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FUNCTIONS WHICH FOLLOW INNER FUNCTIONS

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If $\mathcal{U}$ denotes the unit disc and $A \subset \mathcal{U}$ has (logarithmic) capacity zero, Frostman proved that the universal covering map ϕ_A: $\mathcal{U} \to \mathcal{U} \setminus A$ is an inner function. ϕ_A has some very nice properties: On the one hand, any analytic function f mapping $\mathcal{U}$ into $\mathcal{U} \setminus A$ is subordinate to ϕ_A, that is, there exists an analytic function g on $\mathcal{U}$ so that $f = \phi_A \circ g$. On the other hand, there is a group Γ_A of Möbius transformations of $\mathcal{U}$ such that any function f which is automorphic with respect to Γ_A can be realized as a composition with ϕ_A, that is, there exists a function g on $\mathcal{U} \setminus A$ so that $f = g \circ \phi_A$. In this paper we exploit these properties to obtain results about the composition operators $C_\phi: f \to f \circ \phi$ for general inner functions ϕ.

We consider these operators on the Hardy spaces, the Smirnov class, and the space of meromorphic functions of bounded characteristic. X will denote any one of these spaces. An obvious *necessary* condition for a function f in X to be in the range of C_ϕ is that $f(\alpha) = f(\beta)$ whenever $\phi(\alpha) = \phi(\beta)$. We describe this property by saying that f *follows* ϕ. We prove in Section 3 that this surprisingly simple condition is also *sufficient.* In Section 4 we use this condition to characterize the range of C_ϕ as a linear submanifold of X, extending and simplifying a characterization due to Ball [4]. We conclude with comments and questions in Section 5.

The composition operators C_ϕ have been much studied, particularly in relation to Toeplitz operators. For work in this direction, see Abrahamse [1], Abrahamse and Ball [2], Ball [4], Nordgren [7], [8], and Thomson [12], [13]. Certain properties of the inner functions ϕ_A were studied by the author in [10]. We assume that the reader is familiar with the basic theory of functions of bounded characteristic, the notion of logarithmic capacity for plane sets, and the elementary properties of universal covering surfaces for plane regions. Appropriate references would be Duren [5], Tsuji [14], and Ahlfors and Sario [3], respectively.

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1. Preliminaries

All functions have domain $\mathcal{U}$, the unit disc, unless stated otherwise.

1.1. *Factorization.* H^∞ denotes the space of bounded analytic functions.

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A nonconstant function ϕ in H^∞ is an *inner function* if

$$\lim_{s\to 1} |\phi(se^{i\theta})| = 1$$

for almost all $\theta \in [-\pi, \pi]$. We say ϕ is a *factor* of $f \in H^\infty$ if $f/\phi \in H^\infty$. An analytic function f is an *outer function* if $f = f_1/f_2$ where $f_1, f_2 \in H^\infty$ and

$$\log |f_j(0)| = \sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f_j(re^{i\theta})| \, d\theta, \quad j = 1, 2.$$

A meromorphic function f is said to be of *bounded characteristic* (*BC*) if $f = f_1/f_2$ with $f_1, f_2 \in H^\infty$. If we define

$$\log^+ t = \max\{0, \log t\}, \quad t \in [0, \infty],$$

then $f \in BC$ if and only if $\log^+ |f|$ has a superharmonic majorant in $\mathcal{U}$. If f is analytic, this is equivalent to the condition

$$\sup_{r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta < \infty.$$

If $f \in BC$, then f can be factored as

$$f = \frac{I_1 F_1}{I_2 F_2} \tag{1}$$

where the factors satisfy:

(a) I_1 and I_2 are inner functions or unimodular constants. If both are inner functions, they have no nonconstant common inner factors.

(b) F_1 and F_2 are outer functions in H^∞ with $\|F_1\|_\infty = 1$, $\|F_2\|_\infty \le 1$.

We will call (1) the *canonical factorization* of f because the factors are unique up to multiplication by unimodular constants. I_1 is termed the *inner factor* of f.

