ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 3, Fall 1980

A CONSISTENT FUBINI-TONELLI THEOREM FOR NONMEASURABLE FUNCTIONS

BY

HARVEY FRIEDMAN[1]

The Fubini-Tonelli theorem asserts that if a nonnegative function of two real variables is measurable, then the iterated integrals exist and are equal (where infinite integrals are allowed). What if measurability is not assumed, but instead only that the iterated integrals exist?

It is not hard to see that this is a weaker hypothesis. For example, one can construct a permutation of **R** whose graph is nonmeasurable, and consider its characteristic function. The iterated integrals both exist and are 0.[2]

It is well known that using the continuum hypothesis, one can construct a nonnegative function of two real variables such that the iterated integrals exist but are not equal. The usual example is the characteristic function of the graph of a well ordering of **R**. Actually, this counterexample works under the milder hypothesis that every set of reals of power less than c is of measure 0.

Here we construct a model of ZFC in which there is no such counterexample. In other words, in this model, if the iterated integrals for a nonnegative function of two real variables exist, they are equal.

We fix M to be a countable transitive model of ZFC, and we use c to denote the Von Neumann cardinal of the continuum in the sense of M.

Within M, we can consider σ-algebra K of subsets of $2^{(c^+)}$ (the set of 0, 1-valued functions from c^+) which are generated by the sets $\{f\colon f(\alpha) = 0\}$, for ordinals $\alpha < c^+$. We consider K as equipped with the usual product measure which assigns each set $\{f\colon f(\alpha) = 0\}$ measure 1/2.

Our forcing conditions will be the elements of K of positive measure in M, and p extends q if and only if $p \subset q$. This notion of forcing is commonly referred to as "adding c^+ random reals".[3] In the usual way via Borel codes, there is a one-one correspondence between elements of K (which are in the ground model M) and subsets of actual $2^{(c^+)}$ (i.e., all functions from the c^+ of M into $\{0, 1\}$). In the standard way, one verifies that if G is a generic set of conditions, then $\bigcap G$ (defined using the one-one correspondence) consists of one element $\mathbf{f}$, and G is the set of all conditions which include $\mathbf{f}$. Such an $\mathbf{f}$ is said to be

Received July 14, 1978.

[1] This research was partially supported by the National Science Foundation.

[2] We are grateful to Gerald Edgar for this example, and for conversations which lead to this paper.

[3] Random real forcing was first presented in [2]. In [3], uncountably many random reals are added to a ground model.

K-generic over M. In addition, one verifies that $\mathbf{f}$ is K-generic if and only if it is a member of all elements of K of measure 1. We let $N = M[\mathbf{f}] = M[G]$.

For $A \subset c$, we will consider the submodel $M[A]$. Here it will be convenient to consider $M[A]$-Borel sets of *real numbers*, and $M[A]$-random *real numbers*. The former are the Borel sets in $(-\infty, \infty)$ which have a Borel code in $M[A]$, and the latter consists of those reals in the intersection of all $M[A]$-Borel sets of real numbers of full measure.

Observe that our notion of forcing has the countable chain condition, and so preserves cardinals. The following is proven by straightforward antichain arguments.

LEMMA 1. *If x is a set of integers in N, then there is an M-random real y in N and a set of integers z in M such that x is arithmetic in (y, z). If y is an M-random real and x is a set of integers in $M[y]$, then there is a set of integers z in M such that x is arithmetic in (y, z).*

The following is in the folklore, and is implicit in [3].

LEMMA 2. *In N, let $A \subset c$. Then N is obtainable from $M[A]$ by generically adding c^+ random reals.*

Proof. We indicate the main ideas. The Boolean algebra corresponding to adding c^+ random reals is the homogeneous measure algebra $\mathscr{B}$ of size c^+. From the general theory of factoring in Boolean-valued models, N is given by forcing with respect to a factor of $\mathscr{B}$ by an ideal of power $\leq c$, over $M[A]$. This factor algebra lies in $M[A]$, and can be verified to be a measure algebra, and by cardinality considerations, to be of power c^+. Furthermore, every $\{x\colon x \leq a\}$ for $a \neq 0$, has power c^+. By Maharam's theorem [1], we see that this factor algebra is isomorphic to $\mathscr{B}_K$.

The following is immediate from Lemmas 1 and 2.

