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INVARIANCE PROPERTIES OF FINITELY ADDITIVE MEASURES IN R^n

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0. Introduction

After the existence of a non-Lebesgue measurable set had been proved, the problem of the existence of a total (i.e., defined for all sets), finitely additive, congruence invariant measure (with values in $[0, \infty]$) in $\mathbf{R}^n$ was considered. By congruence (or isometry) of $\mathbf{R}^n$ we mean a distance-preserving bijection from $\mathbf{R}^n$ to $\mathbf{R}^n$; in $\mathbf{R}^1$ there are only translations and reflections about a fixed point. In 1914, Hausdorff [9, p. 469] (see also [21, pp. 74 and 97]) constructed a paradoxical decomposition of the 2-sphere (a construction central to the more well-known Banach-Tarski paradox [3] (see also [24])) which implies that no such measure exists in $\mathbf{R}^n$ if $n \geq 3$. And in 1923, Banach [2] proved that such measures do indeed exist in $\mathbf{R}$ and $\mathbf{R}^2$, and can be chosen to extend Lebesgue measure. To prove this for $\mathbf{R}$, Banach developed the ideas of the Hahn-Banach Theorem to extend Lebesgue measure to a total, finitely additive, translation invariant measure ν. Then it suffices to define $\mu(A) = \frac{1}{2}(\nu(A) + \nu(-A))$. (See [19, p. 193] or [12, p. 359] for details.) The measure in $\mathbf{R}^2$ was obtained by applying a clever integration technique to the measure in $\mathbf{R}$.

The modern approach to these classical theorems of Banach uses the notion of an amenable group, invented by von Neumann [17]. In that paper von Neumann realized that Banach's techniques generalize to any group bearing an appropriate measure; such groups are called amenable. This yielded total measures in higher dimensions: if G is an amenable group of congruences of $\mathbf{R}^n$, then a total, finitely additive, G-invariant extension of Lebesgue measure exists.

Banach's theorem for $\mathbf{R}$ raises the following question, which motivated the work of this paper. Can a total, finitely additive extension of Lebesgue measure be invariant under translations, but not reflections? More generally, letting Inv (μ), for μ a total finitely additive extension of Lebesgue measure in $\mathbf{R}^n$, be the group of congruences with respect to which μ is invariant, we have the question of which groups arise as Inv (μ), for some μ. The following theorem, which is proved in Section 3, gives a necessary condition for a group to be realized in this way, which is applicable to most of the interesting cases (in particular, it follows that all groups of congruences of $\mathbf{R}$ or $\mathbf{R}^2$ are realizable).

THEOREM 1. *If G is an amenable group of congruences of $\mathbf{R}^n$ then there is a*

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total, finitely additive measure μ *in* $\mathbf{R}^n$ *which extends Lebesgue measure, and is such that* Inv $(\mu) = G$.

The converse to this theorem is true as well, with a further assumption on G (see Theorem 5).

Since any group of congruences in $\mathbf{R}$ or $\mathbf{R}^2$ is amenable, this shows that, in dimensions 1 and 2, all groups are realizable in the sense described above. Note that, in other contexts, things may turn out quite differently in that invariance with respect to one group is sufficient to imply invariance with respect to a larger group. For instance, one may ask whether a countably additive, translation invariant measure on the Borel subsets of $\mathbf{R}$ that assigns measure one to the unit interval is necessarily invariant under reflections. In this case the answer is yes, because Lebesgue measure is the only such measure.

There are still some unresolved problems related to the classical theory. Foremost is the problem of formulating a converse to von Neumann's theorem, e.g., if G is a group of isometries of the n-sphere ($\mathbf{R}^n$) and a finitely additive G-invariant measure on the n-sphere ($\mathbf{R}^n$) exists, then what can be said about G? Must G be amenable? Part of the importance of this question stems from two results, a recent theorem of Tits [27] and a classical result of Tarski [25]. The former provides an excellent characterization of amenable groups of Euclidean isometries (namely, such a group is amenable iff it has no free non-Abelian subgroup) while the latter asserts that the existence of a G-invariant measure on the n-sphere is equivalent to the non-existence of a paradoxical decomposition of the sphere using transformations in G. Thus the lack of a converse to von Neumann's theorem is a major gap in our knowledge of which groups of isometries of the n-sphere allow a paradoxical decomposition to be constructed.

After summarizing the classical theory in Section 1, we prove, in Section 2, two results relating to these problems. First, we show that if a G-invariant measure on S^{n-1} extends surface Lebesgue measure, then G must be amenable. Then we show how the extraneous condition on Lebesgue measure may be eliminated if $n = 3$, and speculate on the possibilities for $n > 3$.

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1. Amenable groups

For a group G let $B(G)$ denote the set of bounded functions from G to $\mathbf{R}$.

Definition. A group G is *amenable* if there exists a left-invariant mean on $B(G)$, i.e., a function $M\colon B(G) \to \mathbf{R}$ such that:

(1) $M(af + bg) = aM(f) + bM(g)$ for all $a, b \in \mathbf{R}$ and $f, g \in B(G)$.

(2) $\inf\{f(\sigma)\colon \sigma \in G\} \le M(f) \le \sup\{f(\sigma)\colon \sigma \in G\}$.
(3) $M(f) = M({}_\tau f)$ for any $\tau \in G$ where ${}_\tau f(\sigma) = f(\tau^{-1}\sigma)$.

The following theorem states some characterizations of amenable groups; there are others (see [4], [5], [7], [11] for further equivalences and more information on amenability and invariant means).

