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LACUNARY SPHERICAL MEANS

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0. Introduction and statement of results

Professor E. M. Stein introduced in [4] (see also [6]) the maximal function

$$S(f)(x) = \sup_{\varepsilon>0} \left| \int_\Sigma f(x-\varepsilon\alpha)\, d\sigma \right| \tag{0.1}$$

where f is any Borel measurable function defined on R^n, α is a point on the unit sphere Σ of R^n and $d\sigma$ stands for its "area" element. In the above paper Professor Stein proves the following result: If $n \geq 3$ and $p > n/(n-1)$, then

$$\|S(f)\|_p < C_p \|f\|_p. \tag{0.2}$$

If $p \leq n/(n-1)$ and $n \geq 2$ the result is false; what happens for $n = 2$ and $p > 2$ remains an open problem. Throughout this paper, we shall be concerned with the lacunary version of Stein's theorem. Define

$$\sigma(f)(x) = \sup_{k>0} \left| \int_\Sigma f(x-2^{-k}\alpha)\, d\sigma \right| \tag{0.3}$$

where k takes all the natural values. We have the following result:

0.4. THEOREM. *If $n \geq 2$, $p > 1$ and f is Borel measurable in R^n then*

(i) $$\|\sigma(f)\|_p < C_p \|f\|_p, \quad p > 1.$$

Moreover, we have the following inequality "near" L^1: If Q is a cube in R^n and $\lambda > 1/|Q|$ then

(ii) $$|Q \cap E(\sigma(f) > \lambda)| < \frac{C_1}{\lambda}|Q| + C_2 \frac{|\log \lambda|}{\lambda} \int_{R^n} |f|\,[1 + (\log^+ |f|) \log^+ \log^+ |f|]\, dx.$$

The constants C_1 and C_2 depend on n and Q but not on λ or f.

In particular, (ii) implies differentiability a.e. by lacunary spherical means in the Orlicz Class $L(\log^+ L) \log^+ \log^+ L$. Professor S. Wainger communicated to me that part (i) of the above theorem has been obtained also by R. R. Coifman and G. Weiss.

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1. Auxiliary lemmas

1.1. LEMMA. *Let $\hat{K}(x)$ be a radial function defined on R^n. Let*

$$w(s) = \sup_{0<r<s} |\hat{K}(r) - \hat{K}(0)| \quad \textit{and} \quad v(s) = \sup_{r_1>s,\, r_2>s} |\hat{K}(r_1) - \hat{K}(r_2)|.$$

Assume that $w(s)$ and $v(s)$ satisfy

$$\text{(a)} \qquad \int_1^\infty \frac{v^2(s)}{s}\, ds < \infty, \qquad \text{(aa)} \quad \int_0^1 \frac{w^2(s)}{s}\, ds < \infty.$$

Then the operator

$$\overset{*}{T}(f) = \sup_{k\ge 1} \left| \int_{R^n} e^{i\langle x,y\rangle} \hat{K}[2^{-k}|y|]\hat{f}(y)\, dy \right|$$

(k takes natural values only) satisfies

$$\text{(i)} \qquad \|\overset{*}{T}(f)\|_2 < C_0\left(1 + \int_0^1 \frac{w^2(s)}{s}\, ds + \int_1^\infty \frac{v(s)}{s}\, ds\right)^{1/2} \|f\|_2.$$

C_0 is independent from f and K if $\hat{K}(0) = 1$.

Proof. Let $\varphi(x)$ be a C^∞ radial function such that $\hat{\varphi}$ is C_0^∞ and $\hat{\varphi}(0) = \hat{K}(0)$. Let

$$\text{(1)} \qquad T_k(f)(x) = \int_{R^n} e^{i\langle x,y\rangle} (\hat{\varphi}[2^{-k}|y|] - \hat{K}[2^{-k}|y|]) \hat{f}(y)\, dy$$

and

$$M(f) = \sup_k \left| \int_{R^n} 2^{kn} \varphi[2^k(x-y)] f(y)\, dy \right|.$$

Then we have

$$\text{(1.1.1)} \qquad |\overset{*}{T}(f)(x)|^2 \le 4\{M^2(f)(x) + \sum_1^\infty |T_k(f)(x)|^2\}.$$

Integrating and using Plancherel's inequality and estimates (a) and (aa) we get the thesis.

1.2. *Remark.* The above lemma is a version of the tauberian condition in L^2 (see [4] and [6]).