LEMMA 3. *In N, let $A \subset c$. If x is a set of integers in N, then there is an $M[A]$-random real y in N and a set of integers z in $M[A]$ such that x is arithmetic in (y, z). If y is an $M[A]$-random real and x is a set of integers in $M[A, y]$, then there is a set of integers z in $M[A]$ such that x is arithmetic in (y, z).*

LEMMA 4. *In N, let $A \subset c$ and B be a Borel set of measure 0. Then there is an $M[A]$-Borel set of measure 0 which includes $B \cap M[A]$.*

Proof. By Lemma 3, the code for B is arithmetic in a real in $M[A]$ and an $M[A]$-random real. Therefore B is of the form $\{y\colon (x, y) \in C\}$, for some $M[A]$-random real x and $M[A]$-Borel set C. Note that $M[A, x] \models$"$\{y\colon (x, y) \in C\}$ is of measure 0". Consequently, let D be an $M[A]$-Borel set of positive measure such that $x \in D$ and for all $M[A]$-random reals $x^* \in D$, $\{y\colon (x^*, y) \in C\}$ is of measure

0. Hence for almost all $x^* \in D$, $\{y: (x^*, y) \in C\}$ is of measure 0. By Fubini's theorem, for almost all y, $\{x^* \in D: (x^*, y) \in C\}$ is of measure 0. Consider

$$\{y: \{x^* \in D: (x^*, y) \in C\} \text{ is of positive measure}\}.$$

It is clear that this set includes $\{y: (x, y) \in C\} \cap M[A]$ and is $M[A]$-Borel.

For partial real functions g and total real functions F, we say that F *almost extends* g if $\{x: g(x)$ and $F(x)$ are both defined and unequal$\}$ has measure 0.

LEMMA 5. *In N, let $g \in M[A]$, where $A \subset c$, be a partial function on $\mathbf{R}$. If there is a measurable function on all of $\mathbf{R}$ which almost extends g, then there is an $M[A]$-Borel function H on all of $\mathbf{R}$ which almost extends g. Furthermore, $\{x: g(x) \neq H(x)\}$ is of measure 0 from the point of view of $M[A]$.*

Proof. Let F almost extend g. Without loss of generality, we may assume that F is Borel. By Lemma 4, let B be an $M[A]$-Borel set of measure 0 which includes $\{x: g(x) \neq F(x)\}$.

By Lemma 3, let $G: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be an $M[A]$-Borel function and x be an $M[A]$-random real such that $(\lambda z)(G(x, z)) = F$. Since $\{z: G(x, z) \neq g(z)\} \subset B$, there is an $M[A]$-Borel set C with $x \in C$ such that for all $M[A]$-random reals $x^* \in C$, $\{z: G(x^*, z) \neq g(z)\} \subset B$.

Consider the partial function $H: \mathbf{R} \to \mathbf{R}$ given by $H(y) = z$ if and only if for almost all $x \in C$, $G(x, y) = z$. Then H is a partial $M[A]$-Borel function, and H extends g off of B. Extend H to any $M[A]$-Borel function on $\mathbf{R}$. This completes the proof.

We now fix $F: \mathbf{R} \times \mathbf{R}$ in N, and define $F_x(y) = F(x, y)$ and $F^x(y) = F(y, x)$. In N, we assume that for almost all x, F_x and F^x are both measurable. In N, let B be a Borel set of full measure (i.e., its complement is of measure 0) such that for all $x \in B$, F_x and F^x are measurable.

For each $x \in B \cap M$, choose a Borel function G_x and a Borel set C_x of measure 0 such that $(\forall x \notin C_x)(F_x = G_x)$. For each $x \in B \cap M$, choose a Borel code $g(x)$ for G_x and a Borel code $h(x)$ for C_x. These choices are to be made within N, and constitute c^M choices. Fix $A \subset c$ so that $g, h \in M[A]$ and B is $M[A]$-Borel.

LEMMA 6. *In N, let y be an $M[A]$-random real. Then $F(x, y) = G_x(y)$ for all $x \in B \cap M$. Hence $F^y \restriction B \cap M$ is in $M[A, y]$.*

Proof. Let $x \in B \cap M$. Since $\{y: F(x, y) \neq G_x(y)\}$ is included in the $M[A]$-Borel set C_x of measure 0, we must have $F(x, y) = G_x(y)$.

LEMMA 7. *In N, let y be an $M[A]$-random real. Then there is an $M[A, y]$-Borel function $H: \mathbf{R} \to \mathbf{R}$ which almost extends $F^y \restriction B \cap M$. Furthermore, $\{z: H(z) \neq F^y \restriction B \cap M(z)\}$ is included in some $M[A, y]$-Borel set of measure 0.*

Proof. By Lemma 6, $F^y \restriction B \cap M$ is in $M[A, y]$. Since $y \notin B$, F^y is measurable. Therefore, Lemma 5 applies.

