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ASYMPTOTIC EXPANSIONS FOR THE COMPACT QUOTIENTS OF PROPERLY DISCONTINUOUS GROUP ACTIONS

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1 Introduction

Let M be a connected Riemannian manifold and Γ a group acting isometrically, effectively, and properly discontinuously on M with compact quotient space $\bar{M}=\Gamma\backslash M$. The orbit space $\bar{M}$ is not necessarily a manifold. Suppose that $\pi: M \to \bar{M}$ denotes the associated projection. A function f defined on $\bar{M}$ is said to be of class $C^l(\bar{M})$ if $f \circ \pi \in C^l(M)$. Since Γ acts isometrically, the Laplacian Δ of M is Γ-invariant and Δ induces an operator $\bar{\Delta}$ on $C^2(\bar{M})$.

The Laplacian $\bar{\Delta}$ has a self-adjoint extension to $L^2(\bar{M})$ with pure point spectrum $\lambda_1 \leq \lambda_2 \leq \cdots | \uparrow \infty$. Our main result, Theorem 4.8, is the asymptotic formula as $t \downarrow 0$:

$$\sum_{i=1}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-n/2} \sum_{i=0}^{\infty} a_i t^i$$

where $n = \dim(M)$. Here $a_0 = \mathrm{vol}(\bar{M})$, the volume of $\bar{M}$. The higher order terms a_i may be computed by the method of the author's earlier paper [5].

If $M = G/K$, a symmetric space, and $\Gamma \subset G$ is as above, then the first term of our asymptotic formula was obtained by N. Wallach [13]:

$$\sum_{i=1}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-n/2}\, \mathrm{vol}(\bar{M}).$$

His method relied on the algebraic fact that for symmetric spaces the Γ-action may be factored through a finite group action on a compact Riemannian manifold [3]. As shown below, this technique is not available in the general case, when M need not be symmetric.

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The results of this paper generalize easily to the Laplacian with coefficients in a bundle.

2 Properly discontinuous actions

Let M be a connected manifold and Γ a group acting differentiably and properly discontinuously on M with compact quotient $\bar{M}=\Gamma\backslash M$. Recall that

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a group is said to act properly discontinuously if each compact set in M intersects only a finite number of its translates.

We will need to use some elementary facts concerning properly discontinuous actions [3]:

PROPOSITION 2.1. *There is a relatively compact open set C in M such that $\Gamma C = M$.*

PROPOSITION 2.2. *The group Γ is finitely generated. In particular Γ is a countable set.*

One may apply Proposition 2.1 to construct a Γ-invariant Riemannian metric on M. We now assume that M is endowed with a suitable metric so that Γ acts by isometries.

A function $\phi(x) \in C_0^\infty(M)$ such that for all $x \in M$ one has $\sum_{\gamma\in\Gamma} \phi(\gamma x) = 1$ is called a partition of unity relative to Γ. To obtain such a ϕ, let $\psi \in C_0^\infty(M)$ be a non-negative function whose support contains C with $\Gamma C = M$, as in Proposition 2.1. Then choose

$$\phi(x) = \frac{\psi(x)}{\sum\limits_{\gamma\in\Gamma} \psi(\gamma x)}.$$

We may define a measure on $\bar{M}$ by using a partition of unity relative to Γ:

DEFINITION 2.3. Let f be a continuous function on $\bar{M}$ and ϕ a partition of unity relative to Γ. Then set $\int_{\bar{M}} f(\bar{x})\, d\bar{x} = \int_M \phi(x) f(\pi x)\, dx$ where $\pi: M \to \bar{M}$ is the usual projection. The integral on the right is taken with respect to the measure induced by the Riemannian metric of M.

Properly discontinuous actions on symmetric spaces $M = G/K$ were studied in [3]. A subgroup $\Gamma \subset G$ is discrete if and only if it acts properly discontinuously on M [3, p. 112]. If $\bar{M} = \Gamma\backslash G/K$ is compact then Γ has a normal subgroup Γ_0 of finite index so that $\Gamma_0\backslash G/K$ is a manifold M'. Consequently, the projection $\pi: M \to \bar{M}$ factors as $\pi = \pi_1\pi_2$ where $\pi_1: M' \to \bar{M}$, $\pi_2: M \to M'$. In particular, $\bar{M}$ is the orbit space of the finite group Γ/Γ_0 acting on the manifold M'.

