0. Introduction and statement of results

Professor E. M. Stein introduced in [4] (see also [6]) the maximal function

\[ S(f)(x) = \sup_{\varepsilon > 0} \left| \int_{\Sigma} f(x - \varepsilon \alpha) \, d\sigma \right| \]

where \( f \) is any Borel measurable function defined on \( \mathbb{R}^n \), \( \alpha \) is a point on the unit sphere \( \Sigma \) of \( \mathbb{R}^n \) and \( d\sigma \) stands for its “area” element. In the above paper Professor Stein proves the following result: If \( n \geq 3 \) and \( p > n/(n-1) \), then

\[ \|S(f)\|_p < C_p \|f\|_p. \]

If \( p \leq n/(n-1) \) and \( n \geq 2 \) the result is false; what happens for \( n = 2 \) and \( p > 2 \) remains an open problem. Throughout this paper, we shall be concerned with the lacunary version of Stein’s theorem. Define

\[ \sigma(f)(x) = \sup_{k > 0} \left| \int_{\Sigma} f(x - 2^{-k} \alpha) \, d\sigma \right| \]

where \( k \) takes all the natural values. We have the following result:

0.4. Theorem. If \( n \geq 2, p > 1 \) and \( f \) is Borel measurable in \( \mathbb{R}^n \) then

(i) \[ \|\sigma(f)\|_p < C_p \|f\|_p, \quad p > 1. \]

Moreover, we have the following inequality “near” \( L^1 \): If \( Q \) is a cube in \( \mathbb{R}^n \) and \( \lambda > 1/|Q| \) then

(ii) \[ |Q \cap E(\sigma(f) > \lambda)| < \frac{C_1}{\lambda} |Q| \]

\[ + C_2 \frac{|\log \lambda|}{\lambda} \int_{\mathbb{R}^n} |f| \left( 1 + (\log^+ |f|) \log^+ \log^+ |f| \right) dx. \]

The constants \( C_1 \) and \( C_2 \) depend on \( n \) and \( Q \) but not on \( \lambda \) or \( f \).

In particular, (ii) implies differentiability a.e. by lacunary spherical means in the Orlicz Class \( L(\log^+ L) \log^+ \log^+ L \). Professor S. Wainger communicated to me that part (i) of the above theorem has been obtained also by R. R. Coifman and G. Weiss.
I would like to express my gratitude to Professors A. Zygmund and Y. Sagher for helpful discussions concerning the matters of this paper.

1. Auxiliary lemmas

1.1. Lemma. Let \( \hat{K}(x) \) be a radial function defined on \( \mathbb{R}^n \). Let

\[
w(s) = \sup_{0 < r < s} |\hat{K}(r) - \hat{K}(0)| \quad \text{and} \quad v(s) = \sup_{r_1 > s, r_2 > s} |\hat{K}(r_1) - \hat{K}(r_2)|.
\]

Assume that \( w(s) \) and \( v(s) \) satisfy

(a) \( \int_1^{\infty} \frac{v^2(s)}{s} \, ds < \infty \), \quad (aa) \( \int_0^1 \frac{w^2(s)}{s} \, ds < \infty \).

Then the operator

\[
\mathcal{T}(f) = \sup_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}^n} e^{i(x,y)} \hat{K}[2^{-k}|y|]\hat{f}(y) \, dy \right|
\]

\((k \text{ takes natural values only})\) satisfies

(i) \( \|\mathcal{T}(f)\|_2 < C_0 \left( 1 + \int_0^1 \frac{w^2(s)}{s} \, ds + \int_1^{\infty} \frac{v^2(s)}{s} \, ds \right)^{1/2} \|f\|_2 \).

\(C_0\) is independent from \( f \) and \( K \) if \( \hat{K}(0) = 1 \).