THEOREM 2. *For a group G the following are equivalent:*

(a) *G is amenable.*
(b) *There is a finitely additive measure μ defined on all subsets of G such that $\mu(G) = 1$ and $\mu(A) = \mu(\sigma A)$ for all $A \subseteq G$, $\sigma \in G$.*
(c) *G does not admit a paradoxical decomposition, i.e., there do not exist pairwise disjoint sets $A_1, \ldots, A_r, B_1, \ldots, B_m \subseteq G$ and $\rho_1, \ldots, \rho_r, \sigma_1, \ldots, \sigma_m \in G$ such that $G = \rho_1 A_1 \cup \cdots \cup \rho_r A_r = \sigma_1 B_1 \cup \cdots \cup \sigma_m B_m$.*
(d) *If V is a vector space with subspace V_0, G is a group of linear operators on V such that $\sigma(V_0) \subseteq V_0$ for each $\sigma \in G$, $p\colon V \to \mathbf{R}$ is a positive-homogeneous subadditive functional on V satisfying $p(\sigma v) \le p(v)$ for $\sigma \in G$, $v \in V$, and $F\colon V_0 \to \mathbf{R}$ is a G-invariant linear functional on V_0 that is dominated by p, then F has a G-invariant extension to a linear functional $\bar{F}$ on all of V such that $\bar{F}(v) \le p(v)$ for all $v \in V$.*

Proof. (a) $\Leftrightarrow$ (b). Condition (b) is the original definition of amenability used by von Neumann [17]. For the forward implication use characteristic functions: the reverse is just the familiar construction of an integral from a measure.

(b) $\Leftrightarrow$ (c). This striking equivalence follows from a more general theorem, due to Tarski [25], which we discuss below. The forward implication is easy (see Theorem 3(b) below), as the measure precludes the existence of a paradoxical decomposition. The reverse implication is rather difficult.

(a) $\Leftrightarrow$ (d). Condition (d) is based on the work of Banach [2], who pioneered the investigation of invariant linear functionals. In this generality the forward implication is due to Agnew and Morse [1] and Silverman [23] (the proof consists of an application of the Hahn-Banach Theorem, followed by an averaging procedure similar to that used in the proof of Theorem 8 below). The reverse implication is first stated in Silverman [22] (let $V = B(G)$, $V_0 = \{f \in V\colon f$ is constant$\}$, $p(f) = \sup f$, and $F(f) = c$ if $f \in V_0$ has constant value c).

Part (a) of the following theorem contains the first important result using the general notion of amenability. Part (b) shows that a paradoxical decomposition is the only obstacle to the existence of an invariant measure. For the rest of this paper we use "measure" or "total measure" to mean a finitely additive measure with values in $[0, \infty]$ which is defined for all subsets of the set in question.

THEOREM 3. *Let G be a group acting on a set S.*

(a) (von Neumann [17]) *If G is amenable then S bears a G-invariant measure of total measure one.*

(b) (Tarski [25, Satz 3.8; 26, Theorem 16.12]) *S bears a G-invariant measure of total measure one iff S admits no paradoxical decomposition using mappings in G, i.e., there do not exist pairwise disjoint sets $A_1, \ldots, A_r, B_1, \ldots, B_m \subseteq S$ and $\rho_1, \ldots, \rho_r, \sigma_1, \ldots, \sigma_m \in G$ such that*

$$S = \bigcup \rho_i A_i = \bigcup \sigma_i B_i.$$

Proof. (a) Condition (d) of Theorem 2 may be used to construct a G-invariant linear functional on the space of bounded real-valued functions on S, which assigns value 1 to the function with constant value 1. This functional then induces, via characteristic functions, an appropriate measure.

(b) The forward implication is easy, for suppose μ is a G-invariant measure with $\mu(S) = 1$, and a paradoxical decomposition exists. Let $d_1 = \mu(\cup A_i)$, $d_2 = \mu(\cup B_i)$. Then $d_1 + d_2 \leq \mu(S) = 1$, but

$$1 = \mu(S) = \mu(\cup \rho_i A_i) \leq \sum \mu(\rho_i A_i) = \sum \mu(A_i) = d_1.$$

Similarly $1 \leq d_2$, a contradiction. The reverse implication is an intricate argument using results from the theory of decomposition types.[1] Slightly weaker versions of this result have been rediscovered recently [6], [20]. Emerson's approach [6] is noteworthy since he uses the von Neumann-Dixmier criterion for amenability to obtain his results. Moreover, since the cancellation law stated on page 226 of [6] is in fact valid in the discrete case (see [13, p. 131]; this has been observed independently by Emerson), so is Conjecture 3.1 of [6], from which the full theorem of Tarski follows.

The problem of formulating and proving a converse to Theorem 3(a) is intriguing (see Greenleaf [8, p. 18]). The straightforward converse fails, since if there are any $s \in S$ that are fixed by the action of G, then any measure on the set of such points s is G-invariant, whether or not G is amenable. The following proposition, the method of proof of which appears in von Neumann [17, p. 99], gives a condition on the action which is sufficient for the converse to hold. It appears in [8] for actions assumed, in addition, to be transitive.

PROPOSITION 1. *If G acts on S freely, i.e. $gs \neq s$ for any $s \in S, g \in G$-$\{1\}$, then G is amenable iff there is a G-invariant measure μ on S with $\mu(S) = 1$.*

Proof. By Theorem 3(a) we need only prove the reverse direction, so let μ be a G-invariant measure on S. Let $E \subseteq S$ be a choice set for the orbits of the action and define $\psi: S \to G$ by setting $\psi(s) = \sigma$, where $s \in \sigma(E)$. The freeness of the action implies that ψ is well defined; moreover, $\psi(\sigma s) = \sigma\psi(s)$. Now, it is easy to check that the measure ν on G defined by $\nu(A) = \mu(\psi^{-1}(A))$ is left-invariant, and hence G is amenable.

We now summarize some more elementary facts on the structure of amenable groups.

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[1] Jan Mycielski (*Finitely additive invariant measures* (*III*), to appear) has recently obtained a very neat and short proof of Tarski's theorem.

THEOREM 4.

(a) (Banach [2]) *Abelian groups are amenable.*
(b) (von Neumann [17]) *Finite groups are amenable. If H is a normal subgroup of G, and H, G/H are both amenable, then G is amenable; thus any solvable group is amenable. A free group on two generators is not amenable.*
(c) (Day [4], Følner [7]) *A subgroup of an amenable group is amenable.*
(d) (Day [4]) *A group is amenable iff all of its finitely generated subgroups are amenable.*

These results suggest the following conjecture, which is still unresolved.