It will be important to see that not every projection $\pi: M \to \bar{M}$, associated to a properly discontinuous action, factors through a finite group action on a compact manifold. This fact follows from Propositions 2.4, 2.5 below. For the construction of these examples, the author is indebted to John Boardman.

PROPOSITION 2.4. *There exists a finitely generated group Γ and an element $\omega \in \Gamma$ such that:*

(i) $\omega^3 = 1$

(ii) *Any subgroup of finite index in Γ contains ω.*

Proof. We may take Γ to be the subgroup of permutations of the integers generated by α, β where α is the shift $\alpha(j)=j+1$ and β is the cycle $\beta(1)=2$, $\beta(2)=1$, $\beta(j)=j$ if $j\neq 1,2$.

The group Γ contains as a normal subgroup the infinite alternating group A which is generated by all three cycles. Now let $R\subset\Gamma$ be any subgroup of finite index, $[\Gamma\colon R]<\infty$. Intersecting R with all its conjugates we find a normal subgroup $P\subset\Gamma$ with $[\Gamma\colon P]<\infty$, $P\subset R$. By the Second Isomorphism Theorem of Group Theory [11, p. 26] one has

$$\frac{A}{P\cap A}=\frac{AP}{P}.$$

So $[A\colon P\cap A]<\infty$. Since A is simple [11, p. 46] this forces $A=P\cap A$, so $A\subset P$. Let ω be the three-cycle $(1\ 2\ 3)$.

Suppose that H is a cyclic group of order p with generator ω. Then H acts on the standard sphere $S(2k)$ by

$$\omega(z,x)=\left(\exp\left(\frac{2\pi\sqrt{-1}}{p}\right)z,x\right)$$

where we regard $S(2k)$ as the set of points $(z,x)\in C^k\times R$ satisfying $|z|^2+x^2=1$. If $k>1$, the orbit space $H\backslash S(2k)$ is not a manifold. This is because of the fixed points $(0,\pm1)$ for the H action on $S(2k)$.

PROPOSITION 2.5. *Let Γ be a finitely generated group containing a cyclic group H of order p. Then Γ acts properly discontinuously on a connected $2k$-manifold M with $\Gamma\backslash M$ not a manifold.*

Proof. Let $\{\gamma_i H\}$ be an enumeration of the cosets of H in Γ. We may assume that the γ_i, $i\leq l$, generate Γ.

Suppose that X is the countable disjoint union of copies of $S=S(2k)$, indexed by the various cosets $\gamma_i H$. Then Γ acts on X by $\gamma(s,\gamma_i H)=(\gamma_j^{-1}\gamma\gamma_i s,\gamma_j H)$ for $\gamma\gamma_i H=\gamma_j H$ and $s\in S$. Clearly $\Gamma\backslash X=H\backslash S$.

One obtains M by adding suitable tubes to X. Choose $2l$ points Q_i, $1\leq i\leq 2l$, on $S-(0,\pm1)$ so that no two Q_i lie in the same orbit of H. For $i\leq l$ cut out a small neighborhood of Q_i, $\gamma_i Q_{i+l}$ and join $(S,1)$ to (S,γ_i) by a basic tube starting about Q_i and ending about $\gamma_i Q_{i+l}$. Now add in all Γ translates of these basic tubes, near the translates of Q_i. This gives a connected manifold M and the Γ action extends in a natural way. The orbit space $\Gamma\backslash M$ is $H\backslash S$ with tubes attached away from the singular points.