Proof. Let \( \varphi(x) \) be a \( C^\infty \) radial function such that \( \hat{\varphi} \) is \( C_0^\infty \) and \( \hat{\varphi}(0) = \hat{K}(0) \). Let

\[
T_k(f)(x) = \int_{\mathbb{R}^n} e^{i(x,y)}(\hat{\varphi}[2^{-k}|y|] - \hat{K}[2^{-k}|y|])\hat{f}(y) \, dy
\]

and

\[
M(f) = \sup_k \left| \int_{\mathbb{R}^n} 2^{kn}\varphi[2^k(x - y)]f(y) \, dy \right|.
\]

Then we have

\[
\|\mathcal{T}(f)\|_2 \leq 4\{M^2(f)(x) + \sum_1^\infty |T_k(f)(x)|^2\}.
\]

Integrating and using Plancherel’s inequality and estimates (a) and (aa) we get the thesis.

1.2. Remark. The above lemma is a version of the tauberian condition in \( L^2 \) (see [4] and [6]).

1.3. Lemma. Let \( K(x) \) be a \( L^1 \) function supported on the unit ball of \( \mathbb{R}^n \).
Let \( w_1(t) \) denote its \( L^1 \)-modulus of continuity. Suppose that \( w_1(t) \) satisfies the Dini condition

\[
(a) \quad \int_0^1 w_1(t) \frac{dt}{t} < \infty.
\]

Then the maximal operator

\[
T(f)(x) = \sup_{k \geq 0} \left| 2^{nk} \int_{\mathbb{R}^n} K(2^k(x-y))f(y) \, dy \right|
\]

satisfies

\[
(i) \quad |E(T(f) > \lambda)| < C_0 \left( 1 + \int_0^1 \frac{w_1(t)}{t} \frac{1}{\lambda} \|f\|_1 \right)
\]

where \( C_0 \) depends on the dimension only if \( \|K\|_1 = 1 \).

**Proof.** Consider the Calderón–Zygmund partition for \( f, f \geq 0: f = f_1 + f_2 \) where \( 0 \leq f_1 \leq 2^n \lambda \) a.e. and \( f_2 = \sum_{j=1}^\infty (f - \mu_j)\varphi_j(x) \). Here, \( \varphi_j(x) \) stands for the characteristic function of \( Q_j \) and the \( \mu_j \) are the mean values:

\[
(1.3.1) \quad \mu_j = \frac{1}{|Q_j|} \int_{Q_j} f(t) \, dt, \quad \lambda < \mu_j \leq 2^n \lambda, \quad j = 1, 2, \ldots
\]

and

\[
(1.3.2) \quad \left| \bigcup_{1}^{\infty} Q_j \right| \leq \frac{1}{\lambda} \int_{\mathbb{R}^n} f \, dt.
\]

for details see [5, pp. 17, 18].

Let \( G_\lambda = \bigcup_{1}^{\infty} 5Q_j \) where \( 5Q_j \) stands as usual for the dialation of \( Q_j \) 5 times about its center. Let \( x \) be a point in \( \mathbb{R}^n \) – \( G_\lambda \) and consider the convolutions

\[
(1.3.3) \quad (K_k * f_2)(x) \quad \text{where} \quad K_k(y) = 2^{nk}K[2^ky].
\]

Let \( y_j \) be the center of \( Q_j \). The above convolution can be written as

\[
(1.3.4) \quad (K_k * f_2)(x) = \sum_{j=1}^{\infty} \int_{Q_j} \{K_k(x-y) - K_k(x-y_j)\}f_2(y) \, dy.
\]