DAY'S CONJECTURE. A group is amenable iff it does not contain, as a subgroup, a free group on two generators.

However, it is a consequence of the main theorem of Tits [27] that this conjecture is valid for any matrix group over a field. For connections between Day's Conjecture and the Burnside Problem is group theory, see Mycielski [15], [16].

2. Amenability and Euclidean spaces

DEFINITION. Let G_n denote all congruences (i.e., distance-preserving bijections) in $\mathbf{R}^n$. Let $\mathcal{O}_n$ be the subgroup of G_n consisting of those $\sigma \in G_n$ that fix the origin; $\mathcal{O}_n$ consists of all orthogonal linear transformations of $\mathbf{R}^n$. Each congruence is an affine map (see [10, p. 307]) and so is a composition of a linear map and a translation. Hence we may define a canonical homomorphism $\pi: G_n \to \mathcal{O}_n$ by $\pi(\sigma) = l$ where $\sigma = \tau l$ for some translation τ. The identity in G_n or any other group is denoted by 1.

It is easy to see that G_1 and G_2 are solvable, and hence amenable. The next theorem summarizes many of the known results for amenability of groups of congruences and contains two new results ($c \to f$ and $b \to c$). If μ is a measure on $\mathbf{R}^n$, Inv (μ, G_n) denotes $\{\sigma \in G_n\colon \mu \text{ is } \sigma\text{-invariant}\}$; similarly we use Inv $(\mu, \mathcal{O}_n)$. We use m to denote Lebesgue measure as well as the surface measure on the sphere induced by Lebesgue measure (where a set A is assigned the Lebesgue measure of $\{\alpha x\colon x \in A, 0 \le \alpha \le 1\}$, suitably normalized so the whole surface has measure one). Finally F_2 is used to denote the (isomorphism class of the) free group on two generators.

THEOREM 5. *Let G be a subgroup of $\mathcal{O}_n$. Then the following are equivalent (and are all true if $n = 1$ or 2).*

(a) *There is a total G-invariant measure in $\mathbf{R}^n$ that extends Lebesgue measure.*
(b) *There is a total, G-invariant measure on S^{n-1}, the surface of the unit sphere in $\mathbf{R}^n$, that extends surface Lebesgue measure.*

(c) *G is amenable.*
(d) *G admits no paradoxical decomposition (see Theorem* 2(c)).
(e) *G has no free subgroup on two generators.*
(f) *There is a total measure μ on $\mathbf{R}^n$ such that μ extends Lebesgue measure and* Inv $(\mu, \mathcal{O}_n) = G$.

Proof. (a) $\Rightarrow$ (b). The measure on $\mathbf{R}^n$ induces one on S^{n-1} by the adjunction of radii, as described above.

(b) $\Rightarrow$ (c). Suppose G is not amenable. Then by Theorem 4(d), some finitely generated (and hence countable) subgroup H of G fails to be amenable. Let

$$D = \{x \in S\colon \text{there exists } \sigma \in H\text{-}\{1\} \text{ such that } \sigma(x) = x\}.$$

Since the fixed points (in $\mathbf{R}^n$) of any $\sigma \in \mathcal{O}_n$-$\{1\}$ form a linear subspace of $\mathbf{R}^n$ of dimension at most $n - 1$, and since H is countable, it follows that $m(D) = 0$.

Now, let μ be a measure as in (b). Then $\mu(D) = 0$, so $\mu(S - D) = 1$. But H acts freely on $S - D$, so Proposition 1 implies that H is amenable, a contradiction.

(c) $\Rightarrow$ (d). This is due to Tarski and was stated in Theorem 2 above.

(c) $\Rightarrow$ (e). This follows from Theorem 4. To see that the free group F_2, generated by a, b is not amenable, let $W(a)$ consist of those (reduced) words beginning with a; similarly define $W(a^{-1})$, $W(b)$, and $W(b^{-1})$. These four sets are pairwise disjoint, and $F_2 = a^{-1}W(a) \cup W(a^{-1}) = b^{-1}W(b) \cup W(B^{-1})$, whence F_2 fails to satisfy (d) and so is not amenable.

(e) $\Rightarrow$ (c). This follows from Theorem 4 and Tits' proof [27] of the conjecture of Bass and Serre that a group of matrices over a field of characteristic 0 either has a free subgroup on two generators, or has a normal solvable subgroup of finite index.

(c) $\Rightarrow$ (f). This is just Theorem 1, to be proved below.

(f) $\Rightarrow$ (a). Trivial.

There are two directions for possible generalizations of this theorem. First, assume G is a subgroup of G_n. We then have (c) $\Leftrightarrow$ (d) $\Rightarrow$ (e) (which is true for all groups) and the trivial implications (f) $\Rightarrow$ (a) $\Rightarrow$ (b). Theorem 1 shows that (c) $\Rightarrow$ (f) is valid here, and (e) $\Rightarrow$ (c) follows from the previous case (if G satisfies (e), so does $\pi(G)$; hence $\pi(G)$ is amenable, and since Ker π, which consists of translations, is Abelian and $G/\text{Ker } \pi \cong \pi(G)$, G is amenable too). We are left with the question of whether the existence of a measure as in (a) implies that G is amenable. So long as this is unknown, we can ask whether Theorem 1 can be improved to yield (a) $\Rightarrow$ (f), rather than the ostensibly weaker (c) $\Rightarrow$ (f).

Or, we may remove the requirement that the measures extend Lebesgue measure (an assumption which was central in the proof of (b) $\Rightarrow$ (c)). Note that, by Theorem 3(b), this question is equivalent to asking for which subgroups G of $\mathcal{O}_n$ does S^{n-1} admit a paradoxical decomposition using transformations in G. Hausdorff's original theorem [9, p. 469] (combined with Tarski's Theorem 3(b)) implies that such decompositions exist if $G = \mathcal{O}_n^+$, where $n \geq 3$. Moreover, von Neumann observed [17, p. 80] that Hausdorff's proof implies that $\mathcal{O}_n^+$, $n \geq 3$, has a subgroup isomorphic to F_2. It is tempting, in view of Theorem 5, to conjecture that containment of F_2 is a (necessary and) sufficient condition for the existence of a paradoxical decomposition. However, a simple counterexample shows that, for $n \geq 4$, this is not the case.