1.3. LEMMA. *Let $K(x)$ be a L^1 function supported on the unit ball of R^n.*

Let $w_1(t)$ denote its L^1-modulus of continuity. Suppose that $w_1(t)$ satisfies the Dini condition

(a) $$\int_0^1 w_1(t)\frac{dt}{t}<\infty.$$

Then the maximal operator

$$\overset{*}{T}(f)(x)=\sup_{k>0}\left|2^{nk}\int_{R^n} K(2^k(x-y))f(y)\,dy\right|$$

satisfies

(i) $$|E(\overset{*}{T}(f)>\lambda)|<C_0\left(1+\int_0^1\frac{w_1(t)}{t}\right)\frac{1}{\lambda}\|f\|_1$$

where C_0 depends on the dimension only if $\|K\|_1=1$.

Proof. Consider the Calderón–Zygmund partition for $f, f\geq 0$: $f=f_1+f_2$ where $0\leq f_1\leq 2^n\lambda$ a.e. and $f_2=\sum_1^\infty(f-\mu_j)\varphi_j(x)$. Here, $\varphi_j(x)$ stands for the characteristic function of Q_j and the μ_j are the mean values:

(1.3.1) $$\mu_j=\frac{1}{|Q_j|}\int_{Q_j} f(t)\,dt,\quad \lambda<\mu_j\leq 2^n\lambda,\quad j=1,2,\ldots$$

and

(1.3.2) $$\left|\bigcup_1^\infty Q_j\right|<\frac{1}{\lambda}\int_{R^n} f\,dt.$$

for details see [5, pp.17, 18].

Let $G_\lambda=\bigcup_1^\infty 5Q_j$ where $5Q_j$ stands as usual for the dialation of Q_j 5 times about its center. Let x be a point in R^n-G_λ and consider the convolutions

(1.3.3) $$(K_k*f_2)(x)\quad\text{where}\quad K_k(y)=2^{nk}K[2^ky].$$

Let y_j be the center of Q_j. The above convolution can be written as

(1.3.4) $$(K_k*f_2)(x)=\sum_{j=1}^\infty\int_{Q_j}\{K_k(x-y)-K_k(x-y_j)\}f_2(y)\,dy.$$

In the above summation we have made use of the fact that f_2 has mean value zero over Q_j. Notice also that

(1.3.5) $$\int_{Q_j}\{K_k(x-y)-K_k(x-y_j)\}f_2(y)\,dy=0$$

provided that $2^k \operatorname{diam}(Q_j) \geq 1$. Thus, if $x \in R^n - G_\lambda$ we have

$$(1.3.6) \quad \left| \sum_{j=1}^{\infty} \int_{Q_j} \{K_k(x-y) - K_k(x-y_j)\} f_2(y)\, dy \right|$$
$$\leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_{Q_j} |K_k(x-y) - K_k(x-y_j)|\, |f_2(y)|\, dy$$
$$\leq \sum_{j=1}^{\infty} \sum_{2^k < d_j^{-1}} \int_{Q_j} |K_k(x-y) - K_k(x-y_j)|\, |f_2(y)|\, dy$$

where $d_j \geq \operatorname{diam}(Q_j)$. Notice that the second and third members of the above inequality do not depend on k; consequently, they constitute a bound for $T^*(f_2)$ on $R^n - G_\lambda$. Integrating the third member of (1.3.6) over $R^n - G_\lambda$ we get

$$(1.3.7) \quad \int_{R^n - G_\lambda} \sum_{j=1}^{\infty} \sum_{2^k < d_j^{-1}} \int_{Q_j} |K_k(x-y) - K_k(x-y_j)|\, |f_2(y)|\, dy$$
$$\leq \sum_{j=1}^{\infty} \int_{Q_j} |f_2(y)| \sum_{2^k < d_j^{-1}} \int_{R^n - 5Q_j} |K_k(x-y) - K_k(x-y_j)|\, dx$$
$$\leq C\left(\int_0^1 w_1(t) \frac{dt}{t}\right) \sum_1^{\infty} \int_{Q_j} |f_2(y)|\, dy.$$

Inequalities (1.3.5), (1.3.6) and (1.3.7) show that

$$(1.3.8) \qquad \int_{R^n - G_\lambda} \overset{*}{T}(f_2)\, dx \leq C\left(\int_0^1 w_1(t) \frac{dt}{t}\right) \|f\|_1.$$

Assuming that $\int_{R^n} |K|\, dx = 1$ and using the fact that $0 \leq f_1 \leq 2^n\lambda$ we get

$$(1.3.9) \qquad E\{\overset{*}{T}(f_1) > 2^n\lambda\} = \emptyset.$$

We get the thesis by using (1.3.8), (1.3.9), and the fact that

$$|G_\lambda| \leq \frac{5^n}{\lambda} \|f\|_1.$$

The following lemma is related to a one dimensional result due to R. Fefferman (see [1]).