In the above summation we have made use of the fact that \( f_2 \) has mean value zero over \( Q_j \). Notice also that

\[
(1.3.5) \quad \int_{Q_j} \{K_k(x-y) - K_k(x-y_j)\}f_2(y) \, dy = 0
\]
provided that \(2^k \text{diam}(Q_j) \geq 1\). Thus, if \(x \in \mathbb{R}^n - G_\lambda\) we have

\[
\left(1.3.6\right) \left| \sum_{j=1}^{\infty} \int_{Q_j} \{K_k(x-y) - K_k(x-y_j)\} f_2(y) \, dy \right| \\
\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_{Q_j} |K_k(x-y) - K_k(x-y_j)| |f_2(y)| \, dy \\
\leq \sum_{j=1}^{\infty} \sum_{2^k < d_i} \int_{Q_j} |K_k(x-y) - K_k(x-y_j)| |f_2(y)| \, dy
\]

where \(d_i \equiv \text{diam}(Q_i)\). Notice that the second and third members of the above inequality do not depend on \(k\); consequently, they constitute a bound for \(T^*(f_2)\) on \(\mathbb{R}^n - G_\lambda\). Integrating the third member of (1.3.6) over \(\mathbb{R}^n - G_\lambda\) we get

\[
\left(1.3.7\right) \int_{\mathbb{R}^n - G_\lambda} \sum_{j=1}^{\infty} \sum_{2^k < d_i} \int_{Q_j} |K_k(x-y) - K_k(x-y_j)| |f_2(y)| \, dy \\
\leq \sum_{j=1}^{\infty} \int_{Q_j} |f_2(y)| \sum_{2^k < d_i} \int_{\mathbb{R}^n - Q_j} |K_k(x-y) - K_k(x-y_j)| \, dx \\
\leq C \left( \int_0^1 w_1(t) \frac{dt}{t} \right) \sum_{j=1}^{\infty} \int_{Q_j} |f_2(y)| \, dy.
\]

Inequalities (1.3.5), (1.3.6) and (1.3.7) show that

\[
\left(1.3.8\right) \int_{\mathbb{R}^n - G_\lambda} T^*(f_2) \, dx \leq C \left( \int_0^1 w_1(t) \frac{dt}{t} \right) \|f\|_1.
\]

Assuming that \(\int |K| \, dx = 1\) and using the fact that \(0 \leq f_1 \leq 2^n \lambda\) we get

\[
\left(1.3.9\right) E\{T(f_1) > 2^n \lambda\} = \emptyset.
\]

We get the thesis by using (1.3.8), (1.3.9), and the fact that

\[
|G_\lambda| \leq \frac{5^n}{\lambda} \|f\|_1.
\]

The following lemma is related to a one dimensional result due to R. Fefferman (see [1]).

1.4. LEMMA. Let \(K(x)\) be a non-negative monotonic radial function supported on the unit ball. Then, there exists \(F \succeq K\) such that

(i) \[\|F\|_1 + \int_0^1 w_1(F, t) \frac{dt}{t} < C_1 + C_2 \int_{|x| \leq 1} K \log^+ K \, dx.\]

Here, \(w_1(F, t)\) denotes \(L^1\)-modulus of continuity of \(F\).

Proof. If \(K(r)\) is non-decreasing, it is possible to find a domination of the
form

\[(1.4.1) \quad K(r) \leq \sum_{i} 2^{i} \phi_{i}(x) = F(x)\]

where the \( \phi_{i}(x) \) are characteristic functions of annuli \( E_{j} \) of the form

\[\{x; 0 < r_{j} \leq |x| < 1\}, \quad j = 1, 2, \ldots .\]

If \( K(r) \) is non-increasing, it is possible to find a domination of the form

\[(1.4.2) \quad K(r) \leq \sum_{i} 2^{i} \varphi_{i}(x) = F(x)\]

where the \( \varphi_{i}(x) \) are characteristic functions of balls

\[B_{j} = \{x; 0 \leq |x| \leq r_{j} < 1\}, \quad j = 1, 2, \ldots .\]

We are going to assume that we are in the first case since the second one can be dealt with in a similar manner.