Using Propositions 2.4, 2.5 together one deduces:

PROPOSITION 2.6. *There exists a properly discontinuous action of a group Γ on a connected manifold M so that the projection $\pi\colon M\to\Gamma\backslash M$ does not factor as $\pi=\pi_1\pi_2$ where* (i) *π_2 is the projection associated to a regular covering $M\to M'$ with deck group $\Gamma_0\subset\Gamma$ of finite index and* (ii) *π_1 is the projection associated to the action of the finite group Γ/Γ_0 on M'.*

3 Heat equation—the total space

The usual construction of a fundamental solution for the heat equation [1, pp. 204–215] uses repeatedly the hypothesis that one is working on a compact manifold. However, if M is any connected Riemannian manifold admitting a properly discontinuous group of isometries Γ with compact quotient $\Gamma\backslash M$, the method of [1] with small modifications gives a good fundamental solution on M. The main point is that $M=\Gamma C$ with C relatively compact, according to Proposition 2.1, and this fact can be utilized to obtain the necessary estimates.

Let Δ be the Laplacian for M. A function $E(t, x, y)$ on $(0,\infty)\times M\times M$ is called a fundamental solution of the heat equation if it satisfies the following properties:

P1. $E(t, x, y)$ is C^1 in t and C^2 in (x, y).

P2. $(\partial/\partial t+\Delta_2)E(t, x, y)=0$ where Δ_2 is the Laplacian acting in the second variable.

P3. $\lim_{t\to 0} E(t, x, y)=\delta(x, y)$ where $\delta(x, y)$ is the Dirac measure.

P4. For $T>0$ arbitrary and $0<t\le T$ one has, when M is of dimension n,

$$\|d_t^i d_x^j d_y^k E(t, x, y)\|\le C_1 t^{-n/2-i-j-k}\exp\left(\frac{-d^2(x, y)}{4t}\right)$$

for $0\le i, j, k\le 1$ and d denoting the exterior derivative. Here C_1 depends only on T.

We now outline the construction of E by following through the steps in [1, pp. 204–215] but indicating the necessary modifications since M may not be compact. Since $M=\Gamma C$ with C relatively compact, we may choose $\varepsilon>0$ so that $d(x, y)<\varepsilon$ implies that y lies in a normal coordinate neighborhood of x.

If we let $U_\varepsilon=\{(x, y)\in M\times M \mid d(x, y)<\varepsilon\}$ and $r=d(x, y)$ then we may define

$$S_l(t, x, y)=(4\pi t)^{-n/2}\exp\left(\frac{-r^2}{4t}\right)\left(\sum_{i=0}^{l} u_i(x, y)t^i\right)$$

where $u_i(x, y)$ are smooth functions on U_ε. We obtain a parametrix for $E(t, x, y)$ by requiring the u_i to satisfy:

$$\begin{aligned} &u_0(x, x)=1,\\ &\frac{\partial u_0}{\partial r}+\left(\frac{1}{2}\frac{\theta'}{\theta}\right)u_0=0, \qquad (3.1)\\ &\frac{\partial u_i}{\partial r}+\left(\frac{1}{2}\frac{\theta'}{\theta}+\frac{i}{r}\right)u_i+\frac{1}{r}\Delta_2 u_{i-1}=0. \end{aligned}$$

for $i\ge 1$. Here we use the notation of [1, p. 208]. The equations (3.1) guarantee that

$$\left(\frac{\partial}{\partial t}+\Delta_2\right)S_l(t, x, y)=(4\pi)^{-n/2}t^{l-n/2}\exp\left(\frac{-r^2}{4t}\right)\Delta_2 u_l.$$

Let $\eta\in C^\infty(M\times M)$ be equal to 1 on $U_{\varepsilon/4}$ and 0 on $M\times M-U_{\varepsilon/2}$. Then if $H_l=\eta S_l$, $l>n/2$, the function H_l is a parametrix for E, using the terminology of [1, p. 209].

For $A, B\in C^0((0,\infty)\times M\times M)$, one defines

$$A * B(t, x, y)=\int_0^t ds\int_M A(s, x, z)B(t-s, z, y)\,dz$$

where the interior integral is with respect to the volume element induced by the Riemannian metric of M. To guarantee convergence of the integral we may assume that for each fixed y the functions $B_t(z, y)=B(t, z, y)$ are all supported in a fixed compact set for z. We will use the notation $A^{*j}=A * A * A\cdots * A$ where the convolution is taken j times.

Now set $R_l(t, x, y)=(\partial/\partial t+\Delta_2)H_l(t, x, y)$.

