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LOCAL DILATIONS

BY

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1. Introduction

A local dilation is an embedding of a metric space which "stretches" in all small regions. The concept was introduced by the author in [6, p. 309] where Theorem 3.1 of this paper was proved. Local dilations were also used in [7], especially a special case of Corollary 3.4 below. These papers have applied local dilations by noting that a convergent sequence of local dilations, since close points are pushed apart, is likely to converge to a one-to-one function. Considered in the setting of elastic behavior, local expansions can be considered as locally stretching transformations, although the "strain" (see Fritz John [4]) may well be infinite.

In this paper we examine several properties of local dilations. We show that a local dilation from any closed manifold, with any "reasonable" metric, into itself is an isometry in the "path-metric". We show that a strictly starlike region in a hyperplane in E^n can be "pushed out" along a right cylinder (and not along a slanting cylinder) with a local dilation. Convexity properties and fixed point properties are considered. And we introduce and use the concept of path-metric. In addition, many counterexamples are included.

2. Basic properties

A *global dilation* is a continuous function (map) between metric spaces,

$$f\colon \langle X, \rho\rangle \to \langle Z, d\rangle,$$

such that for x and y in X, $d(f(x), f(y)) \geq \rho(x, y)$. (It's just the opposite of a contractive map.) A *local dilation* is an embedding between metric spaces, $h\colon X \to Z$, such that any point of X has a neighborhood N with $h \mid N$ a global dilation. An embedding is a map which is a homeomorphism when the target space is properly restricted.

LEMMA 2.1. *If $h\colon \langle X, \rho\rangle \to \langle Z, d\rangle$ is a local dilation and X is compact, then h is a uniform local dilation. In other words, there is a $\delta > 0$ such that for $x, y \in X$ with $\rho(x, y) < \delta$, we have $d(h(x), h(y)) \geq \rho(x, y)$.*

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Proof. For each $x \in X$, let N_x be a neighborhood of x in which h is an expansion. Let $N_1, \ldots, N_m$ be a finite subcover of X. Then let δ be a Lebesgue number for that cover (i.e., for $f(x)$ the maximal radius for an open ball about x but contained in some N_i, f is continuous and positive on X. Let δ be the minimum of $f(x)$.) ∎

Note that this can be restated as follows: With the same hypotheses, there is a $\delta > 0$ such that if $d(h(x), h(y)) < \rho(x, y)$ then $\rho(x, y) \geq \delta$.

One of the most fundamental properties of local dilations is that the length of curves are increased or unchanged. This is our next result.

DEFINITION. For $\alpha\colon [a, b] \to X$ a path in a metric space, let $l(\alpha)$, the *length of* α, be

$$\sup \left\{ \sum_{i=1}^{n} d(\alpha(t_i), \alpha(t_{i-1})) \,\middle|\, a = t_0 < t_1 < \cdots < t_n = b \right\},$$

where the supremum is over all such collections $\{t_i\}$. It will be convenient to call the image, $G = \alpha([a, b])$, a curve, although we really have in mind the map α. We say that $\{t_i\}_{i=0}^n$ as above are *in order* and if $x_i = \alpha(t_i)$ we say that $\{x_i\}_{i=0}^n$ are *in order* (or ordered).

THEOREM 2.2 (THE PATH LENGTH THEOREM). *If h is a local dilation and G is a path in the domain of h, then $l(G) \leq l(h(G))$.*

Proof. Each point of G has a neighborhood in which h "expands". G is compact, so we choose a finite subcover. Given $e > 0$ (or in the case $l(G) = \infty$, $N > 0$) there exist $x_0, \ldots, x_m$ in order so that

$$\sum d(x_n, x_{n-1}) > l(G) - e \text{ (or } > N).$$

For $\{x_0, \ldots, x_m\} \subset \{y_0, \ldots, y_k\}$ an ordered subset of G,

$$\sum d(y_n, y_{n-1}) \geq \sum d(x_n, x_{n-1})$$

by the triangle inequality. By an easy argument, we can add more points to $\{y_0, \ldots, y_k\}$, so that each y_{n-1} and y_n are in the same neighborhood of our finite cover. Now $\{h(y_0), \ldots, h(y_k)\}$ is an ordered subset of $h(G)$ and

$$\sum d(h(y_n), h(y_{n-1})) \geq \sum d(y_n, y_{n-1}) \geq \sum d(x_n, x_{n-1}) > l(G) - e \text{ (or } > N).$$

Hence $l(h(G)) \geq l(G)$. ∎

COROLLARY 2.3. *For G an open-ended (half-open-ended) curve, a map of (a, b) ($[a, b)$), we may use a similar definition of length. Just don't require $t_0 = a$ or $t_n = b$ ($t_n = b$). Then for h a local dilation $l(G) \leq l(h(G))$.*

Proof. Use essentially the same proof as for Theorem 2.2. The only problem is that G does not necessarily have a finite subcover. But the sub-curve of G determined by t_0 and t_m does have a finite cover and this suffices. ∎

The following amusing example indicates that there are fewer local dilations than one might expect.

Example 2.4. There does *not* exist a local dilation pulling a right triangular region (the domain) onto a rectangular region (the range) with 2 sides of the rectangle equal to 2 fixed sides of the triangle.

Proof. From the Path Length Theorem (2.2) we see that if G is an arc in the *range* of h, a local dilation, then $l(h^{-1}(G)) \leq l(G)$. Suppose there were a local dilation (hence a homeomorphism) as stated in Example 2.4. Let G be the diagonal of the rectangle corresponding to the hypotenuse of the given right triangle. So $l(h^{-1}(G)) \leq l(G)$. But the endpoints of G are fixed by h, and G is the shortest arc between them. Hence $h^{-1}(G) = G$. This is a contradiction, since a homeomorphism cannot take an interior point to a non-interior point. ∎

This example shows that if we have an elastic sheet with the proper elastic properties and in the shape of a right triangle, pulling on the hypotenuse to get a rectangle would cause "puckering".

3. Pushing out part of a hyperplane

It was found to be quite useful in [6] and [7] to be able to "push out" part of a hyperplane using a local dilation. The first example of this is:

THEOREM 3.1 (in E^3). *Given a square in a horizontal plane, consider a right cylinder with the square as base and height at least double the length of a side of the square. Then there exists a local dilation, h, of the plane onto the plane together with the surface of the cylinder, minus the original square's interior, such that h is the identity outside the square (see Figure 1).*

A proof of this theorem can be found in [6]. Here, it will follow from the much more general Theorem 3.3. Before considering that theorem however, we show that local dilations can only push out hyperplanes in perpendicular directions.

THEOREM 3.2. *Suppose K is a compact set in $E^{n-1}(= \{x \in E^n : x_n = 0\})$ and C is the surface of a cylinder in E^n with K as the base. If the cylinder is not a right cylinder, then there is* **no** *local dilation from E^{n-1} to $E^{n-1} \cup C - (\text{int } K)$ which is the identity on $E^{n-1} - (\text{int } K)$.*

Proof. (see Figure 2). It is easy to see there is an $(n-1)$-hyperplane P, containing at least one of the generating lines of C, so that C is contained in one of the two closed half-spaces determined by P. Then P and E^{n-1} divide E^n into 4 parts and since C is not a right cylinder, we can choose P so that the part of E^n containing C is a dihedral angle of more than 90°.