Choose $\sigma, \tau \in \mathcal{O}_3^+$ to be a free basis for the group they generate and define σ', $\tau' \in \mathcal{O}_n^+$ $(n \geq 4)$ to be extensions of σ, τ which are equal to the identity in dimensions other than the first three. Then $G = \langle \sigma', \tau' \rangle$ is isomorphic to F_2, but G leaves (pointwise) fixed the points on S^{n-1} beginning with three zeros. Thus any total measure on this latter set (e.g., one giving measure one to sets containing a fixed point) is G-invariant.

However, for $n = 3$, containment of F_2 does characterize the groups with respect to which S^2 admits a paradoxical decomposition. In order to prove this, we first state and prove a modification of a theorem due to von Neumann [17] (see also [8, p. 18]).

THEOREM 6. *If a group G acts on a set S, and G has finitely many nonamenable subgroups $H_1, \ldots, H_n$ with the property that for no $s \in S$ do there exist $h_i \in H_i$-$\{1\}$, $i \leq n$, such that $h_i(s) = s$, then there is no total G-invariant measure on S.*

Proof. Suppose μ is a G-invariant measure on S. Let D_i consist of the points in S that are fixed by a non-identity element of H_i. Then the hypothesis ensures that $S = (S - D_1) \cup (S - D_2) \cup \cdots \cup (S - D_n)$, whence $\mu(S - D_i) > 0$ for some $i \leq n$. But H_i acts freely on $S - D_i$, which contradicts Proposition 1.

THEOREM 7. *For $G \leq \mathcal{O}_3$ the following are equivalent:*

(a) *G is amenable (equivalently, G has no subgroup isomorphic to F_2).*
(b) *There is a total G-invariant measure on S^2 (equivalently, S^2 admits no paradoxical decomposition using transformations in G).*

Proof. (a) $\Rightarrow$ (b). See Theorem 3(a) above.

(b) $\Rightarrow$ (a).[2] First, assume $G \leq \mathcal{O}_3^+$, i.e., G consists of rotations of S^2 about an axis through the origin. We prove the contrapositive. Suppose G is not amenable; then, by Theorem 5, there are two rotations $\sigma, \tau \in G$ such that σ, τ form a free basis of $\langle \sigma, \tau \rangle$, which is isomorphic to F_2. It is easy to see that $\sigma\tau$, $\sigma^2\tau^2$, $\sigma^3\tau^3$, $\sigma^4\tau^4$ freely generate a free group of rank 4 (see [14, p. 43, example 12]), and hence

$$H_1 \cong H_2 \cong F_2 \quad \text{where } H_1 = \langle \sigma\tau, \sigma^2\tau^2 \rangle \text{ and } H_2 = \langle \sigma^3\tau^3, \sigma^4\tau^4 \rangle.$$

[2] T. J. Dekker has informed the author that (b) $\Rightarrow$ (a) of Theorem 7 follows from the methods of T. J. Dekker and J. de Groot, *Decompositions of a sphere*, Fund. Math., vol. 43 (1956), pp. 185–194. Actually, their methods show that if $F_2 \leq G$ then S^2 admits a paradoxical decomposition with respect to G. My proof above, that if $F_2 \leq G$ then S^2 bears no G-invariant measure, is simpler, but then Tarski's theorem must be used to conclude that a paradoxical decomposition exists.

Moreover, no element of H_1 commutes with any element of H_2, unless one of them is the identity. It follows that no rotation in H_1-$\{1\}$ has the same axis as a rotation in H_2-$\{1\}$, whence the fixed points, on S^2, of H_1-$\{1\}$ and H_2-$\{1\}$ are disjoint. Thus Theorem 6 implies that S^2 bears no G-invariant measure.

If $G \leq \mathcal{O}_3$ then $G \cap \mathcal{O}_3^+$ is the kernel of the homomorphism into $\{-1, +1\}$ given by the determinant map. If there is a G-invariant measure, then it is $G \cap \mathcal{O}_3^+$-invariant, whence, by the above paragraph, $G \cap \mathcal{O}_3^+$ is amenable. Since $G/G \cap \mathcal{O}_3^+$ is finite, it follows from Theorem 4 that G is amenable.

In light of Theorems 2 and 3(b), one may interpret this theorem as stating that S^2 admits a paradoxical decomposition under the action of G iff G admits a paradoxical decomposition under the (natural) action of G on itself.

The counterexample to this result for $n \geq 4$ given above suggests the following conjecture:

CONJECTURE. Suppose $G \leq \mathcal{O}_n$ is such that whenever $T \subseteq S^{n-1}$ is closed under the action of G then $G \mid T$ (i.e., $\{\sigma \mid T\colon \sigma \in G\}$) is nonamenable (equivalently, contains F_2). Then S^{n-1} bears no G-invariant measure.

We conclude this section with a result which isolates the use of amenability in the proof of Theorem 1 ((c)$\Rightarrow$(f) of Theorem 5).

THEOREM 8. *Suppose $\mathcal{A}$ is a G_n-invariant family of subsets of $\mathbf{R}^n$* (i.e., *$\sigma(A) \in \mathcal{A}$ whenever $A \in \mathcal{A}$, $\sigma \in G_n$*), *G is an amenable subgroup of G_n, and ν is a total measure on $\mathbf{R}^n$ with* $\operatorname{Inv}(\nu \mid \mathcal{A}, G_n) = G$. *Then there is a total measure μ on $\mathbf{R}^n$ such that $\mu \mid \mathcal{A} = \nu \mid \mathcal{A}$ and* $\operatorname{Inv}(\mu, G_n) = G$.