The *Smirnov class* $N_* \subset BC$ consists of those functions f for which the canonical factorization (1) has $I_1 \equiv c$, $|c| = 1$. In particular, $f \in N_*$ implies f is analytic. For $0 < p < \infty$, the *Hardy Space* $H^p \subset N_*$ consists of those functions f for which $|f|^p$ has a harmonic majorant. The following result is a synthesis of known properties of the composition operation. See Nordgren [7] and Stephenson [11].

1.2. THEOREM. *Let X be any one of the spaces BC, N_*, or H^p, $0 < p \le \infty$. If ϕ is an inner function and $g \in X$, then $g \circ \phi \in X$. Furthermore, if g has canonical factorization*

$$g = \frac{I_1 F_1}{I_2 F_2},$$

then $g\circ\phi$ *has canonical factorization*

$$g\circ\phi=\frac{(I_1\circ\phi)(F_1\circ\phi)}{(I_2\circ\phi)(F_2\circ\phi)}.$$

1.3. *Automorphic functions.* $\mathcal{M}$ denotes the group, under composition, of Möbius transformations of $\mathcal{U}$ onto $\mathcal{U}$. A *Fuchsian group* is a subgroup $\Gamma\subseteq\mathcal{M}$ such that for $z\in\mathcal{U}$ the *orbit* $\{\gamma(z): \gamma\in\Gamma\}$ has no cluster point in $\mathcal{U}$. A function f is said to be *automorphic with respect to* Γ if $f\circ\gamma=f$ for all $\gamma\in\Gamma$.

2. The function ϕ_A

2.0. Frostman [6] proved that for $A\subseteq\mathcal{U}$ to be the set of values omitted by an inner function, it is necessary and sufficient that A be a closed subset of (logarithmic) capacity zero. For the sufficiency he showed that if A is such a set, then $\mathcal{U}$ is the universal covering space for $\mathcal{U}\setminus A$ and the *universal covering map* $\phi_A: \mathcal{U}\to\mathcal{U}\setminus A$ is an inner function. Associated with any universal covering map is a group of covering transformations. Since $\mathcal{U}$ is our covering space, this will be a Fuchsian group and we denote it Γ_A. The following proposition gives the essential properties of ϕ_A and Γ_A. Note that ϕ_A is unique only up to composition with a Möbius transformation; all statements referring to ϕ_A, and the associated Γ_A, assume that some particular choice has been made.

2.1. PROPOSITION. *Let* $A\subseteq\mathcal{U}$ *be a closed set of capacity zero, let* $(\mathcal{U},\phi_A)$ *be the universal covering surface of* $\mathcal{U}\setminus A$, *and let* Γ_A *be the associated group of covering transformations.*

(a) *If* $f\in H^\infty$ *and* $f(\mathcal{U})\subseteq\mathcal{U}\setminus A$, *then there exists* $g\in H^\infty$, $\|g\|_\infty\le 1$, *such that* $f=\phi_A\circ g$. f *is an inner function if and only if* g *is an inner function.*

(b) *For* $\alpha\in\mathcal{U}$, $\{z: \phi_A(z)=\phi_A(\alpha)\}=\{\gamma(\alpha): \gamma\in\Gamma_A\}$.

(c) *If* f *is automorphic with respect to* Γ_A, *then there exists a function* g *on* $\mathcal{U}\setminus A$ *such that* $f=g\circ\phi_A$. *If* f *is meromorphic or superharmonic, then* g *is meromorphic or superharmonic, respectively, on* $\mathcal{U}\setminus A$.

Proof. (a) is the usual subordination property associated with conformal maps; g is simply the "lifting" of the map $f: \mathcal{U}\to\mathcal{U}\setminus A$ to the covering surface $\mathcal{U}$. Checking radial limits, it is clear that if g is not an inner function, then $\phi_A\circ g$ cannot be an inner function. Conversely, it is well known that the composition of two inner functions is an inner function.

(b) and (c) are standard properties relating a conformal covering map to its covering transformations (see [3]).