The dominant function \( F(x) \) can be constructed so that the following two inequalities hold:

\[(1.4.3) \quad \sum_{i} 2^{k} |E_{k}| \leq 4 \left( \int_{|x| = 1} K(x) \, dx + |B_{0}| \right),\]

\[\sum_{i} 2^{k} k |E_{k}| \leq C \left( \int_{|x| = 1} K \log^{+} K \, dx + |B_{0}| \right).\]

Here, \( B_{0} \) stands for the unit ball in \( \mathbb{R}^{n} \) and \( |B_{0}| \) for its measure. Assume without loss of generality that \( 2^{k} |E_{k}| < 1 \) and \( r_{k} > \frac{1}{2} \). Our first task will be to estimate \( w_{i}(F, s) \). We have the trivial inequality

\[(1.4.4) \quad w_{i}(F, s) \leq \sum_{i} 2^{k} w_{i}(\phi_{k}, s),\]

thus

\[(1.4.5) \quad \int_{0}^{1} \frac{w_{i}(F, s) \, ds}{s} \leq \sum_{i} 2^{k} \int_{0}^{1} \frac{w_{i}(\phi_{k}, s) \, ds}{s}.\]

In the above inequalities we have used the notation \( w_{i}(\phi_{k}, s) \) for the moduli of continuity of the \( \phi_{k} \).

The following estimates can be easily verified:

\[(1.4.6) \quad w_{i}(\phi_{k}, s) \leq 2 |E_{k}| \quad \text{if} \quad s > \frac{1}{4} (1 - r_{k}),\]

\[w_{i}(\phi_{k}, s) \leq 2n |B_{0}| s \quad \text{if} \quad s < \frac{1}{4} (1 - r_{k}).\]

Consequently

\[(1.4.7) \quad \int_{0}^{1} \frac{w_{i}(\phi_{k}, s) \, ds}{s} \leq 2n^{n+1} |E_{k}| + 2 |E_{k}| \log \frac{1}{|E_{k}|}.\]
From (1.4.5) and (1.4.7) we get

\[ (1.4.8) \quad \int_0^1 w_1(F, s) \frac{ds}{s} \leq C\|F\|_1 + \sum_{k=1}^{\infty} 2^k |E_k| \log \frac{1}{|E_k|}. \]

Now consider the two families of subindices, \{k'\} and \{k''\}, defined as follows:

\[ (1.4.9) \quad \{k'\} \text{ is the set of } k\text{'s for which } 2^k |E_k| < 3^{-k}, \]
\[ \{k''\} \text{ is the set of } k\text{'s for which } 2^k |E_k| \geq 3^{-k}. \]

Thus

\[ (1.4.10) \quad \sum_{k=1}^{\infty} 2^k |E_k| \log \frac{1}{|E_k|} \leq \sum_{k=1}^{\infty} 2^k |E_k| \log 2^k |E_k| + \int_{B_0} F \log^+ F \, dx \]
\[ \leq \int_{B_0} F \log^+ F \, dx + \sum_{k'} 3^{-k/2} + \log 3 \sum_{k'} 2^k |E_k| \]
\[ \leq \frac{3}{2} + 2 \int_{B_0} F \log^+ F \, dx. \]

By combining (1.4.10), (1.4.8), (1.4.5) and (1.4.3) we get the desired result.

Remark. Lemmas 1.3 and 1.4 provide a generalization of Theorem 3 in Zo's paper; see [8].

The following lemma is essentially due to L. Carleson and P. Sjölin (see [3, p. 563]). This, however, is a different type of proof.

1.5. LEMMA (Carleson-Sjölin). Let \(T\) be a sublinear operator mapping \(L^p(R^n), p > 1,\) into weak \(L^p(R^n)\) such that

\[ (a) \quad |E(|T(f)| > \lambda)| \leq \frac{C_0}{(p-1)^p \lambda^p} \|f\|_p^p, \quad p > 1, \]

where \(C_0\) and \(\rho\) are independent from \(f\) and \(p\). Let \(Q\) be a cube in \(R^n\) and \(\lambda > 1/|Q|;\) then

\[ (i) \quad |Q \cap \{|T(f)| > \lambda\}| \leq \frac{C_1}{\lambda} |Q| + C_2 \frac{|\log \lambda|}{\lambda} \int_{R^n} |f| [1 + (\log^+ |f|)^\rho \log^+ \log^+ f] \, dx \]

Here, \(C_1\) and \(C_2\) do not depend on \(f\) or \(\lambda\).