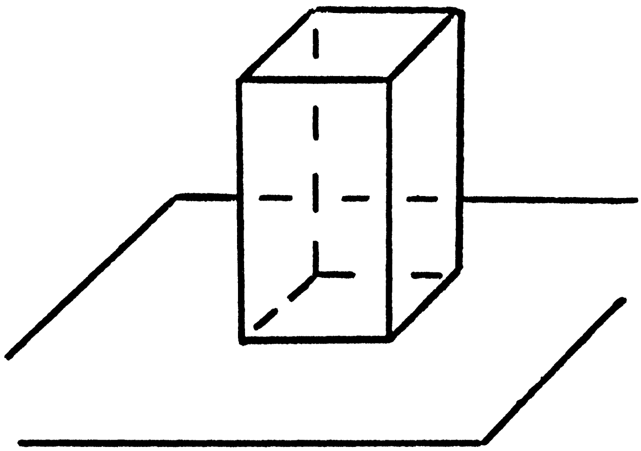


FIG. 1 (Refer to Theorem 3.1)

Suppose h is as desired. Let $x \in P \cap E^{n-1} \cap C$. Then $h(x) = x$ and x has a neighborhood in which h is a dilation. But there are points y and z in any neighborhood of x (see Figure 2) such that:

(1) $h(y) \in C \cap P - E^{n-1}$. Hence $y \in \text{int } K \subset C - \{h(y)\}$.
(2) $z \in E^{n-1} - C$. So $h(z) = z$.
(3) No element of $C - \{h(y)\}$ is as close to z as $h(y)$.

But then $d(z, y) > d(z, h(y)) = d(h(z), h(y))$ a contradiction. ∎

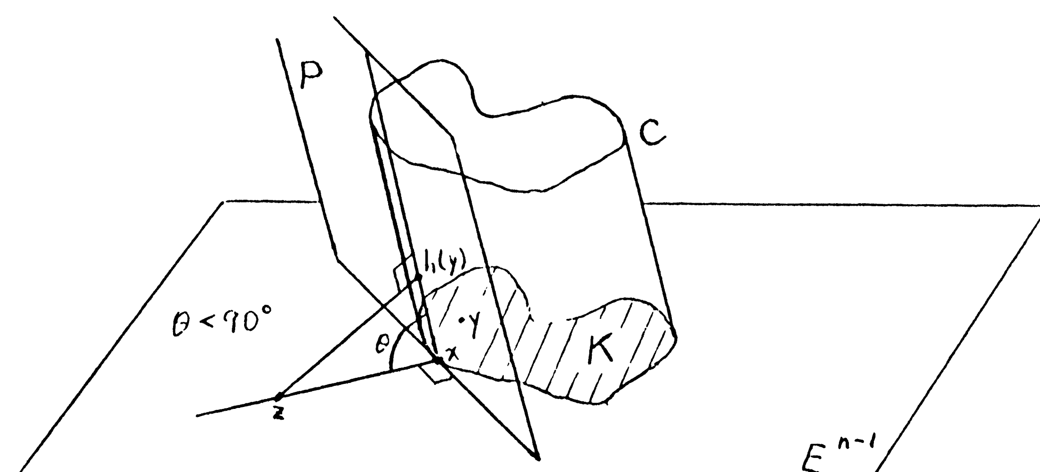
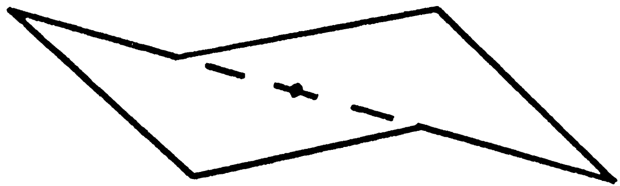


FIG. 2 (Refer to Theorem 3.2)



Starlike but not Strongly Starlike

FIG. 3

Following a definition we will prove our most general result for pushing out a hyperplane.

DEFINITION. An $(n-2)$-sphere S in E^{n-1} is called *strongly starlike* if there exists a point x inside S so that for $y \in S$, the segment from x to y meets S only in y. x is called a *central point*. See Figure 3 for a sketch of a starlike but not strongly starlike 1-sphere.

THEOREM 3.3. *Given S strongly starlike in E^{n-1}, consider a right cylinder whose base is the closure of the interior of S, of height H in the direction of the* nth *axis of E^n ($E^{n-1} \subset E^n$ in the standard position). Then there is a local dilation h of E^{n-1} onto the union of E^{n-1} and the surface of the cylinder, minus the interior of S. And h is the identity outside S ($n \geq 2$).*

Proof. We can assume 0 is a central point for S. For $0 \leq t < 1$, let $S_t = (1-t)S$, $S_0 = S$. Since S is strongly starlike, $S_t \cap S_r = \emptyset$ if $t \neq r$. The map $S_t \to S$ given by $s \mapsto s/(1-t)$ is a local dilation since for s_1 and s_2 in S_t,

$$|h(s_1) - h(s_2)| = \frac{1}{1-t}|s_1 - s_2| \geq |s_1 - s_2|.$$

We may further assume that $1 = \inf\{|s| : s \in S\}$. Let

$$M = \sup\{|s| : s \in S\}.$$

Let C be a fixed constant, $C > H/2 + M\sqrt{12}$. Let $T = H^2/4C^2, 0 < T < 1$. Let

$$f(t) = C(\sqrt{T} + \sqrt{t} - \sqrt{T-t}), \quad 0 \leq t \leq T.$$

So $f'(t) = \frac{1}{2}C((1/\sqrt{t}) + (1/\sqrt{T-t})) \geq C/(2\sqrt{t})$.

Let A_1 be the closed region inside S_T, A_2 the closed region between S_T and S, and A_3 the closed region outside S. (Refer to Figure 4.)

Define h: $E^{n-1} \to E^n$ by

$$h(s) = \begin{cases} (s, 0) & \text{for } s \in E^{n-1} \text{ and outside } S \quad (s \in A_3), \\ (s/(1-t), f(t)) & \text{for } s \in S_t, 0 \leq t \leq T \quad (s \in A_2), \\ (s/(1-T), f(T)) & \text{for } s \text{ inside } S_T \quad (s \in A_1). \end{cases}$$

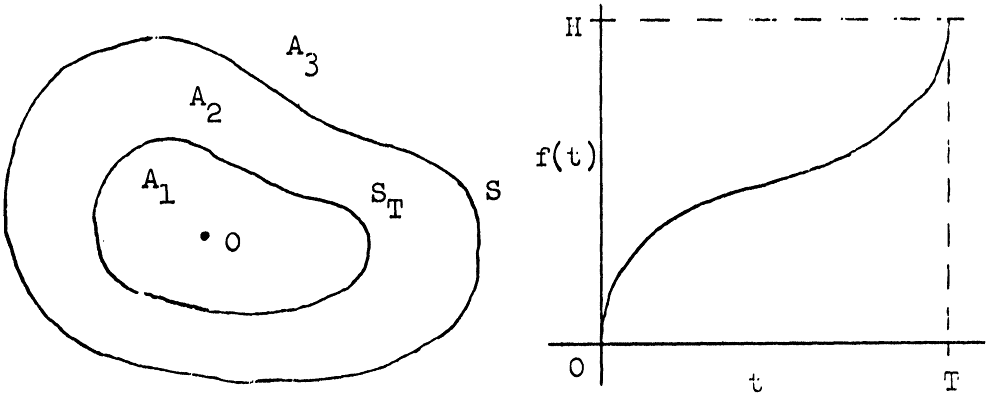


FIG. 4

Note that $f(T) = 2C\sqrt{T} = H$. We use the notation $(x, t) = (x_1, x_2, \ldots, x_{n-1}, t)$. It is clear that h is an embedding as we desire—we only need check that it is a local dilation. We can use neighborhoods so small they do not contain points of both A_1 and A_3 (in Figure 4). Just take the diameter less than $1 - T$.