3. Functions in the range of C_ϕ

3.1. DEFINITION. Let ϕ and f be functions with domain $G\subseteq\mathbf{C}$. f is said to *follow* ϕ if, whenever $\alpha,\beta\in G$ with $\phi(\alpha)=\phi(\beta)$, then $f(\alpha)=f(\beta)$. Equivalently, $\phi^{-1}(\phi(\alpha))\subseteq f^{-1}(f(\alpha))$ for all $\alpha\in G$.

3.2. THEOREM. *Let ϕ be an inner function and let X denote any one of the spaces BC, N_*, or H^p, $0<p\leq\infty$. A function $f\in X$ follows ϕ if and only if there exists $g\in X$ such that $f=g\circ\phi$.*

Sufficiency is clear from Theorem 1.2. To prove necessity, we shall first show in Proposition 3.3 that if there exists a meromorphic function g on $\mathcal{U}$ such that $f=g\circ\phi$, then $g\in X$. Thereafter, we can assume $X=BC$. As for existence, we shall construct g on the range of ϕ using Lemma 3.4 and then prove that it can be extended to be meromorphic on all of $\mathcal{U}$. For this last step we reduce to the case that ϕ is one of our covering maps ϕ_A.

3.3. PROPOSITION. *Let X be any one of the spaces BC, N_*, or H^p, $0<p\leq\infty$. If ϕ is an inner function and g is meromorphic on $\mathcal{U}$, then $g\circ\phi\in X$ implies $g\in X$.*

Proof. Using Theorem 1.2 we can make some simplifying assumptions. First, assume $\phi(0)=0$; for otherwise replace ϕ by $\gamma\circ\phi$ where $\gamma\in\mathcal{M}$ is appropriately chosen. Next, assume $g\circ\phi$ is analytic: For suppose $A\subseteq\mathcal{U}$ is the set of poles of $g\circ\phi$. A is closed and countable, hence of capacity zero. Replace ϕ by the inner function $\psi=\phi\circ\phi_A$. Since $g\circ\phi$ is in BC, $g\circ\psi=g\circ\phi\circ\phi_A$ is in BC with no poles.

Our first aim is to prove that $g\in BC$. It is enough to show

(2) $$\sup_{r<1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|g(re^{i\theta})|\,d\theta<\infty.$$

Let v be the least harmonic majorant of $\log^+|g\circ\phi|$. For $0<r<1$, let $D_r=\{w:|w|<r\}$ and let u_r be the continuous function on $\bar{D}_r$ which agrees with $\log^+|g|$ on ∂D_r and is harmonic in D_r. (At most a countable number of radii r will have to be avoided because $\log^+|g|$ is not continuous on ∂D_r.) (2) will follow if we prove $u_r(0)\leq v(0)$, $0<r<1$.

Claim. $u_r(0)=(u_r\circ\phi)(0)\leq v(0)$, $0<r<1$.

Proof of claim. Fix r and let Ω_r be the component of $\phi^{-1}(D_r)$ which contains 0. Let $E=\partial\Omega_r\cap\mathcal{U}$. If $z\in E$, then $|\phi(z)|=r$ so

(3) $$(u_r\circ\phi)(z)=\log^+|(g\circ\phi)(z)|\leq v(z),\quad z\in E.$$

For each ρ, $0<\rho<1$, define $G_\rho=\Omega_r\cap\{z:|z|<\rho\}$ and split ∂G_ρ into

$$E_\rho=E\cap\{z:|z|<\rho\},\qquad J_\rho=\partial G_\rho\cap\{z:|z|=\rho\}.$$

Let $\varepsilon>0$ be given. ϕ is an inner function so $\lim_{s\to1}|\phi(se^{i\theta})|=1$ for almost all $\theta\in[-\pi,\pi]$. By Egoroff's theorem we can choose $\rho=\rho(\varepsilon)$ so large that

$$m\{e^{i\theta}:|\phi(\rho e^{i\theta})|\leq r\}<\varepsilon/\beta,$$

where m denotes Lebesgue measure on $\partial\mathcal{U}$ and $\beta=\sup_{z\in\bar{D}_r}|u_r(z)|<\infty$. In particular,