Proof. Let M be a left-invariant mean on $B(G)$. For $C \subseteq \mathbf{R}^n$, define $f^C\colon G \to [0, \infty]$ by $f^C(\sigma) = \nu(\sigma^{-1}C)$. Define μ by setting $\mu(C) = M(f^C)$ if f^C is bounded, and hence in $B(G)$, and letting $\mu(C) = \infty$ otherwise. Then μ is a total measure and, since $f^{\tau C} = {}_\tau(f^C)$, the G-invariance of μ follows from the left-invariance of M. Moreover, if $A \in \mathcal{A}$ then f^A is a constant function with value $\nu(A)$; therefore μ extends $\nu \mid \mathcal{A}$. Finally, since $\nu \mid \mathcal{A}$ is not invariant under any congruence not in G, the same is true of μ, whence $\operatorname{Inv}(\mu, G_n) = G$ as required.

3. Proof of Theorem 1

In order to produce a family $\mathcal{A}$ as in Theorem 8, we need a set A such that $\{\chi_{\sigma A}\colon \sigma \in G_n\}$ is linearly independent over the vector space of Lebesgue measurable functions. Not every non-measurable set has this property, for if $A \subseteq [0, 1]$ is non-measurable and symmetric about $\frac{1}{2}$, then $\chi_A - \chi_{1-A} = 0$. To obtain A we prove the following lemma, which generalizes the construction of a Bernstein set (which is a set of reals such that it and its complement meets every uncountable closed set of reals—see [18, p. 23]).

In what follows, $\mathfrak{c}$ denotes $2^{\aleph_0}$, the cardinality of the continuum, m^* denotes Lebesgue outer measure, and I^n denotes $[0, 1]^n$, the unit cube in $\mathbf{R}^n$. If $\bar{\sigma} \subseteq G_n$, we use $\bigcup \bar{\sigma}(A)$ to denote $\bigcup \{\sigma(A): \sigma \in \bar{\sigma}\}$; similarly for $\bigcap \bar{\sigma}(A)$. Finally, $\bar{\sigma}^*$ denotes $\bar{\sigma} \cup \{1\}$.

LEMMA. *There is a set $A \subseteq I^n$ such that for any closed subset C of I^n with $m(C) > 0$, and any disjoint, finite (possibly empty) subsets $\sigma, \rho \subseteq G_n\text{-}\{1\}$,*

(i) $(C - A) - \bigcup \bar{\rho}(A) \equiv \emptyset$,
(ii) *if $C \subseteq \bigcap \bar{\sigma}(I^n)$, then $C \cap \bigcap \bar{\sigma}^*(A) - \bigcup \bar{\rho}(A) \neq \emptyset$.*

Proof. Let $\{(C_\alpha, \bar{\sigma}_\alpha, \bar{\rho}_\alpha): \alpha < \mathfrak{c}\}$ enumerate all triples $(C, \bar{\sigma}, \bar{\rho})$ such that $C \subseteq I^n$ is a closed set of positive Lebesgue measure, and $\bar{\sigma}, \bar{\rho}$ are disjoint, finite subsets of $G_n\text{-}\{1\}$. Such an enumeration exists because there are only $\mathfrak{c}$ closed sets (see [18, p. 23]) and because $|G_n| = \mathfrak{c}$ (there are only $\mathfrak{c}$ affine transformations on $\mathbf{R}^n$). Note that, by the Cantor-Bendixson Theorem (see [12, p. 72]), any uncountable closed set, and hence each C_α, has cardinality $\mathfrak{c}$. For each $\alpha < \mathfrak{c}$, we shall define, by induction, two members of C_α: x_α and y_α. However, if C_α is not contained in $\bigcap \bar{\sigma}_\alpha(I^n)$, then x_α will not be defined. The set A will then be defined as $\{\tau^{-1}(x_\alpha): x_\alpha \text{ is defined}, \tau \in \bar{\sigma}^*_\alpha, \alpha < \mathfrak{c}\}$ and will be disjoint from $\{y_\alpha: \alpha < \mathfrak{c}\}$. Conditions (i), (ii) will be verified by y_α, x_α respectively.

For $\alpha = 0$, if $C_0 \subseteq \bigcap \bar{\sigma}_0(I^n)$, choose $x_0 \in C_0$ so that $x_0 \neq \rho\tau^{-1}(x_0)$ for any $\rho \in \bar{\rho}_0$, $\tau \in \bar{\sigma}^*_0$. Choose $y_0 \in C_0$ so that $y_0 \neq \rho\tau^{-1}(x_0)$ for any $\rho \in \bar{\rho}^*_0, \tau \in \bar{\sigma}^*_0$; if x_0 is undefined, any $y_0 \in C_0$ will do.

In general, if $C_\alpha \subseteq \bigcap \bar{\sigma}_\alpha(I^n)$ then choose $x_\alpha \in C_\alpha$ to satisfy:

(1) $x_\alpha \neq \tau\rho^{-1}(y_\beta)$, any $\beta < \alpha$, $\rho \in \bar{\rho}^*_\beta$, $\tau \in \bar{\sigma}^*_\alpha$.
(2) $x_\alpha \neq \rho\tau^{-1}(x_\beta)$, any $\beta < \alpha$, $\rho \in \bar{\rho}_\alpha$, $\tau \in \bar{\sigma}^*_\beta$.
(3) $x_\alpha \neq \rho\tau^{-1}(x_\alpha)$, any $\rho \in \bar{\rho}_\alpha$, $\tau \in \bar{\sigma}^*_\alpha$.
(4) $x_\alpha \neq \tau\rho^{-1}(x_\beta)$, any $\beta < \alpha$, $\rho \in \bar{\rho}_\beta$, $\tau \in \bar{\sigma}^*_\alpha$.

Then, even if x_α is undefined, choose $y_\alpha \in C_\alpha$ satisfying:

(5) $y_\alpha \neq \rho\tau^{-1}(x_\beta)$, any $\beta \leq \alpha$, $\rho \in \bar{\rho}^*_\alpha$, $\tau \in \bar{\sigma}^*_\beta$.