Let x and y be in E^{n-1}. For $d(x, y) < 1 - T$, we must show that

$$d(x, y) \leq d(h(x), h(y)).$$

If x and y are in A_3, then $h(x) = x$ and $h(y) = y$, so we are done. If x and y are in A_1, then $d(h(x), h(y)) = (1/(1 - T)) \cdot d(x, y) > d(x, y)$ and we are done. There are three cases remaining:

(1) Suppose x and y are in A_2. If x and y both lie on S_r,

$$d(h(x), h(y)) = (1/(1 - r))\, d(x, y)$$

and we are done. So we may assume x is on S_r and y is on S_t with $r > t$. Let z be the point of S_r on the radius from 0 to y ($z = (1 - r)y/(1 - t)$). Let α be the segment from y to z, β the segment from x to z (refer to Figure 5). Let α' be the segment in E^n from $h(y)$ to $h(z)$, β' the segment from $h(x)$ to $h(z)$.

Then we have α' perpendicular to β', $|\beta'| = |\beta|/(1 - r)$, and $|\alpha'| = f(r) - f(t)$. We drop the $|\cdot|$, so that α, α', β, β' also represent the lengths of their respective segments. Now $\alpha' = (r - t)f'(\eta)$ for some $\eta \in (t, r)$, so

$$\alpha' \geq (r - t)C/2\sqrt{\eta} \geq (r - t)C/2\sqrt{r}.$$

For $y' = y/(1 - t)$ in S, $\alpha = |(1 - t)y' - (1 - r)y'| = |r - t|\,|y'|$, so

$$(r - t) \leq \alpha \leq M(r - t).$$

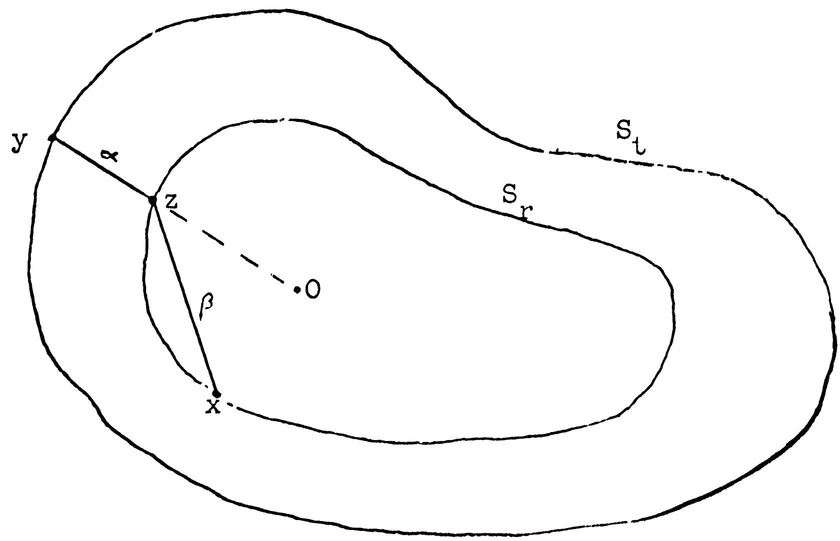


FIG. 5 (enlarged from 4)

Let $d = d(x, y)$, $d' = d(h(x), h(y))$. Then

$$\begin{aligned} d^2 &\leq \alpha^2 + \beta^2 + 2\alpha\beta \\ &\leq M^2(r-t)^2 + \beta^2 + 2M\beta(r-t) \\ &\leq M^2(r-t)^2/r + \beta^2 + 2M\beta(r-t), \end{aligned}$$

and

$$\begin{aligned} d'^2 = \alpha'^2 + \beta'^2 &\geq (r-t)^2C^2/4r + \beta^2/(1-r)^2 \\ &\geq C^2(r-t)^2/4r + \beta^2 + 2r\beta^2, \end{aligned}$$

where we use the fact that $1/(1-r)^2 \geq 1 + 2r$ since $r < 1$. It suffices to show

$$(C^2/4 - M^2)(r-t)^2/r + 2r\beta^2 \geq 2M\beta(r-t).$$

Assuming $\beta \geq (r-t)M/r$ and using the fact that $C^2/4 > M^2$, the left hand side of the inequality is not less than

$$2r\beta^2 \geq 2r\beta(r-t)M/r = 2M\beta(r-t).$$

On the other hand, let us assume that $\beta \leq (r-t)M/r$ and use the fact that $C \geq M\sqrt{12}$ (and so $C^2/4 - M^2 \geq 2M^2$). Then the left hand side of the inequality we are trying to establish is not less than

$$(C^2/4 - M^2)(r-t)^2/r \geq 2M^2(r-t)^2/r \geq 2M\beta(r-t).$$

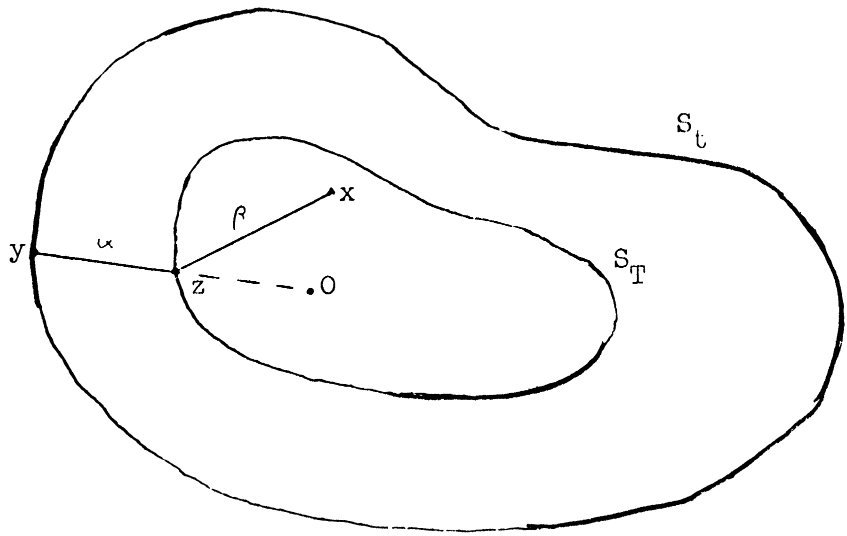


FIG. 6

(2) Suppose we have x in A_1 and y in A_2, say with y on S_t. Let z be the intersection of the segment from 0 to y with S_T, and let α, α', β, β' be as in (1) (see Figure 6). We have α' perpendicular to β' and $\beta' \geq \beta$. Let d and d' be as in (1). Then

$$\alpha'^2 = (f(T) - f(t))^2 = C^2(\sqrt{T} - \sqrt{t} + \sqrt{T-t})^2 \geq C^2(T-t).$$

As in (1), $\alpha \leq M(T-t)$. It is clear that $\alpha \leq M$ and $\beta \leq 2M$. So

$$d'^2 - d^2 \geq \alpha'^2 + \beta'^2 - \alpha^2 - \beta^2 - 2\alpha\beta \geq \alpha'^2 - (\alpha^2 + 2\alpha\beta),$$

since $\beta' \geq \beta$. Hence it suffices to show $\alpha^2 + 2\alpha\beta \leq \alpha'^2$. But

$$\alpha^2 + 2\alpha\beta = \alpha(\alpha + 2\beta) \leq M(T-t)3M \leq 12M^2(T-t) \leq C^2(T-t) \leq \alpha'^2.$$