Proof. Let \(E_k\) be the set where \(2^k < |f| \leq 2^{k+1}, k \geq 1.\) Let \(f_k\) be the function that equals \(f\) on \(E_k\) and is zero otherwise. Let \(Q\) be a given cube in
and choose $\lambda > 1/|Q|$. From (a), taking $p = 1 + 1/k$ we have

\[
|E(|T(f_k)| > \lambda)| \leq \frac{C_0}{\lambda^1 + 1/k} k^p 2^k |E_k|
\]

\[
\leq \frac{C}{\lambda} |Q|^{1/k} k^p 2^k |E_k|
\]

\[
\leq \frac{C(Q)}{\lambda} k^p 2^k |E_k|.
\]

Let us consider the sets $X_k(\lambda) = E(|T(f_k)| > \lambda$ and the exceptional set $X(\lambda) = \bigcup_{k=1}^{\infty} X_k(\lambda)$. By (1.5.1) we have

\[
|X(\lambda)| \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| (|f|)^p \, dx.
\]

Let $D_k(s)$ be the distribution function of $|T(f_k)|$ on $Q - X(\lambda)$. We have the estimates

\[
\int_{Q - X(\lambda)} \sum_{k=1}^{\infty} |T(f_k)| \, dx = \sum_{k=1}^{\infty} \int_{0}^{\lambda} D_k(s) \, ds
\]

\[
\leq \sum_{k; k^2 > 1/\lambda} \int_{0}^{\lambda} D_k(s) \, ds + \sum_{k; k^2 > 1/\lambda} \int_{0}^{1/k^2} D_k(s) \, ds + \int_{1/k^2}^{\lambda} D_k(s) \, ds
\]

\[
\leq |Q| \sum_{k=1}^{\infty} \frac{1}{k^2} + C \sum_{k=1}^{\infty} \int_{1/k^2}^{\lambda} k^p 2^k |E_k| \frac{ds}{s}.
\]

Let $\tilde{f}$ be the function that equals $f$ if $|f| = 2$ and zero otherwise. Decompose $f$ as $\tilde{f} + \sum_k f_k$ and use (a) for $\tilde{f}$ with $p = 1 + 1/k_0$ for some fixed $k_0$. In order to deal with $\sum_k f_k$ use inequalities (1.5.2) and (1.5.3). This finishes the proof.

1.6. Following E. Stein (see [4]) let us introduce the following kernels:

\[
K_\alpha(r) = \frac{(1 - r^2)^{\alpha - 1}}{\Gamma(\alpha)}, \quad R(\alpha) > 0,
\]

and their Fourier transforms

\[
\hat{K}_\alpha(r) = \pi^{-\alpha} r^{-(n/2) - \alpha + 1} J_{(n/2) + \alpha - 1}(2\pi r).
\]

Consider the maximal operators

\[
S^*_\alpha(f) = \sup_{k=1} \left| \int_{\mathbb{R}^n} e^{i(x,y)} \hat{K}_\alpha(2^{-k} |y|) \tilde{f}(y) \, dy \right|, \quad R(\alpha) > 1/2 - n/2
\]

If $f$ is a step function we have (see [4])

\[
(1.6.4) \quad \sigma(f) = S^*_0(f).
\]
2. Proof of the main result

Write \( \alpha = u + iv \) and consider \( 1/2 - n/2 < u < M \). Using the procedure in [7, pp. 158–159], and the formulas

\[
(2.1.1) \quad \Gamma\left(\frac{n}{2} + \alpha - \frac{1}{2}\right) \sim \sqrt{2\pi} |v|^{(n/2)+u-1} e^{-(\pi|v|/2)}, \quad v \to \infty,
\]

\[
\Gamma(z) = \frac{1}{z} \Gamma(z+1), \quad R(z) > 0,
\]