To see that these choices can always successfully be made, note that all conditions except condition (3) exclude $\mathfrak{c}$ possibilities from C_α which, as noted above, has cardinality $\mathfrak{c}$. To handle condition (3), note that the points of C_α which are not a fixed point of any congruence $\rho\tau^{-1}$ with $\rho \in \bar{\rho}_\alpha, \tau \in \bar{\sigma}^*_\alpha$ have the same Lebesgue measure as C_α (see proof of (b) $\Rightarrow$ (c) of Theorem 5), and hence contain a closed set D with $m(D) > 0$. Since $|D| = \mathfrak{c}$, the desired point x_α may be found in D.

Now, if

$$A = \{\tau^{-1}(x_\alpha): x_\alpha \text{ is defined}, \tau \in \bar{\sigma}^*_\alpha, \alpha < \mathfrak{c}\},$$

then, since $x_\alpha \in C_\alpha \subseteq \bigcap \bar{\sigma}^*_\alpha(I^n)$, each $\tau^{-1}(x_\alpha) \in I^n$ and $A \subseteq I^n$. Moreover, A satisfies (i) and (ii). For suppose $C, \bar{\sigma}, \bar{\rho}$ as in the lemma are given. Choose $\alpha < \mathfrak{c}$ such that $C = C_\alpha$, $\bar{\sigma} = \bar{\sigma}_\alpha$, and $\bar{\rho} = \bar{\rho}_\alpha$. For (i), note that $y_\alpha \in C$. If $y_\alpha \in \bigcup \bar{\rho}^*_\alpha(A)$, then either $y_\alpha = \rho\tau^{-1}(x_\beta)$ for some $\beta \leq \alpha$, $\rho \in \bar{\rho}^*_\alpha$, $\tau \in \bar{\sigma}^*_\beta$, contradicting condition (5), or $y_\alpha = \rho\tau^{-1}(x_\beta)$ for some $\beta > \alpha$, $\rho \in \bar{\rho}^*_\alpha$, $\tau \in \bar{\sigma}^*_\beta$, contradicting (1). For (ii), if $C \subseteq \bigcap \bar{\sigma}(I^n)$ then x_α is defined and lies in $C \cap \bigcap \bar{\sigma}^*(A)$, since for any $\tau \in \bar{\sigma}^*$, $\tau^{-1}(x_\alpha) \in A$ whence $x_\alpha \in \tau(A)$. Finally, if $x_\alpha \in \bigcup \bar{\rho}(A)$ than either $x_\alpha = \rho\tau^{-1}(x_\beta)$ for some $\beta < \alpha$, $\rho \in \bar{\rho}$, $\tau \in \bar{\sigma}^*_\beta$, contradicting (2); or $x_\alpha = \rho\tau^{-1}(x_\alpha)$ for some $\rho \in \bar{\rho}$, $\tau \in \bar{\sigma}^*$, contradicting (3); or $x_\alpha = \rho\tau^{-1}(x_\beta)$ for some $\beta \geq \alpha$, $\rho \in \bar{\rho}$, $\tau \in \bar{\sigma}^*_\beta$, contradicting (4). This completes the proof.

COROLLARY. *If A is as in the lemma, then for any finite disjoint sets $\bar{\sigma}, \bar{\rho} \subseteq G_n$,*

(1) $m^*(\bigcap \bar{\sigma}(A) - \bigcup \bar{\rho}(A)) = m(\bigcap \bar{\sigma}(I^n))$, *and*

(2) $m^*(\bigcup \bar{\rho}(I^n) - \bigcup \bar{\rho}(A)) = m(\bigcup \bar{\rho}(I^n))$.

Proof. Let A be as in the lemma. To show (1), assume $\bar{\sigma} \neq \emptyset$, for otherwise (1) reduces to $\infty = \infty$. Now suppose $\bigcap \bar{\sigma}(A) - \bigcup \bar{\rho}(A) \subseteq E$ where E is open and $m(E) < m(\bigcap \bar{\sigma}(I^n))$. Let $C = \bigcap \bar{\sigma}(I^n) - E$ and choose any $\sigma \in \bar{\sigma}$. Then $\sigma^{-1}(C) \subseteq I^n$ and so we may apply property (ii) of the lemma to

$$\sigma^{-1}(C), \quad \{\sigma^{-1}\tau: \tau \in \bar{\sigma} - \{\sigma\}\} \quad \text{and} \quad \{\sigma^{-1}\rho: \rho \in \bar{\rho}\}$$

(for $C, \bar{\sigma}, \bar{\rho}$) to get a point x. Then $\sigma(x) \in C$, $\sigma(x) \in \bigcap \bar{\sigma}(A)$ and $\sigma(x) \notin \bigcup \bar{\rho}(A)$, a contradiction to $C \cap E = \emptyset$.

For (2), assume $\bar{\rho} \neq \emptyset$, for otherwise (2) reduces to $0 = 0$. Suppose

$$\bigcup \bar{\rho}(I^n) - \bigcup \bar{\rho}(A) \subseteq E$$

where E is open and $m(E) < m(\bigcup \bar{\rho}(I^n))$. Let $C = \bigcup \bar{\rho}(I^n) - E$, choose any $\rho \in \bar{\rho}$, and apply (i) of the lemma to $\rho^{-1}(C)$, $\{\rho^{-1}\tau: \tau \in \bar{\rho} - \{\rho\}\}$ to get a point x. Then $\rho(x) \in C$, $x \notin A$ so $\rho(x) \notin \rho(A)$, and $x \notin \rho^{-1}\tau(A)$ for any $\tau \in \bar{\rho} - \{\rho\}$ so $\rho(x) \notin \tau(A)$ for any $\tau \in \bar{\rho} - \{\rho\}$. Hence $\rho(x) \notin \bigcup \bar{\rho}(A)$, a contradiction.

Now, let X_0 be the real vector space consisting of all Lebesgue integrable real-valued functions on $\mathbf{R}^n$. Let X be the vector space of all real-valued functions on $\mathbf{R}^n$ which are bounded by some function in X_0. For $f, g \in X$, let $f \leq g$ mean that $f(x) \leq g(x)$ for each $x \in R^n$.