LEMMA 3.2. *If $l>n/2$ then the function $Q_l=\sum_{j=1}^{\infty}(-1)^{j+1}R_l^{*j}$ is well defined and lies in $C^m((0,\infty)\times M\times M)$ for $m<l-n/2$. Moreover, one has, when $0<t\le T$, estimate*

$$|Q_l(t, x, y)|\le C_2 t^{l-n/2}\exp\left(\frac{-d^2(x, y)}{4t}\right).$$

Proof. Let V be an upper bound on the volume of a ball of radius ε in M. Using the elementary inequality

$$\frac{d^2(x, z)}{4t}\le\frac{d^2(x, y)}{4s}+\frac{d^2(y, z)}{4(t-s)},\quad 0<s<t$$

which follows from the triangle inequality, and the argument of [1, p. 212] one shows that

$$|R_l^{*j}(t, x, y)|\le\frac{C_3 t^{l-n/2+j-1}C_4^{j-1}V^{j-1}}{\left(l-\frac{n}{2}+1\right)\cdots\left(l-\frac{n}{2}+j-1\right)}\exp\left(\frac{-d^2(x, y)}{4t}\right)$$

for $j\ge 2$. Similarly one estimates the derivatives of the R^{*j}. The lemma follows.

A fundamental solution for the heat equation is obtained by setting $E=H_l-Q_l * H_l$. Moreover, we have:

THEOREM 3.3. *Let M be a connected Riemannian manifold which admits a properly discontinuous group of isometries Γ with compact quotient $\Gamma\backslash M$. There is a unique fundamental solution $E(t, x, y)$ for the heat equation on M.*

In fact $E(t, x, y)$ *is contained in*

$$C^\infty((0,\infty)\times M\times M)$$

and E *satisfies* $E(t, x, y)=E(t, y, x)$. *One has an asymptotic expansion*

$$E(t,x,y)\sim(4\pi t)^{-n/2}\exp\left(\frac{-r^2}{4t}\right)\sum_{i=0}^{\infty}t^iu_i(x,y)$$

valid on a sufficiently small neighborhood of the diagonal in $M\times M$. *Furthermore,* $E(t, x, y)$ *satisfies the semigroup property* $E(t+s, x, y)=\int_M E(t,x,z)E(s,z,y)\,dz$.

Proof. The main point is to establish the uniqueness. Let E_1, E_2 be two fundamental solutions for the heat equation. Set $F_2(t,x,y)=E_2(t,y,x)$. Following [10, p. 49], we may write

$$\begin{aligned}F_2(t,x,y)-E_1(t,x,y)\\ &=\int_0^t ds\frac{\partial}{\partial s}\int_M F_2(s,x,z)E_1(t-s,z,y)\,dz\\ &=\int_0^t ds\int_M [\Delta_zF_2(s,x,z)E_1(t-s,z,y)-F_2(s,x,z)\Delta_zE_1(t-s,z,y)]\,dz\\ &=0.\end{aligned}$$

This shows the uniqueness and symmetry $E(t,x,y)=E(t,y,x)$.

If l is sufficiently large in the above construction one deduces

$$E\in C^m((0,\infty)\times M\times M)$$

for any given m. So uniqueness shows that $E\in C^\infty((0,\infty)\times M\times M)$. The asymptotic expansion and semigroup property follow similarly from uniqueness.

4 Heat equation—the quotient space

Suppose that the group Γ acts properly discontinuously and isometrically on the Riemannian manifold M with compact quotient $\bar{M}=\Gamma\backslash M$. Let $\pi: M\to\bar{M}$ denote the projection. Since Γ does not necessarily act freely, $\bar{M}$ may not be a manifold. However, we can define $C^l(\bar{M})$ by setting a function $f\in C^l(\bar{M})$ if and only if $f\circ\pi\in C^l(M)$. Since Γ acts isometrically on M, the Laplacian Δ of M is Γ-invariant. Thus Δ induces an operator $\bar{\Delta}$ on $C^2(\bar{M})$ which we call the Laplacian of $\bar{M}$.

