}{K^2}\frac{1+\dfrac{K^2}{n}\left|\dfrac{c_1}{c_0}\right|}{\dfrac{1}{n}\left|\dfrac{c_1}{c_0}\right|+1}|np_n(z)-zp_n'(z)| \quad (|z|=1).$$

If $q_n(z):=z^n\overline{p_n(1/\bar{z})}$, then, on $|z|=1$, $|np_n(z)-zp_n'(z)|\equiv|q_n'(z)|$, and therefore

$$|p_n'(z)|\le \frac{1}{K^2}\frac{1+\dfrac{K^2}{n}\left|\dfrac{c_1}{c_0}\right|}{\dfrac{1}{n}\left|\dfrac{c_1}{c_0}\right|+1}|q_n'(z)| \quad (|z|=1).$$

Combining this with the inequality (see for example [9, p. 511])

$$\max_{|z|=1}(|p_n'(z)|+|q_n'(z)|)\le n\max_{|z|=1}|p_n(z)| \tag{26}$$

valid for all polynomials of degree at most n, we get (10).

In order to prove (11), we observe that

$$f'(0)=(n-1)\left\{\frac{2K^2c_2}{n(n-1)c_0}-\left(\frac{Kc_1}{nc_0}\right)^2\right\}=(n-1)(\mu-\lambda^2)$$

and then we use the lemma to conclude that

$$|f(z)|\le\frac{(1-|\lambda|)\,|z|^2+(n-1)\,|\mu-\lambda^2|\,|z|+|\lambda|\,(1-|\lambda|)}{|\lambda|\,(1-|\lambda|)\,|z|^2+(n-1)\,|\mu-\lambda^2|\,|z|+(1-|\lambda|)} \quad (|z|<1).$$

Hence for $|z|=1$, we have

$$|p_n'(z)|\le\frac{1}{K}\frac{(1-|\lambda|)+(n-1)\,|\mu-\lambda^2|\,K+|\lambda|\,(1-|\lambda|)K^2}{|\lambda|\,(1-|\lambda|)+(n-1)\,|\mu-\lambda^2|\,K+(1-|\lambda|)K^2}|q_n'(z)|,$$

and this combined with (26) gives us (11).

Proof of Theorem 2. If $g(z)$ is an entire function of exponential type τ

such that

(i) $$g(z) \neq 0 \quad \text{for} \quad \operatorname{Im} z > -\eta \geq 0,$$

(ii) $$h_g(\pi/2) = \limsup_{r\to\infty} \frac{\log g(re^{i\pi/2})}{r} = 0,$$

(iii) $$\sup_{-\infty<x<\infty} |g(x)| = M < \infty,$$

then ([14], see the proof of (1.7) and in particular p. 592, line 3)

(27) $$|g'(x)| \leq e^{-\tau\eta} \, |g'(x) - i\tau g(x)| \quad \text{for} \quad -\infty < x < \infty.$$

If $f(z)$ satisfies the hypotheses of Theorem 2, then the Phragmén–Lindelöf principle ([1, Theorem 1.4.2]; also see [1, Theorem 6.2.4]) shows that $h_f(-\pi/2)$ must be τ and so $g(z) := e^{i\tau z}\overline{f(\bar{z})}$ satisfies all the three conditions mentioned above. Consequently, $|\tau f(x) + if'(x)| \leq e^{-\tau\eta} \, |f'(x)|$ for $-\infty < x < \infty$ which implies that

$$\tau \, |f(x)| \leq (1 + e^{-\tau\eta}) \, |f'(x)| \quad \text{for} \quad -\infty < x < \infty.$$

This gives the desired result.

Proof of Theorem 3. Let $\sup_{-\infty<x<\infty} |f(x)| = M < \infty$. Then

$$F(z) := f(z)\overline{f(\bar{z})} - \frac{M^2}{2}$$

is an entire function of exponential type $\tau + \sigma$ and $\sup_{-\infty<x<\infty} |F(x)| \leq M^2/2$. Besides, $F(z)$ is real for real z. Hence according to a theorem of Duffin and Schaeffer (see for example [1, p. 215])

$$(\tau+\sigma)^2\left\{|f(x)|^4 + \frac{M^4}{4} - M^2 \, |f(x)|^2\right\} + \left\{\frac{d}{dx} \, |f(x)|^2\right\}^2 \leq \frac{M^4}{4} (\tau+\sigma)^2.$$

From this it follows that at an arbitrary point x_0 of the real axis where $f(x_0) \neq 0$ inequality (23) holds. By continuity, it is true at every point of the real axis other than a simple zero of $f(z)$.

For the proof of Corollary 4 we have only to note that $f(z) \equiv p_n(e^{iz})$ is an entire function of exponential type $\tau \leq n$ and $h_f(\pi/2) \leq 0$.

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