ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 1, Spring 1980

LINK COMPLEMENTS AND COHERENT GROUP RINGS

BY
PHILIP S. HIRSCHHORN

1. Introduction

A link of multiplicity m is an embedding of the disjoint union of m copies of the $(q-2)$-sphere S^{q-2} in the q-sphere S^q. (We will work in the smooth category, but all statements will be true when said for the piecewise linear category as well.) Any attempt to classify links analyzes the complement of the embedded spheres; thus, one would like to have a characterization of the homotopy type of a link complement.

By Alexander duality, a link complement C always has the same homology groups as the wedge $(\vee^m S^1) \vee (\vee^{m-1} S^{q-1})$. Gutierrez [4] has shown that for $q \geq 6$, if $\pi_i C \cong \pi_i(\vee S^1)$ for $i \leq (q-1)/2$ and if $\pi_1 C$ is generated by the meridians, then the link is trivial. We will consider those links for which there is an integer n satisfying $1 \leq n \leq (q-3)/2$, such that $\pi_i C \cong \pi_i(\vee S^1)$ for $i \leq n$. Thus, all fundamental groups will be isomorphic to $F(m)$, the free group on m generators, and all higher homotopy groups will be modules over $\Phi = \mathbf{Z}[F(m)]$, the group ring of the free group on m generators.

If G is a group and M is a $\mathbf{Z}[G]$-module, the *homology groups* $H_*(G; M)$ are defined as the homology groups of a $K(G, 1)$ with (twisted) coefficients in M, and a $\mathbf{Z}[G]$-module M is *acyclic* if $H_i(G; M) \cong 0$ for all $i \geq 0$.

Our main results are then the following two theorems.

THEOREM 1. *Let n and q be integers satisfying $1 \leq n \leq (q-3)/2$, and let C be the complement of a link $f\colon \cup S^{q-2} \to S^q$ such that $\pi_i C \cong \pi_i(\vee S^1)$ for $i \leq n$. Then $\pi_i C$ is a finitely presented acyclic Φ-module for $n+1 \leq i \leq 2n$.*

THEOREM 2. *Let n be a positive integer, and let X be a space for which $\pi_i X \cong \pi_i(\vee S^1)$ for $i \leq n$, $\pi_i X$ is a finitely presented acyclic Φ-module for $n+1 \leq i \leq 2n$, and $\pi_i X \cong 0$ for $i \geq 2n+1$. Then for any $q \geq 4n+3$ there is an embedding $f\colon \cup S^{q-2} \to S^q$ with complement C, and a map $C \to X$ inducing isomorphisms $\pi_i C \cong \pi_i X$ for $i \leq 2n$.*

A key technical point is that Φ is a *coherent* ring, which we discuss in Section 2, along with some properties of acyclic Φ-modules. Theorems 1 and 2 are proved in Sections 3 and 4, respectively.

Received March 14, 1978.

These theorems are extensions to links of results obtained for knots in [6]. I would like to thank Nobuyuki Sato for pointing out that the first nontrivial homotopy group of a link must be finitely presented, and Howard Hiller for many helpful conversations.

2. Coherence

All modules in this paper will be left modules. We will be working with modules over $\Phi = \mathbf{Z}[F(m)]$, the integral group ring of the free group on m generators. Although Φ is not Noetherian, it does have a weaker property, coherence, which will be sufficient for our purposes.

DEFINITION. A ring is *coherent* if any homomorphism between finitely generated free modules has a finitely generated kernel.

THEOREM. *Φ is a coherent ring.*

Proof. This is a result of Waldhausen [7]; see also [2].

The properties of coherent rings that we will need are contained in the next proposition. (For a more complete discussion of coherence, see [3].)

2.1. PROPOSITION. *If $f: M \to N$ is a homomorphism between finitely presented modules over a coherent ring, then the kernel and cokernel of f are finitely presented. Extensions of finitely presented modules by finitely presented modules are finitely presented.*

Proof. The statement for extensions is a straightforward exercise, and so we need only prove the first part of the proposition.

Let I be the image of f. We then have exact sequences

$$0 \to K \to M \to I \to 0 \quad \text{and} \quad 0 \to I \to N \to C \to 0$$

where K and C are respectively the kernel and cokernel of f. Since C is clearly finitely presented, we can let $0 \to G \to F \to C \to 0$ be a finite presentation of C (i.e., F and G are finitely generated and F is free). Since F is free and $N \to C$ is an epimorphism, we can fill in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & G & \to & F & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & I & \to & N & \to & C & \to & 0 \end{array}$$

from which we obtain the exact sequence $0 \to G \to F \oplus I \to N \to 0$. Now G is a finitely generated submodule of the finitely generated free module F, and so G is finitely presented. Since N is also finitely presented, we have that $F \oplus I$, and so also I, is finitely presented. One can now repeat this entire argument to show that K is finitely presented, and so the proof of the proposition is complete.