A function $\bar{E}$ on $(0,\infty)\times\bar{M}\times\bar{M}$ is said to be a fundamental solution for the heat equation on $\bar{M}$ if $\bar{E}$ satisfies properties analogous to P1, P2, P3 of Section 3. To obtain $\bar{E}$ one takes the fundamental solution E on M and sums over Γ to obtain a Γ-invariant expression. This requires some estimates:

LEMMA 4.1. *Let M be a manifold whose sectional curvatures are bounded from below by D_1. Then if $B(x, r)$ is any ball of radius r in M, i.e. $B(x, r) = \{y \in M \mid d(x, y) < r\}$, for some $x \in M$,* one has $\mathrm{vol}(B(x, r)) \leq D_2 e^{D_3 r}$ *for constants D_2, D_3 depending only on D_1 and the dimension of M.*

Proof. Follows by comparison with a space of constant curvature [2, p. 253].

LEMMA 4.2. *Suppose that C is a relatively compact set in M which intersects precisely N of its Γ-translates. Let $W = \mathrm{vol}(C)$ and suppose that $r > \mathrm{diam}(C)$. Then for each $x, y \in C$, $B(x, r)$ contains at most $(N/W)\,\mathrm{vol}(B(x, 2r))$ Γ-translates of y.*

Proof. Suppose that $B(x, r)$ contains p Γ-translates of y. Then $B(x, 2r)$ contains p translates of C since $r > \mathrm{diam}(C)$. However, any point of M is contained in at most N translates of C. Consequently, $pW \leq N\,\mathrm{vol}(B(x, 2r))$.

One may deduce:

THEOREM 4.3. *Let Γ act properly discontinuously on M with compact quotient $\bar{M} = \Gamma \backslash M$. Choose a relatively compact C in M with $M = \Gamma C$. If $\bar{x}$, $\bar{y} \in \bar{M}$ then set*

$$\bar{E}(t, x, y) = \sum_{\gamma \in \Gamma} E(t, x, \gamma y) \tag{4.4}$$

where $x, y \in C$, $\bar{x} = \pi(x)$, and $\bar{y} = \pi(y)$. If E is the fundamental solution for the heat equation of M, the sum on the right converges uniformly on $[t_1, t_2] \times C \times C$, $0 < t_1 \leq t_2$, to the fundamental solution for the heat equation on $\bar{M}$.

The fundamental solution $\bar{E}(t, \bar{x}, \bar{y})$ is unique and satisfies the semigroup property

$$\bar{E}(t+s, \bar{x}, \bar{y}) = \int_{\bar{M}} \bar{E}(t, \bar{x}, \bar{z}) \bar{E}(s, \bar{z}, \bar{y})\, dz.$$

Proof. The existence of C is guaranteed by Proposition 2.1. The main point is to check that the sum converges uniformly on $[t_1, t_2] \times C \times C$. By Property P4, Section 3, of $E(t, x, y)$,

$$\begin{aligned}
\sum_{\gamma \in \Gamma} E(t, x, \gamma y) &\leq C_1 t^{-n/2} \sum_{\gamma \in \Gamma} \exp\left(\frac{-d^2(x, \gamma y)}{4t}\right) \\
&\leq C_1 t^{-n/2} \sum_{i=1}^{\infty} \left(\frac{N}{W}\right) \mathrm{vol}(B(x, 2ir)) \exp\left(\frac{-(i-1)^2 r^2}{4t}\right) \\
&\leq C_1 D_2 t^{-n/2} \sum_{i=1}^{\infty} \left(\frac{N}{W}\right) e^{2iD_3 r} \exp\left(\frac{-(i-1)^2 r^2}{4t}\right)
\end{aligned}$$

using Lemmas 4.1, 4.2. Similarly, one obtains uniform convergence for the derivatives of E. It is then easy to show that $\bar{E}$ is a fundamental solution on $\bar{M}$.

Uniqueness and the semigroup property follows as in the proof of Theorem 3.3.