LEMMA 8 (Solovay). *In N, $M \cap \mathbf{R}$ is of full outer measure, (i.e., its complement is of zero inner measure).*

Proof. Let B be a Borel subset of $[0, 1]$ which includes $M \cap \mathbf{R}$. As in Lemma 4, B is of the form $\{y: (x, y) \in C\}$ for some M-random real x and M-Borel set C. The statement $M \cap \mathbf{R} \subset \{y: (x, y) \in C\}$ holds in $M[x]$, and so holds for all M-random reals x^* in some M-Borel set D of positive measure. Hence, for each $y \in M$, $\{x^* \in D: (x^*, y) \notin C\}$ is of measure 0. So by Fubini's theorem, for almost all $x^* \in D$, $\{y: (x^*, y) \notin C\}$ is of measure 0. Hence $\{y: (x, y) \notin C\}$ is of measure 0, and so B is of full measure.

LEMMA 9. *In N, there is a function H_1 from $M[A]$-random reals into ω, and a function H_2 from $M[A]$-random reals into $M[A] \cap \mathscr{P}(\omega)$, such that for each $M[A]$-random real y, the set of integers which is arithmetic in $(y, H_2(y))$ with index $H_1(y)$ codes a Borel function which is almost equal to F^y. Furthermore, there are formulas $\phi_1(u, g, y, z)$ and $\phi_2(u, g, y, z)$ of set theory such that $H_i(y)$ is the unique z with $M[A, y] \neq \phi_i(u, g, y, z)$. Here u is a parameter from $M[A]$.*

Proof. Let u encode B as well as a well-ordering of $M[A] \cap \mathscr{P}(\omega)$ lying in $M[A]$. By Lemma 7, for $M[A]$-random reals y, choose $H_1(y)$ and $H_2(y)$ to be the least pair (n, z) in $M[A]$ such that the set of integers arithmetic in (y, z) with index n codes a Borel function which almost extends $F^y \restriction B \cap M$ from the point of view of $M[A, y]$. By Lemma 6, the formulas ϕ_1 and ϕ_2 exist. Observe that any Borel (in fact measurable) function which almost extends $F^y \restriction B \cap M$ must be almost equal to F^y. This follows from Lemma 8.

LEMMA 10. *In N, there is a partition of $\mathbf{R}$ into countably many Borel sets $\{E_n\}$ such that for each E_n, the functions H_1 and H_2 are constant on the $M[A]$-random reals in E_n. Furthermore, in N there is a countable ordinal λ such that for any $M[A]$-random real y, the Borel code which is arithmetic in $(y, H_2(y))$ with index $H_1(y)$ is of rank less than λ.*

Proof. This follows from the countable chain condition for random real forcing.

LEMMA 11. *In N, there is a Borel function J: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that for each $M[A]$-random real y, J^y is F^y almost everywhere.*

Proof. To compute $J(x, y)$, first compute the set of integers t which is arithmetic in $(y, H_2(y))$ with index $H_1(y)$. If t is a Borel code of rank less than λ and codes the Borel function t^*, let $J(x, y) = t^*(x)$. Otherwise, let $J(x, y) = 0$.

LEMMA 12 (Solovay). *In N, the intersection of c^M sets of full measure has full outer measure.*

Proof. Without loss of generality we may assume that the c^M sets of full measure are all Borel. Let B be any Borel set of positive measure. It suffices to prove that the intersection of the c^M Borel sets meets B. Let $A^* \subset c$ be such that B and the c^M Borel sets are all $M[A^*]$-Borel. By Lemma 2, there is an $M[A^*]$-random real y. Choose a nonempty interval $[a, b]$ such that $a, b \in M[A^*]$ and $B \cap [a, b]$ has positive measure. Choose an interval $[c, d]$ about y, again with $c, d \in M[A^*]$, and such that $d - c = \mu(B \cap [a, b])$. Let ρ be any measure preserving automorphism of $\mathbf{R}$ which maps $[c, d]$ onto $B \cap [a, b]$, and which is $M[A^*]$-Borel. Then $\rho(y)$ is an $M[A^*]$-random real, and so is an element of each of the c^M Borel sets as well as of B.