We will also need the following properties of acyclic Φ-modules.

2.2. PROPOSITION. *If f: $M \to N$ is a homomorphism of acyclic Φ-modules, then the kernel and cokernel of f are acyclic. Extensions of acyclic Φ-modules by acyclic Φ-modules are acyclic.*

Proof. Let I be the image of f. We then have exact sequences

$$0 \to K \to M \to I \to 0 \quad \text{and} \quad 0 \to I \to N \to C \to 0$$

where K and C are respectively the kernel and cokernel of f. Since $\vee S^1$ is a $K(F(m), 1)$, all Φ-modules have zero homology above dimension one. Thus, by the acyclicity of M and N and the long exact homology sequence of a short exact sequence of coefficient modules,

$$H_i(F(m); K) \cong H_{i+1}(F(m); I) \cong H_{i+2}(F(m); C) \cong 0,$$

and so K is acyclic. This now implies that I is acyclic, and so C is acyclic as well. The statement for extensions follows similarly.

2.3. PROPOSITION. *If D is a finitely presented Φ-module, then so is $H_i K(D, p)$ for $i \leq 2p - 1$. If D is an acyclic Φ-module, then so is $H_i K(D, p)$ for $i \leq 2p - 1$.*

Proof. According to [1, p. 11-11], these groups are finite direct sums of groups of the form $D \otimes \mathbf{Z}/q\mathbf{Z}$ and $D * \mathbf{Z}/q\mathbf{Z}$ for various primes q. But these are respectively the cokernel and kernel of the homomorphism $D \to D$ that multiplies each element by q, and so the result now follows from 2.1 and 2.2.

3. Proof of Theorem 1

Given a link f: $\cup S^{q-2} \to S^q$, if we take the complement of an open tubular neighborhood [5] of the embedded spheres, we obtain a compact manifold (with boundary) homotopy equivalent to C. Thus, C is homotopy equivalent to a finite CW-complex, and so we will assume throughout this section that C is a finite CW-complex.

The proof of Theorem 1 will use the following two lemmas.

LEMMA. *If $\tilde{C}$ is the universal cover of C, then $H_i \tilde{C}$ is a finitely presented Φ-module for all $i \geq 0$.*

Proof. The chain complex of $\tilde{C}$ consists of finitely generated free Φ-modules, with one generator for each cell of C. The lemma now follows from 2.1.

LEMMA. *$H_i \tilde{C}$ is an acyclic Φ-module for $i < q - 2$.*

Proof. Consider the Serre spectral sequence of the fibration $\tilde{C} \to C \to \vee S^1$. Since $\vee S^1$ is one dimensional, $E^2_{st} \cong 0$ for $s > 1$. Thus, since $H_i C \cong H_i(\vee S^1)$ for $i \leq q - 2$, we have

$$H_s(F(m); H_t\tilde{C}) \cong H_s(\vee S^1; H_t\tilde{C}) \cong E^2_{st} \cong 0$$

for $t < q - 2$.

Proof of Theorem 1. One shows by induction on k that for $k \leq 2n$,

(1) $\pi_k C$ is a finitely presented acyclic Φ-module and
(2) $H_i P^k\tilde{C}$ is a finitely presented acyclic Φ-module for $i \leq 2n + 1$

(where $P^k\tilde{C}$ is the *kth* Postnikov approximation of $\tilde{C}$). The induction is begun by Proposition 2.3 together with the fact that $\pi_{n+1} C \cong H_{n+1} \tilde{C}$. The induction step follows by considering the Serre spectral sequence of the fibration $K(\pi_{k+1} C, k + 1) \to P^{k+1}\tilde{C} \to P^k\tilde{C}$ and using 2.1, 2.2 and 2.3.

4. Proof of Theorem 2

We will use the following modification of a theorem of Wall.

THEOREM. *Let W be a finite CW-complex of dimension $2n + 2$, with $\pi_1 W \cong \pi_1(\vee S^1)$ and $H_* W \cong H_*(\vee S^1)$. Then if $q \geq 4n + 3$, there is an embedding $\cup S^{q-2} \to S^q$ with complement C, and a map $C \to W$ inducing $\pi_i C \cong \pi_i W$ for $i \leq 2n$.*

Proof. This is similar to the proof of the corresponding theorem for knots given by Wall in [9, p. 17]. The trivial link has complement homotopy equivalent to $(\vee S^1) \vee (\vee S^{q-1})$, and one attaches the cells of W (as handles) to this.