(see [7, p. 281 bottom note], we get the estimates

\[
(2.1.2) \quad |K_\alpha(x)| \leq \min\left(C_1, C_2 \Gamma\left(\frac{n}{2} + u - 1/2\right) e^{2\pi|v|/n} |v|^{-(n/2)+u} \frac{n+\alpha-1/2}{|x|^{(n/2)+u-1/2}}\right)
\]

where \( C_1 \) and \( C_2 \) are uniform provided \( 1/2 - n/2 < R(\alpha) < M \). (For similar estimates see [6, pp. 60 and 61].) An application of Lemma 1.1 gives

\[
(2.1.3) \quad \|S^*_\alpha(f)\|_{2,\infty} \leq \frac{K}{n+u-1/2} \left|\int \left|K_\alpha\right| \log^+ |K_\alpha| \, dx \right| < \frac{n+\alpha-1/2}{2 - n/2} < R(\alpha) < M.
\]

Here, \( \| \cdot \|_{p,q} \) is the usual notation for Lorentz’s norms. The estimate

\[
\int_{|x|\leq 1} |K_\alpha| \log^+ |K_\alpha| \, dx < \frac{C}{u} e^{\pi(|v|/2)(1+|v|)}
\]

and Lemmas 1.3 and 1.4 give

\[
(2.1.4) \quad \|S^*_\alpha(f)\|_{1,\infty} \leq \frac{C}{u} e^{\pi(|v|/2)(1+|v|)} \|f\|_{1,1}^*.
\]

To end the proof of the main result consider the case \( n = 2 \), a typical one.

Consider step functions \( f \) and the analytic family of operators

\[
(2.1.5) \quad T_{\alpha(z)}(f) = \int_{\mathbb{R}^2} e^{i(x,y)} \hat{K}_{\alpha(z)}(2^{-k(x)}|y|) \hat{f}(y) \, dy
\]

where \( 0 \leq R(z) \leq 1, \alpha(z) = \frac{1}{2}[(u-1) + \varepsilon + iv] \) and \( k(x) \) is a bounded measurable function taking natural values only. (See [7, p. 280]).

The main theorem and definitions in [2] can be formulated in terms of characteristic functions of finite union of intervals and step functions. From this remark and estimate (2.1.2) we see that \( T_{\alpha(z)}(f) \) is admissible (see [2]).
From (2.1.3) and (2.1.4) we have
\[
\| T_{\alpha(1+i\nu)}(f) \|_{L^2}^* < C(|\nu| + 1) \frac{e^{2\pi|\nu|}}{\epsilon^{3/2}} \| f \|_{L^2}^*.
\]
(2.1.6)
\[
\| T_{\alpha(1+i\nu)}(f) \|_{L^1}^* < C(|\nu| + 1) \frac{e^{4|\nu|/4}}{\epsilon} \| f \|_{L^1}^*.
\]

Take \( u = 1 - \epsilon \) and define \( P_u \) by
\[
\frac{1}{P_u} = \frac{\epsilon}{2} + \frac{1-\epsilon}{1}.
\]

Sagher's convexity theorem gives (see [2])
(2.1.7)
\[
\| T_{\alpha(1-\epsilon)}(f) \|_{L^1}^* \leq \frac{K}{\epsilon} \| f \|_{L^1}^*.
\]

Replacing \( P_u \) by its value, \( P_u = 1 + \epsilon/2 - \epsilon \), and using (2.1.7) and the fact that \( k(x) \) is arbitrary we get
(2.1.8)
\[
\| S_0^*(f) \|_{L^1(1+(\epsilon/2-\epsilon),\infty)}^* \leq \frac{K}{\epsilon} \| f \|_{L^1(1+\epsilon/2-\epsilon,1+\epsilon/2-\epsilon)}^*.
\]

An application of Lemma 1.5 gives part (ii) of the thesis and Marcinkiewicz's interpolation theorem gives part (i).

REFERENCES