LEMMA 13. *In N, let $K \subset \mathbf{R} \times \mathbf{R}$ be such that $\{y: \{x: (x, y) \in K\}$ is of full measure$\}$ contains a set of power c^M of full outer measure. Then $\{x: \{y: (x, y) \in K\}$ is of full outer measure$\}$ is of full outer measure.*

Proof. Let S be of power c^M with full outer measure, and assume

$$S \subset \{y: \{x: (x, y) \in K\} \text{ is of full measure}\}.$$

Then by Lemma 12, the set $Q = \bigcap_{y \in S} \{x: (x, y) \in K\}$ is of full outer measure. Note that for $x \in Q$, $\{y: (x, y) \in K\}$ includes S. Hence for $x \in Q$, $\{y: (x, y) \in K\}$ is of full outer measure. Since Q is of full outer measure, we are done.

LEMMA 14. *In N, the set of all $M[A]$-random reals contains a subset which is of power c and of full outer measure.*

Proof. It is sufficient to observe that the $M[A]$-random reals are closed under translation by reals in M, and to invoke Lemma 8.

LEMMA 15. *In N, there is a Borel function J: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that $\{x: F_x = J_x$ a.e.$\}$ and $\{y: F^y = J^y$ a.e.$\}$ are both of full outer measure.*

Proof. By Lemma 11, let J be a Borel function such that for each $M[A]$-random real y, $J^y = F^y$ a.e. It remains to show that $\{x: F_x = J_x \text{ a.e.}\}$ is of full outer measure. Let K be $\{(x, y): J(x, y) = F(x, y)\}$.

We are given that $\{y: \{x: (x, y) \in K\}$ is of full measure$\}$ includes all $M[A]$-random reals. Hence by Lemma 14, the hypothesis of Lemma 13 applies. Therefore

$$\{x: \{y: (x, y) \in K\} \text{ is of full outer measure}\}$$

is of full outer measure. Since for almost all x, F_x is measurable, we see that for almost all x, $\{y: (x, y) \in K\}$ is measurable. Hence $\{x: \{y: (x, y) \in K\}$ is of full measure$\}$ is of full outer measure, and we are done.

Lemma 15 is of independent interest, and we single it out as a theorem.

THEOREM 1. *The following holds in N. Let F: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be such that for almost all x, F_x and F^x are measurable. Then there is a Borel function J: $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that $\{x: J_x = F_x$ a.e.$\}$ and $\{y: J^y = F^y$ a.e.$\}$ are both of full outer measure.*

THEOREM 2. *The following Fubini-Tonelli theorem for nonmeasurable functions holds in N. Let* $F\colon \mathbf{R} \times \mathbf{R} \to [0, \infty)$ *be such that for almost all x,* F_x *and* F^x *are measurable. Furthermore, assume that* $(\lambda x)(\int F_x)$ *and* $(\lambda y)(\int F^y)$ *are both measurable (on their respective domains of full measure) as extended real valued functions. Then* $\int (\lambda x)(\int F_x) = \int (\lambda y)(\int F^y)$.

Proof. By Theorem 1, choose J to be a Borel function such that

$$\{x\colon J_x = F_x \text{ a.e.}\} \quad \text{and} \quad \{y\colon J^y = F^y \text{ a.e.}\}$$

are both of full outer measure. Then $\int F_x = \int J_x$ for all x in a set of full outer measure, and $\int F^y = \int J^y$ for all y in a set of full outer measure. Since $(\lambda x)(\int F_x)$ and $(\lambda y)(\int F^y)$ are measurable functions, we see that $\int F_x = \int J_x$ for almost all x, and $\int F^y = \int J^y$ for almost all x. In other words

$$(\lambda x)\left(\int F_x\right) = (\lambda x)\left(\int J_x\right) \text{ a.e.} \quad \text{and} \quad (\lambda y)\left(\int F^y\right) = (\lambda y)\left(\int J^y\right) \text{ a.e.}$$

Therefore $\int (\lambda x)(\int F_x) = \int (\lambda x)(\int J_x)$, and $\int (\lambda y)(\int F^y) = \int (\lambda y)(\int J^y)$. By the Fubini-Tonelli theorem, $\int (\lambda x)(\int J_x) = \int (\lambda y)(\int J^y)$, and hence

$$\int (\lambda x)\left(\int F_x\right) = \int (\lambda y)\left(\int F^y\right).$$

REFERENCES

1. D. MAHARAM, *On homogeneous measure algebras*, Proc. Nat. Acad. Sci. U.S.A., vol. 28 (1942), pp. 108–111.
2. R. SOLOVAY, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math., vol. 92 (1970), pp. 1–56.
3. ———, "Real valued measurable cardinals" in *Axiomatic set theory*, Amer. Math. Soc. Symposia, XIII, 1971, pp. 397–428.

OHIO STATE UNIVERSITY
COLUMBUS, OHIO