(3) Suppose we have x in A_3 and y in A_2, say with y on S_t. We proceed as before (see Fig. 7). We have $\alpha'^2 = (f(t) - f(0))^2 = (f(t))^2 \geq C^2 t$. By the triangle inequality,

$$\beta \leq d(x, y) + \alpha \leq d(x, y) + M.$$

Since we are assuming x and y are in a neighborhood of diameter less than $(1-T) < 1$, we may conclude that $\beta \leq 1 + M \leq 2M$. Also, as in (1), $\alpha \leq Mt$. So

$$\alpha^2 + 2\alpha\beta = \alpha(\alpha + 2\beta) \leq Mt(3M) \leq 12M^2 t \leq C^2 t \leq \alpha'^2.$$

But as in (2), this implies $d'^2 \geq d^2$. ∎

A special case of the following Corollary was found to be useful in [7].

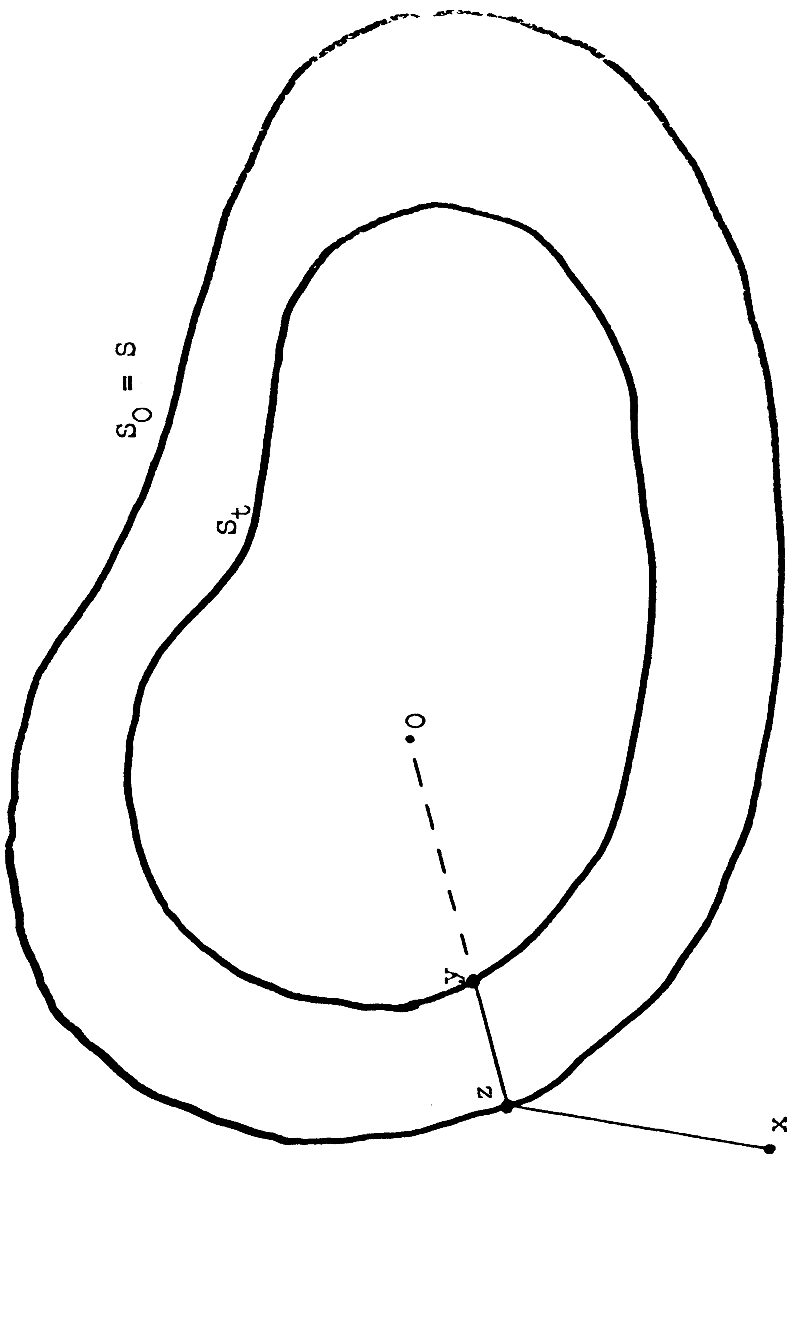


Fig. 7

COROLLARY 3.4. *For h as in the above theorem, h extends to $h'\colon E^n \to E^n$ a homeomorphism $(h' \mid E^{n-1} = h)$. Also, h' can be taken to be the identity outside any given neighborhood of the (solid) cylinder. (The cylinder is the collection of points (x, t) for $x \in A_1 \cup A_2$ and $0 \leq t \leq H$.)*

Proof. Let $\varepsilon > 0$. We assume 0 is a central point for S as in the previous proof. Let A be the closed and bounded region in E^{n-1} determined by $(1 + \varepsilon)S = \{(1 + \varepsilon)x\colon x \in S\}$. Consider the region R in E^n consisting of points of the form (x, t) with x in A, and $0 \leq t \leq H + \varepsilon$, or points lying on a segment from $(0, -\varepsilon)$ to one of these first type points. Let h' be the identity outside R and let $h' \mid E^{n-1} = h$.

Then for r in $R - (0, -\varepsilon)$, r determines a segment from $(0, -\varepsilon)$ to a point (x, t) in ∂R, with either x in ∂A and $t \geq 0$ or x in A and $t = H + \varepsilon$. On this segment lies a point of the form $(y, 0)$. h' has already been defined on these three co-linear points, and so we extend linearly between consecutive pairs (see Figure 8). It is easy to see that h' is a homeomorphism, and we may choose $\varepsilon > 0$ as needed. ∎

Note that since standard spheres and surfaces of cubes are strongly starlike, this is an extension of Theorem 3.1. We next consider some limitations of dilations.

THEOREM 3.5. *No local dilation h with the properties of Theorem 3.3 can be a global dilation.*