THEOREM 9. *Let A be as in the lemma.*

(a) *The set $\{\chi_{\sigma A}: \sigma \in G_n\}$ is linearly independent over X_0*, i.e., *no function in this set lies in the subspace of X spanned by the rest of them together with X_0.*

(b) *If $g \in X_0$ is such that $g \geq \sum_{i=1}^m p_i \chi_{\sigma_i A} + \sum_{j=1}^r q_j \chi_{\rho_j A}$, where $p_i > 0$, $q_j < 0$, and $\bar{\sigma} = \{\sigma_1, \ldots, \sigma_m\}$ is disjoint from $\bar{\rho} = \{\rho_1, \ldots, \rho_r\}$, then $\int g \, dm \geq \sum_{i=1}^m p_i$.*

Proof. (a) Suppose $\chi_{\sigma A}$ is in the space spanned by $X_0 \bigcup \{\chi_{\rho A}: \rho \neq \sigma\}$. Then $\chi_{\sigma A} - \sum_{i=1}^m p_i \chi_{\rho_i A} \in X_0$ for some reals p_i, and congruences ρ_i, where $\rho_i \neq \sigma$. Since a bounded measurable function is uniformly approximable by simple functions (see [12, p. 159]), there is a simple function $\sum_{k=1}^{l} s_k \chi_{E_k}$ (where E_k is Lebesgue measurable) such that, for any $x \in \mathbf{R}^n$,

$$(*) \qquad |\chi_{\sigma A}(x) - \sum p_i \chi_{\rho_i A}(x) - \sum s_k \chi_{Ek}(x)| < \tfrac{1}{2}.$$

Now, choose $k \leq l$ such that $m(E_k \cap \sigma I^n) > 0$; then choose a closed set $C \subseteq E_k \cap \sigma I^n$ such that $m(C) > 0$. Apply the lemma to $\sigma^{-1}(C)$, θ, $\{\sigma^{-1}\rho_i: i = 1, \ldots, m\}$ for C, $\bar{\sigma}$, $\bar{\rho}$ to get, by (ii), a point

$$x \in \sigma^{-1}(C) \cap A - \bigcup \sigma^{-1}\rho_i(A)$$

and, by (i), a point $y \in (\sigma^{-1}(C) - A) - \bigcup \sigma^{-1}\rho_i(A)$. Then $\sigma(x) \in C \cap \sigma(A) - \bigcup \{\rho_i(A): i \leq m\}$, and $\sigma(y) \in (C - \sigma(A)) - \bigcup \{\rho_i(A): i \leq m\}$. We may now apply $(*)$ to $\sigma(x)$ and $\sigma(y)$ to obtain $|1 - s_k| < \frac{1}{2}$ and $|0 - s_k| < \frac{1}{2}$, a contradiction.

(b) *Case 1.* $\bar{\sigma} \neq \theta$. For each nonempty $S \subseteq \{1, \ldots, m\}$, let W_S be the set

$$\bigcap \{\sigma_i I^n: i \in S\} - \bigcup \{\sigma_i I^n: i \leq m, i \notin S\}.$$

Many of these sets may be empty, but they are pairwise disjoint and Lebesgue measurable, so $\int g\, dm \geq \sum_S \int_{W_S} g\, dm$. Note that

$$\left\{x \in W_S: g(x) \geq \sum_{i \in S} p_i\right\} \subseteq W_S \cap \bigcap_{i \in S} \sigma_i A - \bigcup \bar{\rho} A.$$

By property (1) of the corollary above,

$$m^*\left(\bigcap_{i \in S} \sigma_i A - \bigcup \bar{\rho} A\right) = m\left(\bigcap_{i \in S} \sigma_i I^n\right)$$

and since each W_S is a Lebesgue measurable subset of $\bigcap_{i \in S} \sigma_i I^n$, it follows that

$$m^*\left(W_S \cap \bigcap_{i \in S} \sigma_i A - \bigcup \bar{\rho} A\right) = m(W_S).$$

Hence $m(\{x \in W_S: g(x) \geq \sum_{i \in S} p_i\}) = mW_s$, and so $\int_{W_S} g\, dm \geq (\sum_{i \in S} p_i)(mW_s)$. This implies that

$$\int g\, dm \geq \sum_S \left(\sum_{i \in S} p_i\right)(mW_S) = \sum_{i=1}^{m} p_i\left(\sum_{i \in S} mW_S\right) = \sum_{i=1}^{m} p_i,$$

since, for each i, $\sum_{i \in S} mW_S = m(\sigma_i I^n) = 1$.

Case 2. $\bar{\sigma} = \theta$. Simply note that

$$m(\{x \in \bigcup \bar{\rho} I^n: g(x) \geq 0\}) \geq m^*(\{x \in \bigcup \bar{\rho} I^n: x \notin \bigcup \bar{\rho} A\})$$

which by property (2) of the corollary, equals $m(\bigcup \bar{\rho} I^n)$. So

$$m(\{x \in \bigcup \bar{\rho} I^n: g(x) < 0\}) = 0 \quad \text{and} \quad m(\{x \in R^n - \bigcup \bar{\rho} I^n: g(x) < 0\}) = 0$$

whence $\int g\, dm \geq 0$.

We now use Theorem 5 and the Hahn-Banach Theorem to conclude the proof of Theorem 1.

THEOREM 1. *If G is an amenable subgroup of G_n, then there is a total measure μ in $\mathbf{R}^n$ extending Lebesgue measure and such that* Inv $(\mu, G_n) = G$.

Proof. Let A be as in the Lemma and let X_1 be the subspace of X spanned by X_0 and $\{\chi_{\sigma A}: \sigma \in G_n\}$. Define p: $X \to \mathbf{R}$ by $p(f) = \inf\{\int g\, dm : g \in X_0$ and $g \geq f\}$. Note that for each $f \in X$ there is some $g \in X_0$ such that $g \geq f$. It is clear that $p(f_1 + f_2) \leq p(f_1) + p(f_2)$, and that $p(\alpha f) = \alpha p(f)$ for any real $\alpha \geq 0$. Define a linear functional F on X_1 by setting $F(f) = \int f\, dm$ if $f \in X_0$, and letting $F(\chi_{\sigma A}) = 1$ if $\sigma \in G$, and 0 if $\sigma \in G_n - G$. Then extend F linearly to all of X_1; by Theorem 5(a) this successfully defines a linear functional on X_1.