(4) $$m\{e^{i\theta}:\rho e^{i\theta}\in J_\rho\}<\varepsilon/\beta.$$

Now consider the difference $(u_r \circ \phi) - v$. This is harmonic in G_ρ, continuous on $\bar{G}_\rho$, nonpositive on E_ρ by (3), and

$$(u_r \circ \phi)(z) - v(z) \leq \beta, \quad z \in G_\rho. \tag{5}$$

Let h be the harmonic function in $\{z: |z| < \rho\}$ obtained using the boundary values $(u_r \circ \phi) - v$ on J_ρ and zero on $\{z: |z| = \rho, z \notin J_\rho\}$. h majorizes $u_r \circ \phi - v$ and by (4) and (5), $h(0) < \varepsilon$. (This is simply the extension of domain principle.) Therefore $(u_r \circ \phi)(0) < v(0) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves the claim, and hence that $g \in BC$.

Suppose, now, that g has canonical factorization

$$g = \frac{I_1 F_1}{I_2 F_2}.$$

By Theorem 1.2, $g \circ \phi$ has canonical factorization

$$\frac{(I_1 \circ \phi)(F_1 \circ \phi)}{(I_2 \circ \phi)(F_2 \circ \phi)}.$$

In particular, if $g \circ \phi \in N_*$, then $I_2 \circ \phi \equiv c$, $|c| = 1$, so $I_2 \equiv c$, hence $g \in N_*$. If $g \circ \phi \in H^p$, $0 < p < \infty$, then we may repeat the arguments used earlier, replacing $\log^+ |\cdot|$ with $|\cdot|^p$, to prove $g \in H^p$. The H^∞ case is trivial. This completes the proof of the proposition.

3.4. LEMMA. *Let G be a region in $\mathbf{C}$ with ϕ analytic on G. If f is meromorphic on G and f follows ϕ, then there exists g meromorphic on $\Omega = \phi(G)$ such that $f = g \circ \phi$.*

Proof. It is sufficient to prove the result on the subset of G where f is analytic, for the same can then be done with $1/f$; so without loss of generality assume f is analytic. If ϕ is constant, the conclusion is obvious; if ϕ is not constant, let $\mathscr{X}$ be the discrete (possibly empty) set $\mathscr{X} = \{z \in G: \phi'(z) = 0\}$. Define g first on $\Omega' = \phi(G \setminus \mathscr{X})$ as follows: For $w \in \Omega'$ choose $z \in G \setminus \mathscr{X}$ with $\phi(z) = w$, $\phi'(z) \neq 0$, and let ψ be the local inverse for ϕ with $\psi(w) = z$. Define g locally as $f \circ \psi$. Since f follows ϕ, the choice of the local inverse makes no difference. Therefore, g is well-defined and analytic on Ω' with $f = g \circ \phi$ on $G \setminus \mathscr{X}$.

Now $\Omega \setminus \Omega'$ consists of isolated points, and it is easy to see that g is bounded in a neighborhood of each, so g can be extended to be analytic on all of Ω. Continuity implies $f = g \circ \phi$ on all of G.

3.5. *Completion of proof of Theorem* 3.2. We assume that $f \in BC$ and that f follows the inner function ϕ. By the previous lemma there is a function g, meromorphic on $\phi(\mathscr{U})$, with $f = g \circ \phi$. If $\phi(\mathscr{U}) = \mathscr{U}$, then g is meromorphic on all of $\mathscr{U}$ and Proposition 3.3 proves the result. However, in general, $\phi(\mathscr{U}) = \mathscr{U} \setminus A$, A a closed set of capacity zero. We may reduce the problem as follows: By Proposition 2.1(a), there is an inner function ψ such that $\phi = \phi_A \circ \psi$. Fortunately, we do not need to know anything about ψ; what we do know is that

$$f = g \circ \phi = g \circ \phi_A \circ \psi.$$

Since g is meromorphic on $\mathcal{U} \setminus A$, $h = g \circ \phi_A$ is meromorphic on all of $\mathcal{U}$ with $f = h \circ \psi \in BC$. Proposition 3.3 now implies $h \in BC$. This puts us back in the original situation, only with h replacing f and ϕ_A replacing ϕ; that is, $g \circ \phi_A \in BC$ and we wish to show that g (more precisely, an extension of g) is in BC. Therefore, we may assume without loss of generality that ϕ is one of our special inner functions ϕ_A.