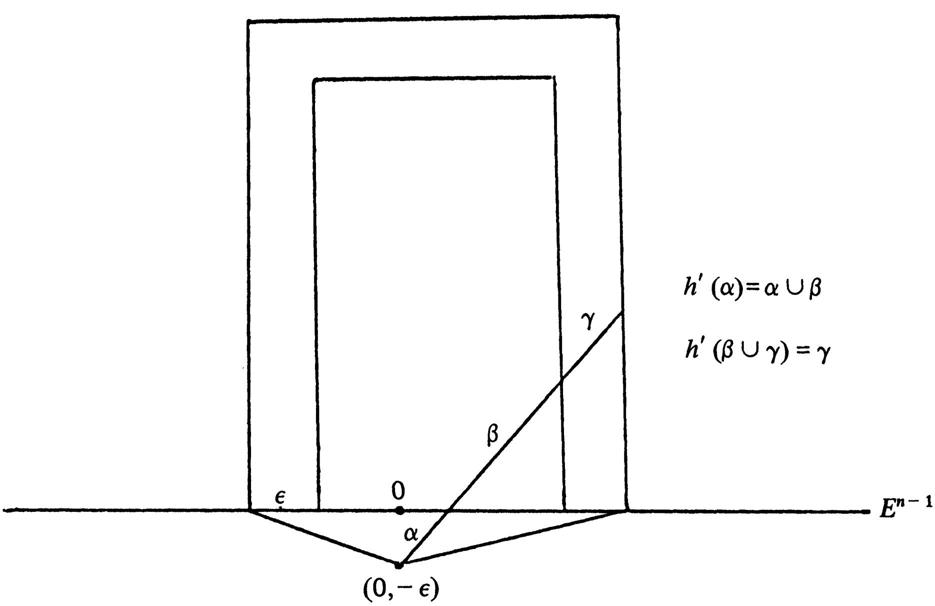


FIG. 8

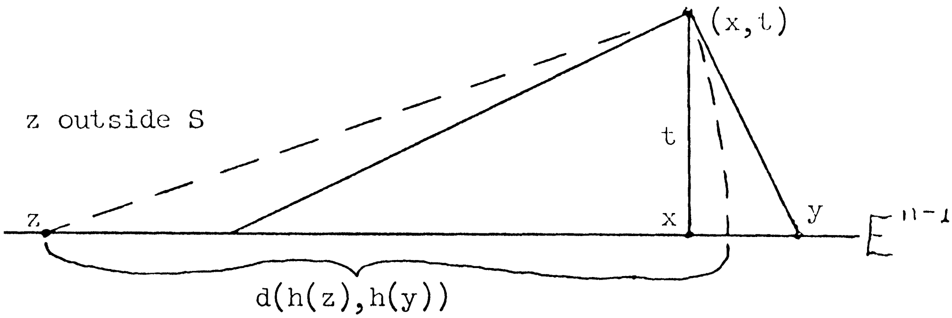


FIG. 9

Proof. Suppose h as stated. For some y in E^{n-1} we have $h((y, 0)) = (x, t)$ with $t \neq 0$. Consider the plane determined by $(y, 0)$, $(x, 0)$, and (x, t). Then there is a z in E^{n-1}, fixed by h, so that $d(z, y) > d(h(z), h(y))$. Figure 9 should be sufficiently explanatory. ▮

Example 3.6. We give an example of a closed body K in E^2 which cannot be pushed out to a right cylinder in E^3 by a local dilation as above. K is a standard annulus (see Figure 10).

Proof. Some simple closed curve G in int (K) would be carried to S_1, the inner boundary of K at the top level of the cylinder. But then $l(G) \leq l(S_1)$ which is not the case. ▮

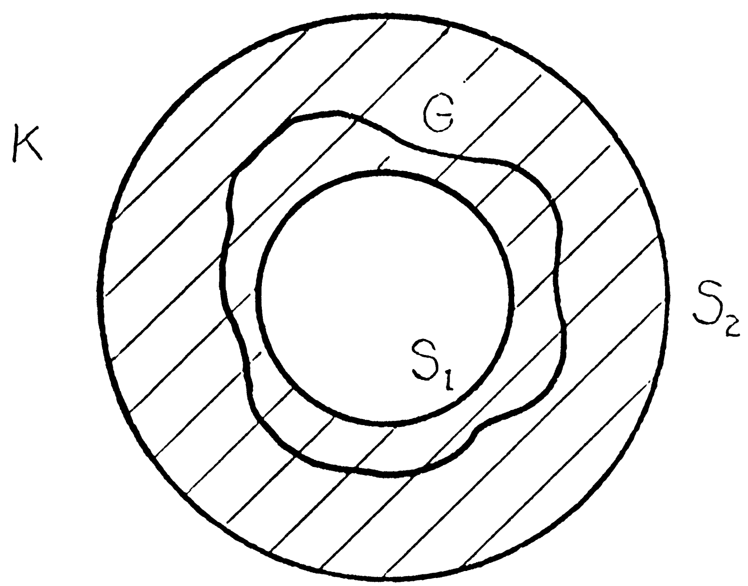


FIG. 10

Conjecture 3.7. The interior of any $(n-2)$-sphere (a homeomorph of the standard one) in E^{n-1} can be pushed out as in Theorem 3.3. ∎

4. Convexity and fixed points

The study of fixed points for dilations is motivated by the great number of fixed point results in the field of contractions (for example, see [8]). We give several results with brief proofs.

THEOREM 4.1. *If $h\colon A \to E^n$ is a local dilation ($A \subset E^n$) and $h(A)$ is convex, then the set of fixed points of h is a convex set.*

Proof. Let F be the set of fixed points with x and y in F. Let G be the line segment connecting x and y. Then, since x and y are in convex set $h(A)$, G lies in $h(A)$. By the Path Length Theorem (2.2), $l(h^{-1}(G)) \leq l(G)$. But G is *the* shortest arc from x to y. So $h^{-1}(G) = G$, i.e., $h(G) = G$. Hence for z in G, $h(z)$ is in G. If $h(z) \neq z$, $h(z)$ is closer to either x or y than z. But then h decreases the length of the segment from x to z or the segment from y to z, contradicting Theorem 2.2. So G lies in F. But this implies F is convex. ∎

More generally, if $A \subset \langle X, \rho\rangle$ with $h\colon A \to \langle X, \rho\rangle$ a local dilation and each two points of $h(A)$ may be connected by a unique segment in X, then the set of fixed points is convex. (See Section 6 for definitions of segment and convex in a metric space.)

Note that in Theorem 3.1, the set of fixed points is not convex. However, $h(A)$ is not convex ($A = E^2$ here).

COROLLARY 4.2. *Example 2.4 also follows from this result.* ∎

THEOREM 4.3. *If $h\colon E^n \to E^n$ is a local dilation, then h is onto E^n.*

Proof (suggested by Robert Osserman). Suppose h is not onto. We may assume $0 \in h(E^n)$. Since h is a homeomorphism of E^n with $h(E^n)$, $h(E^n)$ is open, and there is a point $x \in E^n - h(E^n)$ of minimal norm. Let G be the half-open segment from 0 to x, $x \notin G$. Then G lies in $h(E^n)$. So by Corollary 2.3,

$$l(h^{-1}(G)) \leq l(G) = |x|.$$

Since $h^{-1}(G)$ is a half-open arc of finite length, $h^{-1}(G) \subset E^n$ is bounded. So there exists y in E^n and $t_n \nearrow 1$, so that $h^{-1}(t_n x) \to y$. Hence, by the continuity of h,

$$t_n x = h(h^{-1}(t_n x)) \to h(y).$$

But $t_n x \to x \notin h(E^n)$, a contradiction. ∎

COROLLARY 4.4. *If $h\colon E^n \to E^n$ is a local dilation, the set of fixed points is convex.*

Proof. By Theorems 4.1 and 4.3 we are done. ∎

COROLLARY 4.5. *For $E^2 = \mathbf{C}$, the complex plane, suppose $h: \mathbf{C} \to \mathbf{C}$ is a local dilation fixed on $\{z_0^n\}_{n=1}^{\infty}$, $|z_0| > 1$, $z_0 \notin R$. Then h is the identity map. Both hypotheses on z_0 are necessary.*

Proof. The first claim follows from Theorem 4.4 and the fact that for such a z_0, the convex hull of z_0^n is $\mathbf{C}$. The rest of the proof is elementary. ∎

THEOREM 4.6. *Suppose M is a Riemannian manifold with x and y in M such that there exists unique minimal geodesic G from x to y. If h: $M \to M$ is a local dilation which fixes x and y, then h fixes all points of G.*

Proof. Analogous to Theorem 4.1. ∎

Note that the uniqueness of the minimal geodesic is essential. If $M = S^1$ and x and y are antipodal, then the reflection across the axis through x and y leaves only x and y fixed and is an isometry.