It is now standard to deduce:

THEOREM 4.5. *The Laplacian $\bar{\Delta}$ on $\bar{M}$ is a symmetric operator on $C^\infty(\bar{M})$ which has a self-adjoint extension to $L^2(\bar{M})$. The operator $\bar{\Delta}$ has pure point spectrum $\{\lambda_i\}_{i=1}^\infty$, $\lambda_1 \leq \lambda_2 \leq \cdots \uparrow \infty$, with corresponding eigenfunctions $\phi_i(\bar{x}) \in C^\infty(\bar{M})$. Moreover, we may write*

$$\bar{E}(t, \bar{x}, \bar{y}) = \sum_i e^{-t\lambda_i} \phi_i(\bar{x}) \phi_i(\bar{y}) \tag{4.6}$$

and

$$\sum_i e^{-t\lambda_i} = \int_{\bar{M}} E(t, \bar{x}, \bar{x})\, d\bar{x}. \tag{4.7}$$

Proof. For each fixed t, the kernel $\bar{E}(t, \bar{x}, \bar{y})$ defines a self-adjoint compact operator on $L^2(\bar{M})$, since $\bar{E}$ is continuous in $(\bar{x}, \bar{y})$ and $\bar{E}(t, \bar{x}, \bar{y}) = \bar{E}(t, \bar{y}, \bar{x})$ [7, p. 13]. Moreover, the semigroup property of Theorem 4.3 implies that the $\bar{E}$'s form a commuting family of compact operators, parameterized by t, and can therefore be simultaneously diagonalized [7, p. 12].

Let $\phi_i(\bar{x})$ be an orthonormal basis for $L^2(\bar{M})$ so that

$$\mu_i(t)\phi_i(\bar{x}) = \int_M \bar{E}(t, \bar{x}, \bar{y})\phi_i(\bar{y})\, d\bar{y}.$$

Since $\bar{E}$ is smooth in $t, \bar{x}$, the $\mu_i(t)$, $\phi_i(\bar{x})$ are smooth. Furthermore, since $\bar{E}$ is a fundamental solution of the heat equation on $\bar{M}$, one has

$$\mu_i'(t)\phi_i(\bar{x}) = -\mu_i(t)\bar{\Delta}\phi_i(\bar{x}).$$

Since $\mu_i(t)$ is non-zero for small t we see that $\phi_i(\bar{x})$ is an eigenfunction of $\bar{\Delta}$, $\bar{\Delta}\phi_i = \lambda_i\phi_i$. The fact that $\bar{\Delta}$ has a self-adjoint extension to $L^2(\bar{M})$ is immediate [7, p. 370].

The expansions (4.6), (4.7) follow as in [1, p. 205].

Our main result is:

THEOREM 4.8. *Suppose that Γ acts properly discontinuously isometrically, and effectively on the Riemannian manifold M with compact orbit space $\bar{M} = \Gamma \backslash M$.*

Let $\{\lambda_i\}$ be the spectrum of the Laplacian $\bar{\Delta}$ of $\bar{M}$. Then there is an asymptotic expansion as $t \downarrow 0$:

$$\sum_{i=1}^\infty e^{-t\lambda_i} \sim (4\pi t)^{-n/2} \sum_{i=0}^\infty a_i t^i \tag{4.9}$$

where a_i constants and $n = \dim(\bar{M})$. One has $a_0 = \operatorname{vol}(\bar{M})$.

Proof. Using formulas (4.4) and (4.7) we may conclude that

$$\sum_{i=1}^{\infty} e^{-t\lambda_i} = \int_M \sum_{\gamma\in\Gamma} E(t, x, \gamma x)\phi(x)\, dx$$

where $\phi(x)$ is a partition of unity relative to Γ.

Recall that $\phi(x)$ is compactly supported. Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be the finite collection of $\gamma\in\Gamma$ which have a fixed point contained in the support of ϕ. It follows from estimates similar to those used in the proof of Theorem 4.4 that

$$\sum_{i=1}^{\infty} e^{-t\lambda_i} \sim \sum_{j=1}^{m} \int_M E(t, x, \gamma_j x)\phi(x)\, dx.$$

It is shown in [5] that

$$\int_M E(t, x, \gamma_j x)\phi(x)\, dx \sim (4\pi t)^{-n_j/2} \sum_{i=0}^{\infty} a_{ij} t^i \tag{4.10}$$

where n_j is the maximum dimension of any component of the fixed point set of γ_j. More precisely, one uses the method of [5] and the fact that $\phi(x)$ is compactly supported. This gives existence for the expansion (4.9).