By Proposition 2.1 (b), for all α, $\beta \in \mathcal{U}$, $\phi_A(\alpha) = \phi_A(\beta)$ if and only if $\gamma(\alpha) = \beta$, some $\gamma \in \Gamma_A$. Thus, a function follows ϕ_A if and only if it is automorphic with respect to Γ_A. $f \in BC$ implies $\log^+ |f|$ has a superharmonic majorant in $\mathcal{U}$. Therefore, the function v_f defined by

$$v_f(z) = \inf \{v(z) \colon v \text{ superharmonic with } \log^+ |f| \leq v\}$$

is the unique least superharmonic majorant of $\log^+ |f|$. The uniqueness implies v_f is automorphic with respect to Γ_A, hence there is a superharmonic function u on $\mathcal{U} \setminus A$ so that $v_f = u \circ \phi_A$. Clearly u is a superharmonic majorant of $\log^+ |g|$. By a result of Parreau [9, Théorème 20], g extends to a meromorphic function on $\mathcal{U}$. This, along with Proposition 3.3, completes the proof of Theorem 3.2.

4. The range of C_ϕ

Theorem 3.2 characterizes the functions in the range of the composition operator C_ϕ as those which follow ϕ. Using this, we can characterize the range of C_ϕ as a linear submanifold. This simplifies somewhat a characterization due to Ball [4, Theorem 1] for H^p, $1 \leq p \leq \infty$. Also, since the proof does not depend on the expectation operator of Ball, it extends to H^p, $0 < p < 1$, N_*, and BC. Let *g.c.d.* denote greatest common divisor.

4.1. THEOREM. *Let X be any one of the spaces BC, N_*, or H^p, $0 < p \leq \infty$, and let M be a linear submanifold of X which is closed under uniform convergence on compact subsets of $\mathcal{U}$. Then $M = C_\phi(X)$, some inner function ϕ, if and only if M has the following properties:*

(a) *M contains a nonconstant function.*
(b) *If $f, g \in M$ and $f \cdot g \in X$ (resp. $f/g \in X$), then $f \cdot g \in M$ (resp. $f/g \in M$).*
(c) *If $f \in M$ and χ is the inner factor of f, then $\chi \in M$.*
(d) *M contains* g.c.d. *$\{B \in M \colon B$ inner and $B(0) = 0\}$.*

Proof. First, suppose $M = C_\phi(X)$, and assume without loss of generality that $\phi(0) = 0$. Then $\phi \in M$ and ϕ is the g.c.d. of part (d), so properties (a) and (d) hold. (c) and (d) are easy consequences of Theorem 2.2. Also it is clear that the limit of functions which follow ϕ will likewise follow ϕ, so M is closed under uniform convergence on compact subsets of $\mathcal{U}$.

For the converse, suppose M satisfies properties (a)–(d). By (a) and (b), M contains the constant functions, and along with (c) this implies there exists an inner function $B \in M$ with $B(0) = 0$. Thus the inner function

$$\phi = \text{g.c.d.}\, \{B \in M \colon B \text{ inner}, B(0) = 0\}$$

is well defined. We claim $M = C_\phi(X)$.