5. Spheres and balls

In [5, p. 104, # 12], Kaplansky shows how to prove that a *global* dilation from a compact metric space into itself must be an isometry. It's natural to ask how far we can go with local dilations. Geometric spheres and balls provide two easy applications of The Path Length Theorem.

THEOREM 5.1. *For S the unit n-sphere in E^{n+1}, if h: $S \to S$ is a local dilation, then h is an isometry.*

Proof. First we mention some facts:

(1) For h: $S \to S$ a local dilation, h is a global dilation. For if $d(h(x), h(y)) < d(x, y)$, let G and G' be the minimal geodesics from $h(x)$ to $h(y)$, and from x to y respectively. Then we would have $l(G) < l(G') \leq l(h^{-1}(G))$ a contradiction.

(2) If h: $S \to S$ is a local dilation and h leaves x fixed, then h leaves the antipodal point, x', fixed since x' is the unique point of S a maximal distance from x. Also, for S_0 the equator they determine, $h(S_0) \subset S_0$, since S_0 consists of those points of S a maximal distance from $\{x, x'\}$.

(3) Suppose a local dilation h: $S \to S$ is the identity on two antipodal points, x and x', and on the equator, S_0, which they determine. Then h is the identity on S by Theorem 4.6. This theorem may be applied for the following reason. If $y \in S_0$, then y, x, 0 determine a plane. The plane meets S in a circle. x and y determine a quarter of the circle which is the minimal geodesic from x to y.

(4) If $f\colon S_0 \to S_0$ is an isometry defined on such an equator, f can be naturally extended to S. In particular, for

$$S = \{x \in E^{n+1}\colon |x| = 1\}, \qquad S_0 = \{x \in S\colon x_{n+1} = 0\},$$

$tx + se_{n+1}$ is a general element of E^{n+1}, $x \in S_0$, $t \geq 0$, $s \in \mathbf{R}$. Let F take this element to $tf(x) + se_{n+1}$. One checks that F is an isometry of E^{n+1} and $F(S) \subset S$, hence F can be restricted to S. Also note that $F(e_{n+1}) = e_{n+1}$.

Now we prove Theorem 5.1. For $n = 0$, $S = \{-1, +1\}$ and the result is trivial. Now suppose $n > 1$, and that the result holds for $n - 1$. There is an isometry f of S carrying $h(e_1)$ to e_1. So $f \circ h$ is a local dilation fixed on $\pm e_1$ and carrying the $(n-1)$-sphere equator into itself. But then by the induction hypothesis, $f \circ h|S_0$ is an isometry. We extend it by note 4 to an isometry F on all S, $F(e_1) = e_1$. Then $F^{-1} \circ f \circ h$ is the identity on S_0 and on $\pm e_1$, so by note 3 we are done. ∎

Another proof which applies to more general manifolds will be given in the next section (see Corollary 6.6 and the notes following it). It will use the concept of path-metric and Kaplansky's result for global dilations.

THEOREM 5.2. *For B the open or closed unit ball in E^n, if $h\colon B \to B$ is a local dilation, then h is an isometry.*

Proof. Let $S_r = \{x \in E^n\colon |x| = r\}$, $0 \leq r \leq 1$.

First we show $|h(x)| \leq |x|$. Suppose for some x, $|h(x)| > |x|$. Suppose B is closed. Then let G be the longest radial segment outward from $h(x)$ such that G lies in $h(B)$. Then $l(G) \leq 1 - |h(x)|$. Since h is a homeomorphism with its range, for $h(x)$ and y the endpoints of G, $h^{-1}(y)$ lies in ∂B. Hence

$$l(h^{-1}(G)) \geq d(x, \partial B) = 1 - |x| > 1 - |h(x)| \geq l(G),$$

a contradiction.

Suppose B is open. Then let G the longest radial half-open segment outward from $h(x)$ such that G lies in $h(B)$. Then $h^{-1}(G)$ must approach ∂B. And we use the same argument as in the preceding paragraph.

$$(y_n \in G,\ y_n \to \sim h(B),\ l(h^{-1}(G)) \geq d(x, h^{-1}(y_n)) \to d(x, \partial B) > l(G)).$$

Now we show that $h(S_r)$ lies in S_r. By the above, $h(0) = 0$. So it suffices to show that $|h(x)| \geq |x|$. Let G be the segment from 0 to $h(x)$. Since $h(0) = 0$, we can apply the same argument used in Theorem 4.3 to get that $h(B) = B$. Hence G lies in $h(B)$. Thus $h^{-1}(G)$ is an arc from 0 to x and so $|x| \leq l(h^{-1}(G)) \leq l(G) = |h(x)|$.

By Theorem 5.1, $h|S_r$ is an isometry. We radially extend $h/S_{1/2}$ to an isometry F of B with $F(0) = 0$. Then $F = h$. For $F^{-1} \circ h$ is the identity on 0 and $S_{1/2}$ and $F^{-1} \circ h(S_r)$ lies in S_r. If $F^{-1} \circ h$ fails to be the identity, it fails on some closed ball with center the origin and radius between 1/2 and 1. On this closed ball let $\delta > 0$ be such that $F^{-1} \circ h$ is a global dilation on neighborhoods of diameter δ. There are spheres S_r and S_t in this ball with $|r - t| < \delta$ and $F^{-1} \circ h$ the identity on the first but not the latter. So we have $y = F^{-1} \circ h(x) \neq x$ for some x and y in S_t. Then

$$d(y, y/(2|y|)) = |t - 1/2| < d(x, y/(2|y|)),$$

a contradiction. ∎

It was conjectured in [7] that a local dilation of a path connected space into itself must be a dilation. (A three-point space shows the necessity for some sort of connectivity.) We give below a counter-example.

Example 5.3. A local dilation (in fact a local isometry) from a piecewise linear 1-sphere onto itself which is not an isometry.

Construction. Consider the following points in the plane:

$A(-2, 0)$, $B(-1, 0)$, $C(0, 1)$, $D(1, 0)$, $E(2, 0)$, $F(2, -1)$, $G(1, -1)$, $H(0, 0)$, $I(-1, -1)$, $J(-2, -1)$. (See Figure 11). Let f map $A, B, C, D, E, F, G, H, I, J$ to $J, I, H, G, F, E, D, C, B, A$ respectively and be an isometry on each segment. Then f is an isometry on any two adjacent segments and so a local dilation. But $d(D, H) \neq d(G, C) = d(f(D), f(H))$. ∎

Note, however, that f is onto. This is necessary for certain spaces (such as piecewise linear spheres) as we will show in the next section.