Since Γ acts effectively only the identity element has a fixed point set of dimension n. The formula for a_0 follows from Theorem 3.3 and formula (3.1).

The author's paper [5] was especially concerned with the problem of computing the terms in the equivariant expansions (4.10). This then provides a procedure for computing the terms in (4.9).

If $M = G/K$ is a symmetric space, then the first term asymptotic formula

$$\sum_{i=1}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-n/2} \operatorname{vol}(\bar{M})$$

was obtained by Wallach [13]. His proof relies on the algebraic fact that if Γ is discrete in G with compact quotient $\Gamma\backslash G/K$, then Γ has a torsion free normal subgroup Γ_0 of finite index [3, p. 112]. This allows one to factor $\pi: M \to \Gamma\backslash M$ as $\pi = \pi_1\pi_2$ where $\pi_2: M \to \Gamma_0 M = M'$ and $\pi_1: M' \to \bar{M}$. Then $\bar{M}$ is the orbit space of the compact Riemannian manifold M' by the finite group Γ/Γ_0 of isometries. Wallach then applies the usual spectral theory on the compact manifold M'. This method is not available to prove Theorem 4.8. As shown in Section 2, Proposition 2.6, it is not possible to factor every properly discontinuous action through a finite group action.

5 Computations for the Poincaré upper half plane

Let H be the upper half plane consisting of points $z = x + \sqrt{-1}\ y$, $y > 0$, endowed with its Poincaré metric.

The isometry group of H is $SL(2, R)$ the group of 2×2 real matrices with determinant one, which acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az+b}{cz+d}$$

for $z\in H$. A subgroup $\Gamma\subset SL(2, R)$ acts properly discontinuously if and only if Γ is discrete in $SL(2, R)$ [3, p. 112].

For Γ acting on H with compact quotient $\bar{H}=\Gamma\backslash H$, the terms in the expansion (4.9) can be computed quite explicitly. This example may serve as a nice illustration of our Theorem 4.8. We will need the following lemma:

LEMMA 5.1. *The heat kernel $E(t, z_1, z_2)$ for the Poincaré upper half plane H is given by*

$$E(t, z_1, z_2) = \frac{e^{-t/4}\sqrt{2}}{(4\pi t)^{3/2}} \int_a^\infty \frac{be^{-b^2/4t}\,db}{\sqrt{\cosh b - \cosh a}}$$

where $a = d(z_1, z_2)$ is the Riemannian distance from z_1 to z_2.

Proof. [9, p. 233].

An element $\gamma\in\Gamma$ fixes a point in H only if it is conjugate in $SL(2, R)$ to an elliptic element $\hat{\gamma}=a\gamma a^{-1}$, $a\in SL(2, R)$, [9, p. 228]. Here $\hat{\gamma}$ is given by

$$\hat{\gamma} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$$

where $\omega = \pi/m$ for some integer $m>1$. Then $\hat{\gamma}$ has order m and fixes the unique point $z=i$. Recall that an element $\gamma\in\Gamma$ is said to be primitive if γ is not a power of any other element in Γ and $\gamma\neq 1$.

THEOREM 5.2. *Let Γ be a discrete subgroup of $SL(2, R)$ acting on the Poincaré upper half plane H with compact quotient $\bar{H}=\Gamma\backslash H$. Then in the expansion*

$$\sum_{i=1}^{\infty} e^{-t\lambda_i} \sim (4\pi t)^{-1} \sum_{i=0}^{\infty} a_i t^i$$

of Theorem 5.2, one has

$$a_0 = \mathrm{vol}\,(\bar{H}), \quad a_1 = \frac{-\mathrm{vol}\,(\bar{H})}{3} + \sum_{\gamma^m=1} \sum_{j=1}^{m-1} \frac{1}{4m}\left(\frac{1}{\sin^2 (j/m)}\right)$$

where the sum is over primitive elliptic elements of order m having a fixed point in the support of a suitably chosen partition of unity for Γ. The higher order terms may be obtained by expanding certain elementary functions in their Taylor series.