Clearly $\phi(0) = 0$, and by (b) and (d) the functions $\{\phi^j \colon j = 0, 1, 2, \dots\}$ are in M. These form an orthonormal set in H^2 under the usual inner product, denoted $\langle \cdot, \cdot \rangle$. Fix an inner function $B \in M$. Write

$$B = g + h \circ \phi \tag{6}$$

where $g \in H^2$ with $\langle g, \phi^j \rangle = 0$, $j = 0, 1, 2, \dots,$ and $h \in H^2$ is given by

$$h(z) = \sum_{j=0}^{\infty} \langle B, \phi^j \rangle z^j.$$

Suppose $k \geq 0$ is the largest integer such that $(g/\phi^k) \in H^2$. Rewrite (6) as

$$B - \sum_{j=0}^{k} \langle B, \phi^j \rangle \phi^j = g + \sum_{j=k+1}^{\infty} \langle B, \phi^j \rangle \phi^j.$$

The left side shows that $\psi = g + \sum_{j=k+1}^{\infty} \langle B, \phi^j \rangle \phi^j$ is in $M \cap H^\infty$. Now, by the definition of k, ϕ^k is a factor of ψ, so by (b),

$$\psi/\phi^k = g/\phi^k + \sum_{j=k+1}^{\infty} \langle B, \phi^j \rangle \phi^{j-k} \tag{7}$$

is in $M \cap H^\infty$. $0 = \langle g, \phi^k \rangle = (g/\phi^k)(0)$ implies $(\psi/\phi^k)(0) = 0$, so (d) implies ϕ is a factor of ψ/ϕ^k. But this means $\psi/\phi^{k+1} \in H^2$ and hence by (7), $g/\phi^{k+1} \in H^2$. This contradicts the definition of k. The only possible conclusion is $g \equiv 0$; so by (6), $h \in H^\infty$ and $B \in C_\phi(X)$.

Finally, fix $f \in M$ and suppose $\alpha, \beta \in \mathcal{U}$ with $\phi(\alpha) = \phi(\beta)$. Let B be the inner factor of $f - f(\alpha)$. $B \in M$, so by the above, B follows ϕ. Since $B(\alpha) = 0$, we have $B(\beta) = 0$, hence $f(\beta) - f(\alpha) = 0$ or $f(\beta) = f(\alpha)$. Therefore f follows ϕ and by Theorem 3.2, $f \in C_\phi(X)$. This shows that $M \subseteq C_\phi(X)$. The opposite inclusion follows from the fact that $\{\phi^j \colon j = 0, 1, 2, \dots\} \subset M$ and M is closed under uniform convergence on compact subsets of $\mathcal{U}$.

5. Comments and questions

5.1. If ϕ is any nonconstant function in H^∞ with $\|\phi\|_\infty \leq 1$, then $f \in X$ implies $f \circ \phi \in X$. However, if ϕ is not an inner function, the reverse implication may fail. This is most evident when ϕ has small range; for instance, if $\|\phi\|_\infty < 1$, then $f \circ \phi \in H^\infty$ for *any* function f analytic in $\mathcal{U}$. But a small range is not a necessity: Let $\Omega=\{z:|z|<1, \operatorname{Re} z>0\}$ and let ψ be a conformal map of $\mathcal{U}$ onto Ω. ψ is outer [5, page 51], so $\phi=\psi^3$ is outer and has range $\mathcal{U}\setminus\{0\}$. Let $f(z)=1/z$. Then $f\in BC$, $f\notin N_*$, but $f\circ\phi\in N_*$ (see Section 1.1.). Can it be the case that the conclusion of Theorem 3.2 holds *only* when ϕ is an inner function?

5.2. In Theorem 4.1., condition (b) is stronger than the analogous conditions in the theorem of Ball. However, a much weaker condition was all that was needed in the proof, namely:

(b′) *Let $B\in M$ be an inner function and suppose $f\in M\cap H^\infty$. Then $f\cdot B\in M$ and, if B is a factor of f, $f/B\in M$.*

5.3. It is conceivable that condition (d) of Theorem 4.1 is redundant. One might ask the following question, for example: Given $f\in X$ nonconstant, let M_f be the smallest linear manifold of X containing f satisfying conditions (b) and (c), and closed under uniform convergence on compact subsets of $\mathcal{U}$. If $M_f\subsetneq X$, is there necessarily an inner function ϕ, not a Möbius transformation, such that $f\in C_\phi(X)$?

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