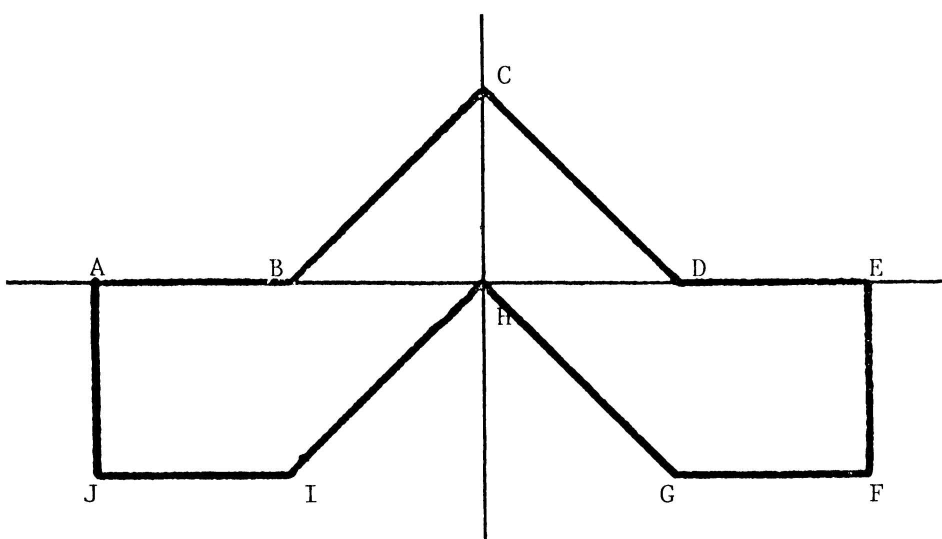


FIG. 11

6. Path-metrics

In this last section we will generalize several of our results by using a new metric on $\langle X, \rho\rangle$, the path-metric.

Define $\rho'\colon X \times X \to R \cup \{\infty\}$ by $\rho'(a, b) = \inf l(\alpha)$ where the infimum is taken over all paths, α, from a to b. Then ρ' is the *path-metric* for $\langle X, \rho\rangle$. (If there is no path from a to b, then $\rho'(a, b) = \infty$.)

We call $\langle X, \rho\rangle$ *convex* if any two points can be connected by an arc isometric to a segment (e.g., see Bing [1]). We refer the reader to Buseman [3, p. 12] for some conditions implying convexity. Note that the intersection of convex subspaces may fail to be convex—let T and B be the top and bottom closed semicircles of S^1. Using the path-metric, T and B are convex, but $T \cap B$ (two points) is not.

The first theorem lists some of the basic properties of path-metrics.

THEOREM 6.1. (1) (a) $\rho \leq \rho'$.

(b) *ρ' is a metric, possibly with infinite values. It is finite-valued if and only if there is a path of finite length between any two points in $\langle X, \rho\rangle$.*

(c) *$\langle X, \rho'\rangle$ has a finer topology than $\langle X, \rho\rangle$.*

(d) *If $\langle X, \rho\rangle$ is convex, then $\rho' = \rho$.*

(e) *If $\rho' = \rho$ and $\langle X, \rho\rangle$ is compact, then $\langle X, \rho\rangle$ is convex. The compactness assumption is needed.*

(f) *If $\rho'(x, y) = \infty$ then x and y lie in distinct components of $\langle X, \rho'\rangle$.*

(g) *If $\rho' < \infty$, then $\langle X, \rho'\rangle$ is convex.*

(2) (a) *If $\langle X, \rho\rangle$ is complete, then so is $\langle X, \rho'\rangle$. The converse is false. In fact, we may have $\langle X, \rho'\rangle$ complete and $\langle X, \rho\rangle$ not even topologically complete.*

(b) *If $\langle X, \rho'\rangle$ is compact, then $\langle X, \rho\rangle$ is compact. The converse is false (in fact see* (2c)*). Compact may not be replaced by locally compact.*

(c) *We may have $\langle X, \rho\rangle$ compact, locally path connected, and such that every pair of points can be connected by a path of length at most* 1, *but $\langle X, \rho'\rangle$ is not even locally compact.*

(d) *$\langle X, \rho'\rangle$ is always locally path-connected.*

(3) (a) *If $\alpha\colon I \to \langle X, \rho\rangle$ is continuous and $l(\alpha) < \infty$, then $\alpha\colon I \to \langle X, \rho'\rangle$ is continuous.* (*I is any standard interval.*)

(b) *If $\langle X, \rho\rangle$ is path connected then $\langle X, \rho'\rangle$ is path connected if and only if $\rho' < \infty$.*

(c) *For α a path in $\langle X, \rho\rangle$, $l(\alpha)$ is independent of whether we use ρ or ρ'.*

(d) $\rho'' = \rho'$.

Proof. (1) All but parts (e) and (f) are straightforward. From the hypotheses of (1e) one can conclude that there is an arc of minimal length between any two points (see Buseman [5, p. 10]). Since $\rho = \rho'$, the length of such an arc is the distance between its endpoints. By the triangle inequality and the definition of length, these minimal arcs are segments. To see that the second hypothesis is needed, consider the plane with a point deleted.

For (1f), it suffices to note that $Z = \{z \mid \rho'(x, z) < \infty\}$ is open and closed. It is closed since for w not in Z, $\{r \mid \rho'(w, r) < \infty\}$ cannot meet it.

(2) (a) Suppose $\langle X, \rho \rangle$ is complete and $\{x_n\}$ is a Cauchy sequence in $\langle X, \rho' \rangle$. Then $\{x_n\}$ is ρ-Cauchy and so converges to some x_0 in $\langle X, \rho \rangle$. Since $\{x_n\}$ is ρ'-Cauchy there exists, for each $k > 0$, an n_k ($> n_{k-1}$) such that for $m, n \geq n_k$, $\rho'(x_m, x_n) \leq 2^{-k}$. Hence there is a path from x_{n_k} to $x_{n_{k+1}}$ of length at most 2^{-k}. Putting all these paths end to end, and throwing in x_0, we get a path of length at most 1 through all the x_{n_k} and ending at x_0. With this path we see that the subsequence $\{x_{n_k}\}$ converges to x_0 in $\langle X, \rho' \rangle$. But a Cauchy sequence with a convergent subsequence is convergent.

To see the converse is false let $\langle X, \rho \rangle$, a subspace of E^2, be the cone from (1, 1) to the rationals in the interval [0, 2] of the x-axis. Then $\langle X, \rho \rangle$ is not Baire, so not topologically complete. But $\langle X, \rho' \rangle$ is complete.

(2) (b) The first statement is true since $\langle X, \rho' \rangle$ is finer than $\langle X, \rho \rangle$. To see that compact cannot be replaced by locally compact consider point A in Figure 12.

(2) (c) Consider the "light bulb" space in Figure 13. There are infinitely many "filaments" each homeomorphic to a circle and with enough "zig-zags" to be of length 1, all joined at a point ($\langle X, \rho' \rangle \approx \bigvee_{n=1}^{\infty} S^1$).

(2) (d) Follows easily from (3a).