1.4. LEMMA. *Let $K(x)$ be a non-negative monotonic radial function supported on the unit ball. Then, there exists $F \geq K$ such that*

$$\text{(i)} \qquad \|F\|_1 + \int_0^1 w_1(F, t) \frac{dt}{t} < C_1 + C_2 \int_{|x| \leq 1} K \log^+ K\, dx.$$

Here, $w_1(F, t)$ denotes L^1-modulus of continuity of F.

Proof. If $K(r)$ is non-decreasing, it is possible to find a domination of the form

$$K(r) \le \sum_{1}^{\infty} 2^i \phi_j(x) = F(x) \tag{1.4.1}$$

where the $\phi_j(x)$ are characteristic functions of annuli E_j of the form

$$\{x; 0 < r_j \le |x| < 1\}, \quad j = 1, 2, \ldots \ .$$

If $K(r)$ is non-increasing, it is possible to find a domination of the form

$$K(r) \le \sum_{1}^{\infty} 2^j \varphi_j(x) = F(x) \tag{1.4.2}$$

where the $\varphi_j(x)$ are characteristic functions of balls

$$B_j = \{x; 0 < |x| \le r_j < 1\}, \quad j = 1, 2, \ldots \ .$$

We are going to assume that we are in the first case since the second one can be dealt with in a similar manner.

The dominant function $F(x)$ can be constructed so that the following two inequalities hold:

$$\begin{aligned} &\sum_{1}^{\infty} 2^k |E_k| \le 4\left(\int_{|x|\le 1} K(x)\,dx + |B_0|\right), \\ &\sum_{1}^{\infty} 2^k k |E_k| \le C\left(\int_{|x|\le 1} K \log^+ K\,dx + |B_0|\right). \end{aligned} \tag{1.4.3}$$

Here, B_0 stands for the unit ball in R^n and $|B_0|$ for its measure. Assume without loss of generality that $2^k|E_k| < 1$ and $r_k > \frac{1}{2}$. Our first task will be to estimate $w_1(F, s)$. We have the trivial inequality

$$w_1(F, s) \le \sum_{1}^{\infty} 2^k w_1(\phi_k, s), \tag{1.4.4}$$

thus

$$\int_0^1 w_1(F, s)\frac{ds}{s} \le \sum_{1}^{\infty} 2^k \int_0^1 w_1(\phi_k, s)\frac{ds}{s}. \tag{1.4.5}$$

In the above inequalities we have used the notation $w_1(\phi_k, s)$ for the moduli of continuity of the ϕ_k.

The following estimates can be easily verified:

$$\begin{aligned} &w_1(\phi_k, s) \le 2\,|E_k| \quad &&\text{if} \quad s > \tfrac{1}{4}(1 - r_k), \\ &w_1(\phi_k, s) \le 2n\,|B_0|\,s \quad &&\text{if} \quad s < \tfrac{1}{4}(1 - r_k). \end{aligned} \tag{1.4.6}$$

Consequently

$$\int_0^1 w_1(\phi_k, s)\frac{ds}{s} \le 2n^{n+1}\,|E_k| + 2\,|E_k| \log\frac{1}{|E_k|}. \tag{1.4.7}$$

From (1.4.5) and (1.4.7) we get

$$\int_0^1 w_1(F, s)\frac{ds}{s} \le C\|F\|_1 + \sum_1^\infty 2^k |E_k| \log\frac{1}{|E_k|}. \tag{1.4.8}$$

Now consider the two families of subindicies, $\{k'\}$ and $\{k''\}$, defined as follows:

$$\begin{aligned} &\{k'\} \text{ is the set of } k\text{'s for which } 2^k|E_k| < 3^{-k},\\ &\{k''\} \text{ is the set of } k\text{'s for which } 2^k|E_k| \ge 3^{-k}. \end{aligned} \tag{1.4.9}$$

Thus

$$\begin{aligned} &\sum_1^\infty 2^k|E_k|\log\frac{1}{|E_k|}\\ &\le \sum_1^\infty 2^k|E_k|\,|\log 2^k|E_k|\,| + \int_{B_0} F\log^+ F\,dx\\ &\le \int_{B_0} F\log^+ F\,dx + \sum_{k'} 3^{-k/2} + \log 3\sum_{k''} k2^k|E_k|\\ &\le \tfrac{3}{2} + 2\int_{B_0} F\log^+ F\,dx. \end{aligned} \tag{1.4.10}$$

By combining (1.4.10), (1.4.8), (1.4.5) and (1.4.3) we get the desired result.