We wish to apply the Hahn-Banach Theorem to extend F to all of X, and so we must verify that $F(F) \leq p(f)$ for any $f \in X_1$. If $f \in X_1$ then $f = h + \sum p_i \chi_{\sigma_i A} + \sum q_j \chi_{\rho_j A}$, where $h \in X_0$, $p_i > 0$, $q_j < 0$, and $\{\sigma_i\} \cap \{p_j\} = \theta$. Then $F(f) \leq \int h\, dm + \sum p_i$, whence it suffices to show that $p(f) \geq \int h\, dm + \sum p_i$. Suppose $g \in X_0$ and $g \geq f$. Then $g - h \geq \sum p_i \chi_{\sigma_i A} + \sum q_j \chi_{\rho_j A}$ and Theorem 9(b) may be applied to yield that $\int g - h\ dm \geq \sum p_i$. Hence $\int g\, dm \geq \int h\, dm + \sum p_i$, showing that $p(f) \geq F(f)$.

Now, the Hahn-Banach Theorem yields a linear functional $\bar{F}$ on X which extends F and is dominated by p. Then $\bar{F}(f) \geq 0$ if $f \geq 0$; for $\bar{F}(f) = -\bar{F}(-f) \geq -p(-f) \geq 0$. Define a total measure ν on $\mathbf{R}^n$ by setting $\nu(B) = \bar{F}(\chi_B)$, if $\chi_B \in X$, and letting $\nu(B) = \infty$ otherwise. Then if we define $\mathscr{A}$ to consist of the Lebesgue measurable subsets of $\mathbf{R}^n$, together with the sets σA for $\sigma \in G_n$, $\mathscr{A}$ and ν satisfy the hypotheses of Theorem 8 (since Inv $(\nu \mid \mathscr{A}, G_n) = G$). Thus there is a total measure μ extending $\nu \mid \mathscr{A}$, and with Inv $(\mu, G_n) = G$. Since $\nu \mid \mathscr{A}$ extends Lebesgue measure, this completes the proof.

It is a consequence of this theorem that, for example, Lebesgue measure on the line has a total, finitely additive extension that is invariant under rational translations, but no others (or all translations, but no reflections). Note that if G is a subgroup of $\mathcal{O}_n$ then, by (a) $\Rightarrow$ (c) of Theorem 5, the existence of a measure as in Theorem 1 implies that G is amenable.

REFERENCES

1. R. P. AGNEW and A. P. MORSE, *Extensions of linear functionals, with applications to limits, integrals, measures, and densities*. Ann. of Math., vol. 39 (1938), pp. 20–30.
2. S. BANACH, *Sur le problème de la mesure*, Fund. Math., vol. 4 (1923), pp. 7–33.
3. S. BANACH and A. TARSKI, *Sur la décomposition des ensembles de points en parties respectivement congruentes*, Fund. Math., vol. 6 (1924), pp. 244–277.
4. M. M. DAY, *Amenable semigroups*, Illinois J. Math., vol. 1 (1957), pp. 509–544.
5. W. R. EMERSON, *Characterizations of amenable groups*, Trans. Amer. Math. Soc., vol. 241 (1978), pp. 183–194.
6. W. R. EMERSON, *The Hausdorff paradox for genreral group actions*, J. Func. Anal., vol. 32 (1979), pp. 213–227.
7. E. FØLNER, *On groups with full Banach mean value*, Math. Scand., vol. 3 (1955), pp. 243–254.
8. F. P. GREENLEAF, *Invariant means on topological groups*, Van Nostrand, New York, 1969.
9. F. HAUSDORFF, *Gründzuge der Mengenlehre*, Chelsea, New York, 1965.
10. M. HAUSNER, *A vector space approach to geometry*, Prentice Hall, Englewood Cliffs, 1965.

11. E. HEWITT and K. ROSS, *Abstract harmonic analysis, Volume I*. Springer, New York, 1963.
12. E. HEWITT and K. STROMBERG, *Real and abstract analysis*. Springer, New York, 1965.
13. D. KÖNIG, *Sur les correspondances multivoques des ensembles*, Fund. Math., vol. 8 (1926), pp. 114–134.
14. W. MAGNUS, A. KARASS and D. SOLITAR, *Combinatorial group theory*. Wiley, New York, 1966.
15. J. MYCIELSKI, "Commentary on a paper of Banach" in S. Banach, *Oeuvres, avec des Commentaires, Volume 1*, Editions scientifiques de Pologne, Warsaw, 1967.
16. J. MYCIELSKI, *Finitely additive invariant measures (I)*, Colloq. Math., to appear.
17. J. VON NEUMANN, *Zur allgemeinen Theorie des Masses*, Fund. Math., vol. 13 (1929), pp. 73–116.
18. J. C. OXTOBY, *Measure and category*, Macmillan, New York, 1971.
19. H. L. ROYDEN, *Real analysis*, second edition, Macmillan, New York, 1968.
20. J. SHERMAN, *A new characterization of amenable groups*, Trans. Amer. Math. Soc., vol. 254 (1979), pp. 365–389.
21. W. SIERPINSKI, *On the congruence of sets and their equivalence by finite decomposition*, Chelsea, New York, 1954.
22. R. J. SILVERMAN, *Invariant linear functions*, Trans. Amer. Math. Soc., vol. 81 (1956), pp. 411–424.
23. ———, *Means on semigroups and the Hahn-Banach extension property*, Trans. Amer. Math. Soc., vol. 83 (1956), pp. 222–237.
24. K. STROMBERG, *The Banach-Tarski paradox*, Amer. Math. Monthly, vol. 86 (1979), pp. 151–161.
25. A. TARSKI, *Algebraische Fassung des Massproblems*, Fund. Math., vol. 31 (1938), pp. 47–66.
26. ———, *Cardinal algebras*, Oxford University Press, New York, 1949.
27. J. TITS, *Free subgroups in linear groups*, J. Alg., vol. 20 (1972), pp. 250–270.

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