Proof. Let $\phi(z)$ be a partition of unity relative to Γ so that for each primitive γ of order m having a fixed point p in the support of ϕ one has $\phi(z)=1/m$ on a neighborhood of p. The existence of such a ϕ follows easily from Proposition 2.1 and the fact that Γ acts properly discontinuously.

We may write

$$\sum_{i=1}^{\infty} e^{-t\lambda_i} \sim \int_H E(t, z, z)\phi(z)\,dz + \sum_{\gamma^m=1} \sum_{j=1}^{m-1} \int_H E(t, z, \gamma^j z)\phi(z)\,dz$$

where the sum is over primitive elements γ of order m having a fixed point in the support of $\phi(z)$.

The integral corresponding to the identity element behaves asymptotically as

$$\int_H E(t, z, z)\phi(z)\,dz = \operatorname{vol}(\bar{H})e^{-t/4}(4\pi t)^{-3/2}\int_0^\infty \frac{be^{-b^2/4t}\,db}{\sinh(b/2)}.$$

Thus

$$\int_H E(t, z, z)\phi(z)\,dz \sim \operatorname{vol}(\bar{H})(4\pi t)^{-1}(1-\tfrac{1}{3}t+O(t^2)) \tag{5.3}$$

The higher order terms may be obtained by expanding $b/\sinh(b/2)$ in its Taylor series about $b=0$ and using the elementary integral formula [12, p. 426]

$$\int_0^\infty x^{2n}e^{-x^2}\,dx = \left(\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}}\right)\sqrt{\pi} \tag{5.4}$$

Now let γ be a primitive element of order m and denote $\omega=\pi/m$. Set

$$I(\gamma, j) = \int_H \phi(x)E(t, z, \gamma^j z)\,dz \quad \text{for} \quad 1\le j\le m-1.$$

Then

$$I(\gamma, j) \sim \frac{1}{m}\int_H E(t, z, \gamma^j z)\,dz$$

since γ has a unique isolated fixed point p and $\phi(z)=1/m$ on a neighborhood of p.

Furthermore

$$I(\gamma, j) \sim \frac{e^{-t/4}\sqrt{2}}{(4\pi t)^{3/2}}\int_0^\infty be^{-b^2/4t}\,db\,\frac{1}{m}\int_H \operatorname{Re}\left(\frac{1}{\sqrt{\cosh b-\cosh a}}\right)dz$$

where $a=d(z, \gamma^j z)$ and Re means to take the real part. The interior integral is computed in [8, p. 245] yielding

$$I(\gamma, j) \sim \frac{e^{-t/4}}{(4\pi t)^{3/2}}\frac{\omega}{\sin(j\omega)}\int_0^\infty\left[\frac{\pi}{2}-\sin^{-1}\left(\frac{\sin^2(j\omega)-\sinh^2(b/2)}{\sin^2(j\omega)+\sinh^2(b/2)}\right)\right]be^{-b^2/4t}\,db.$$

Integrating by parts one obtains

$$I(\gamma, j) \sim \frac{e^{-t/4}}{(4\pi t)^{1/2}} \left(\frac{1}{2m}\right) \int_0^\infty e^{-b^2/4t} \frac{\cosh(b/2)\, db}{\sin^2(j\omega) + \sinh^2(b/2)}.$$

Consequently

$$I(\gamma, j) \sim \frac{1}{m}\left(\frac{1}{4\sin^2(j\omega)}\right) + O(t) \tag{5.5}$$

The higher order terms may be obtained by expanding

$$\cosh(b/2)[\sin^2(j\omega) + \sinh^2(b/2)]^{-1}$$

in its Taylor series about $b = 0$ and using the formula (5.4).

Since

$$\sum_{j=1}^{\infty} e^{-t\lambda_i} \sim \int_H E(t, z, z)\phi(z)\, dz + \sum_{\gamma^m = 1} \sum_{j=1}^{m-1} I(\gamma, j),$$

the theorem follows by summing the expansions (5.3), (5.5).

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