Remark. Lemmas 1.3 and 1.4 provide a generalization of Theorem 3 in Zo's paper; see [8].

The following lemma is essentially due to L. Carleson and P. Sjölin (see [3, p. 563]). This, however, is a different type of proof.

1.5. LEMMA (Carleson–Sjölin). *Let T be a sublinear operator mapping $L^p(R^n)$, $p>1$, into weak $L^p(R^n)$ such that*

$$|E(|T(f)|>\lambda)| < \frac{C_0}{(p-1)^\rho}\frac{1}{\lambda^p}\|f\|_p^p, \quad p>1, \tag{a}$$

where C_0 and ρ are independent from f and p. Let Q be a cube in R^n and $\lambda > 1/|Q|$; then

$$|Q\cap\{|T(f)|>\lambda\}| < \frac{C_1}{\lambda}|Q| + C_2\frac{|\log\lambda|}{\lambda}\int_{R^n} |f|\,[1+(\log^+|f|)^\rho \log^+\log^+ f]\,dx \tag{i}$$

Here, C_1 and C_2 do not depend on f or λ.

Proof. Let E_k be the set where $2^k < |f| \le 2^{k+1}$, $k\ge 1$. Let f_k be the function that equals f on E_k and is zero otherwise. Let Q be a given cube in

R^n and choose $\lambda > 1/|Q|$. From (a), taking $p = 1 + 1/k$ we have

$$|E(|T(f_k)| > \lambda)| < \frac{C_0}{\lambda^{1+1/k}} k^\rho 2^k |E_k| < \frac{C}{\lambda} |Q|^{1/k} k^\rho 2^k |E_k| \leq \frac{C(Q)}{\lambda} k^\rho 2^k |E_k|. \tag{1.5.1}$$

Let us consider the sets $X_k(\lambda) = E(|T(f_k)| > \lambda$ and the exceptional set $X(\lambda) = \bigcup_1^\infty X_k(\lambda)$. By (1.5.1) we have

$$|X(\lambda)| < \frac{C}{\lambda} \int_{R^n} |f|(\log^+ |f|)^\rho \, dx. \tag{1.5.2}$$

Let $D_k(s)$ be the distribution function of $|T(f_k)|$ on $Q - X(\lambda)$. We have the estimates

$$\begin{aligned} \int_{Q-X(\lambda)} \sum_1^\infty |T(f_k)| \, dx &= \sum_1^\infty \int_0^\lambda D_k(s) \, ds \\ &\leq \sum_{k;k^2 \leq 1/\lambda} \int_0^\lambda D_k(s) \, ds + \sum_{k;k^2 > 1/\lambda} \int_0^{1/k^2} D_k(s) \, ds + \int_{1/k^2}^\lambda D_k(s) \, ds \\ &\leq |Q| \sum_1^\infty \frac{1}{k^2} + C \sum_1^\infty \int_{1/k^2}^\lambda k^\rho 2^k |E_k| \frac{ds}{s}. \end{aligned} \tag{1.5.3}$$

Let $\bar{f}$ be the function that equals f if $|f| \leq 2$ and zero otherwise. Decompose f as $\bar{f} + \sum_k^\infty f_k$ and use (a) for $\bar{f}$ with $p = 1 + 1/k_0$ for some fixed k_0. In order to deal with $\sum_1^\infty f_k$ use inequalities (1.5.2) and (1.5.3). This finishes the proof.

1.6. Following E. Stein (see [4]) let us introduce the following kernels:

$$K_\alpha(r) = \frac{(1-r^2)_+^{\alpha-1}}{\Gamma(\alpha)}, \quad R(\alpha) > 0, \tag{1.6.1}$$

and their Fourier transforms

$$\hat{K}_\alpha(r) = \pi^{-\alpha} r^{-(n/2)-\alpha+1} J_{(n/2)+\alpha-1}(2\pi r). \tag{1.6.2}$$

Consider the maximal operators

$$S_\alpha^*(f) = \sup_{k=1} \left| \int_{R^n} e^{i\langle x,y \rangle} \hat{K}_\alpha(2^{-k}|y|) \hat{f}(y) \, dy \right|, \quad R(\alpha) > 1/2 - n/2 \tag{1.6.3}$$