Thus, it is sufficient to construct W as above, together with a map $W \to X$ inducing $\pi_i W \cong \pi_i X$ for $i \leq 2n$. To do this, we will use the following theorem, also a variant of a theorem of Wall.

THEOREM. *Let X be a space for which $\pi_1 X \cong F(m)$ and let $\tilde{X}$ be the universal cover of X. If there is an integer k for which $H_i \tilde{X}$ is a finitely presented Φ-module for all $i \leq k$, then X is homotopy equivalent to a space with a finite k-skeleton.*

Proof. This is the same as the proof of Wall [8, Theorems A and B] for the Noetherian case.

To make use of this theorem, we prove the following proposition.

PROPOSITION. *$H_i \tilde{X}$ is a finitely presented Φ-module for $i \leq 2n + 1$.*

Proof. One shows by induction on k that for $k \leq 2n + 1$ and $i \leq 2n + 1$, the groups $H_i P^k\tilde{X}$ are finitely presented Φ-modules (where $P^k\tilde{X}$ is the *kth* Postnikov approximation of $\tilde{X}$). The induction follows easily from the Serre spectral sequence, using Propositions 2.1 and 2.3.

Thus, we may assume that X^{2n+1}, the $(2n+1)$-skeleton of X, is finite, and we have the following proposition (which we prove at the end of this section).

4.1. PROPOSITION. *$H_i P^{2n} X^{2n+1} \cong H_i(\vee S^1)$ for $i \leq 2n+1$.*

Thus, we also have that $H_i X^{2n+1} \cong H_i(\vee S^1)$ for $i \leq 2n$. Now $H_{2n+1} X^{2n+1}$ is a finitely generated free abelian group, so we can choose a (finite) free basis for this group, and because we have the exact sequence

$$\pi_{2n+1} X^{2n+1} \to H_{2n+1} X^{2n+1} \to H_{2n+1} P^{2n} X^{2n+1},$$

we can lift each element of this basis to $\pi_{2n+1} X^{2n+1}$. If we now use these elements of $\pi_{2n+1} X^{2n+1}$ to attach $(2n+2)$-cells, we get our complex W with $H_* W \cong H_*(\vee S^1)$. The inclusion $X^{2n+1} \to X$ can now be extended over all of W, and so the proof of Theorem 2 is complete (except for Proposition 4.1).

Proof of Proposition 4.1. One shows by induction on k that $H_i P^k X^{2n+1} \cong H_i(\vee S^1)$ for $k \leq 2n$ and $i \leq 2n+1$. The induction follows easily from the Serre spectral sequence, using Proposition 2.3 together with the fact that $H_p(P^k X^{2n+1}; M) \cong 0$ for $2 \leq p \leq n$ and any local coefficient system M on $P^k X^{2n+1}$.

BIBLIOGRAPHY

1. H. CARTAN, *Algebres d'Eilenberg-MacLane et homotopie*, Seminaire Henri Cartan, E.N.S., Paris, 1954–55.
2. K. G. CHOO, K. Y. LAM and E. LUFT, *On free products of rings and the coherence property*, Lecture Notes in Math. no. 342, Springer-Verlag, New York, 1973.
3. J. M. COHEN, *Coherent graded rings and the non-existence of spaces of finite stable homotopy type*, Comment. Math. Helv., vol. 44 (1969), pp. 217–228.
4. M. A. GUTIERREZ, *Boundary links and an unlinking theorem*, Trans. Amer. Math. Soc., vol. 171 (1972), pp. 491–499.
5. M. W. HIRSCH, *Differential topology*, Springer-Verlag, New York, 1976.
6. P. S. HIRSCHHORN, *On the "stable" homotopy type of knot complements*, Illinois J. Math., vol. 23 (1979), pp. 128–134.
7. F. WALDHAUSEN, *Whitehead groups of generalized free products*, Lecture Notes in Math. no. 342, Springer-Verlag, New York, 1973.
8. C. T. C. WALL, *Finiteness conditions for CW-complexes*, Ann. of Math., vol. 81 (1965), pp. 56–69.
9. ———, *Surgery on compact manifolds*, Academic Press, London, 1970.

BRANDEIS UNIVERSITY
WALTHAM, MASSACHUSETTS
WELLESLEY COLLEGE
WELLESLEY, MASSACHUSETTS