If f is a step function we have (see [4])

$$\sigma(f) = S_0^*(f). \tag{1.6.4}$$

2. Proof of the main result

Write $\alpha = u + iv$ and consider $1/2 - n/2 < u < M$. Using the procedure in [7, pp. 158–159], and the formulas

(2.1.1) $$\Gamma\left(\frac{n}{2}+\alpha-\frac{1}{2}\right) \sim \sqrt{2\pi}\,|v|^{(n/2)+u-1}\, e^{-(\pi|v|/2)}, \quad v \to \infty,$$

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1), \quad R(z) > 0,$$

(see [7, p. 281 bottom note], we get the estimates

(2.1.2) $$|\hat{K}_\alpha(x)| \le \min\left(C_1,\, C_2\Gamma\left(\frac{n}{2}+u-1/2\right)\frac{e^{2\pi|v|}|v|^{-(n/2)+u}\left|\frac{n}{2}+\alpha-1/2\right|}{|x|^{(n/2)+u-1/2}}\right)$$

where C_1 and C_2 are uniform provided $1/2 - n/2 < R(\alpha) < M$. (For similar estimates see [6, pp. 60 and 61].) An application of Lemma 1.1 gives

(2.1.3)
$$\|S_\alpha^*(f)\|_{2,\infty}^* \le \frac{K}{\left|\frac{n}{2}+u-1/2\right|^{3/2}}|v|^{-(n/2)+u}e^{2\pi|v|}\|f\|_{2,2}^*, \quad \frac{1}{2}-\frac{n}{2} < R(\alpha) < M.$$

Here, $\|\ \|_{p,q}^*$ is the usual notation for Lorentz's norms. The estimate

$$\int_{|x|\le 1} |K_\alpha| \log^+ |K_\alpha|\, dx < \frac{C}{u} e^{\pi(|v|/2)}(1+|v|)$$

and Lemmas 1.3 and 1.4 give

(2.1.4) $$\|S_\alpha^*(f)\|_{(1,\infty)}^* < \frac{C}{u} e^{\pi(|v|/2)}(1+|v|)\,\|f\|_{(1,1)}^*.$$

To end the proof of the main result consider the case $n = 2$, a typical one.

Consider step functions f and the analytic family of operators

(2.1.5) $$T_{\alpha(z)}(f) = \int_{R^2} e^{i\langle x,y\rangle}\hat{K}_{\alpha(z)}(2^{-k(x)}|y|)\hat{f}(y)\, dy$$

where $0 \le R(z) \le 1$, $\alpha(z) = \frac{1}{2}[(u-1)+\varepsilon+iv]$ and $k(x)$ is a bounded measurable function taking natural values only. (See [7, p. 280]).

The main theorem and definitions in [2] can be formulated in terms of characteristic functions of finite union of intervals and step functions. From this remark and estimate (2.1.2) we see that $T_{\alpha(z)}(f)$ is admissible (see [2]).

From (2.1.3) and (2.1.4) we have

(2.1.6)
$$\|T_{\alpha(iv)}(f)\|^*_{(2,\infty)} < C(|v|+1)\frac{e^{2\pi|v|}}{\varepsilon^{3/2}}\|f\|^*_{(2,2)},$$
$$\|T_{\alpha(1+iv)}(f)\|^*_{(1,\infty)} < C(|v|+1)\frac{e^{|v|/4}}{\varepsilon}\|f\|^*_{(1,1)}.$$

Take $u=1-\varepsilon$ and define P_u by

$$\frac{1}{P_u}=\frac{\varepsilon}{2}+\frac{1-\varepsilon}{1}.$$

Sagher's convexity theorem gives (see [2])

(2.1.7)
$$\|T_{\alpha(1-\varepsilon)}(f)\|^*_{(P_u,\infty)} \leq \frac{K}{\varepsilon}\|f\|^*_{(P_u,P_u)}.$$

Replacing P_u by its value, $P_u=1+\varepsilon/2-\varepsilon$, and using (2.1.7) and the fact that $k(x)$ is arbitrary we get

(2.1.8)
$$\|S_0^*(f)\|^*_{(1+(\varepsilon/2-\varepsilon),\infty)} \leq \frac{K}{\varepsilon}\|f\|^*_{(1+\varepsilon/2-\varepsilon,\ 1+\varepsilon/2-\varepsilon)}.$$

An application of Lemma 1.5 gives part (ii) of the thesis and Marcinkiewicz's interpolation theorem gives part (i).

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