(3) (a) Let $\alpha(c) = C$ and let $t \to c$ from either side, say the left. For each t,

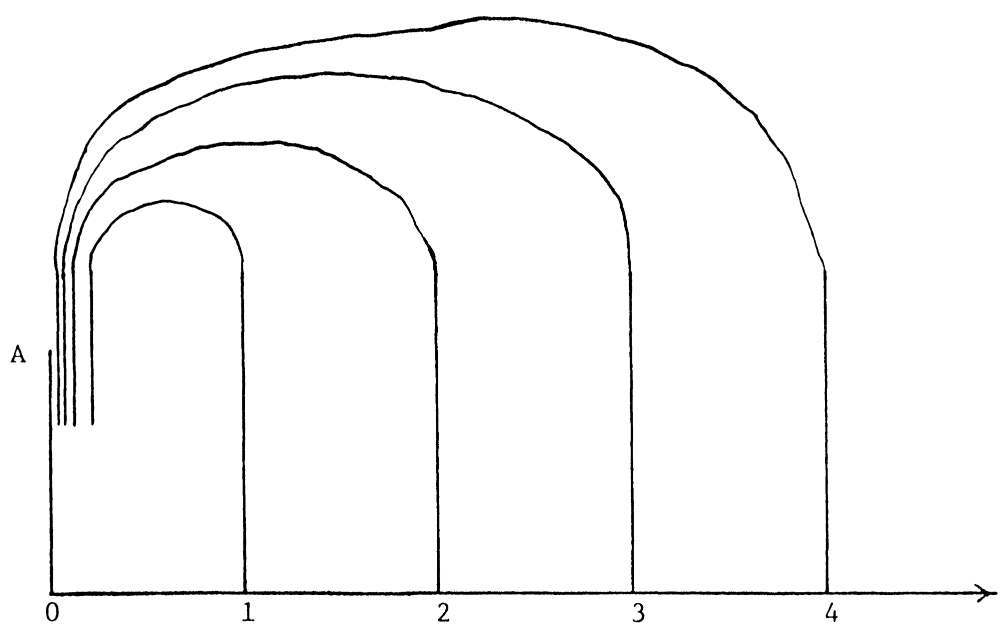
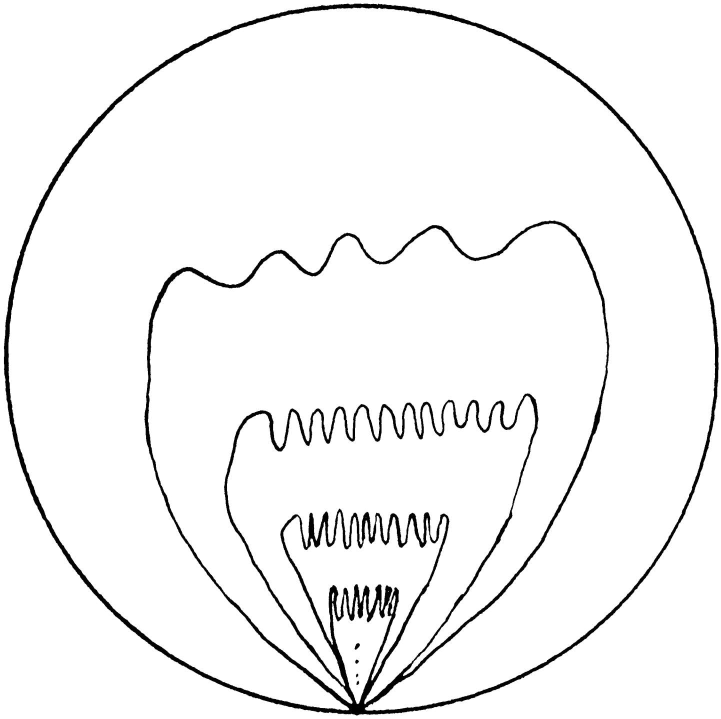


FIG. 12



"Light Bulb Space"

FIG. 13

we get a subpath, $\alpha_t = \alpha \mid [t, c]$. Now $l(\alpha_t)$ is non-increasing as $t \to c$. Suppose $l(\alpha_t) \to L > 0$. Then choose t' such that $L \le l(\alpha_{t'}) < 5L/4$. There are points

$$t' = t_0 < t_1 < \cdots < t_n = c \quad \text{with} \ \sum_{i=1}^{n} \rho(\alpha(t_i), \alpha(t_{i-1})) > 3L/4.$$

By using more points if necessary, we can assume $\rho(\alpha(t_n), \alpha(t_{n-1})) < L/4$ so that

$$\sum_{i=1}^{n-1} \rho(\alpha(t_i), \alpha(t_{i-1})) > L/2.$$

Hence $l(\alpha \mid [t', t_{n-1}]) > L/2$ and $l(\alpha_{t_{n-1}}) < 3L/4$, a contradiction. So $l(\alpha_t) \to 0$. Thus as $t \to c$, $\rho'(\alpha(t), \alpha(c)) \to 0$ and α is continuous.

Note that we may have α continuous in both cases but $l(\alpha) = \infty$. Just let $\langle X, \rho \rangle = E^2$ and $l(\alpha) = \infty$.

(3) (b) If $\langle X, \rho \rangle$ is path connected and $\rho' < \infty$ then it follows from (3a). If $\rho'(x, y) = \infty$, by (1f) there is no path in $\langle X, \rho' \rangle$ from x to y.

(3) (c) Let $l(\alpha)(l'(\alpha))$ be the length of α in $\langle X, \rho\rangle(\langle X, \rho'\rangle)$. Then

$$l'(\alpha) = \sup \sum \rho'(\alpha(t_i), \alpha(t_{i-1})) = \sup l(\alpha) = l(\alpha)$$

where the supremums are over $\{t_i\}$ in order in the domain of α.

(3) (d) Follows from (3a) and (3b). ▮

We now apply path-metrics to local dilations.

THEOREM 6.2. *If f: $\langle X, \rho\rangle \to \langle Z, d\rangle$ is onto and a local dilation, and*

$$f\colon \langle X, \rho'\rangle \to \langle Z, d'\rangle$$

is continuous, then the latter map is a global dilation.

Proof. Given $x, y \in X$, there is a path α from $f(x)$ to $f(y)$ with $l(\alpha)$ arbitrarily close to $d'(f(x), f(y))$. By The Path Length Theorem (2.2), $l(f^{-1}(\alpha)) \leq l(\alpha)$ where $f^{-1}(\alpha)$ is a path from x to y (f is a homeomorphism). Hence $\rho'(x, y) \leq d'(f(x), f(y))$. ▮

To see that we need to assume the map f: $\langle X, \rho'\rangle \to \langle Z, d'\rangle$ is continuous, consider $\langle X, \rho\rangle = [0, 1]$ in E^1 and $\langle Z, d\rangle = \{(x, x \sin(1/x))\}$ in E^2 where x is in $[0, 1]$ (an infinitely long curve) and $f(x) = (x, x \sin(1/x))$. Then in the path-length metric, f fails to be continuous at 0.

To see that we need f onto above, consider a local dilation wrapping a segment almost all the way around a circle.

It follows immediately from Theorem 6.2 that if f: $\langle X, \rho\rangle \to \langle Z, d\rangle$ is a local dilation and f: $\langle X, \rho'\rangle \to \langle Z, d'\rangle$ is a homeomorphism, then the latter map is a local dilation.

Note that the converse to Theorem 6.2 is false: Let $\langle X, \rho\rangle$, a subspace of E^2, consist of the segments, S_n, from (0, 0) to $(1, 1/n)$, $n = 1, 2, 3, \ldots$, and the segment S from (0, 0) to (1, 0). Let $\langle Z, d\rangle = \langle X, \rho\rangle - S_1$. And define f: $X \to X$ by $f(S_n) = S_{n+1}$ and $f(S) = S$, such that f doesn't change x-coordinates.

Since we will repeatedly use the fact that a global dilation from a compact metric space into itself is an isometry [5, p. 104, # 12] we indicate below more details from the proof that are in Kaplansky's book (where it is an exercise with hint). Let a and b lie in X, a compact metric space. Define $a_n = f^n(a)$ and $b_n = f^n(b)$. Then there is a convergent subsequence of $\{a_n\}$. There is a subsequence of this, say $\{a_{n_k}\}$, such that $\{b_{n_k}\}$ is also convergent. So we can find n and m to make a_n and a_{n+m} arbitrarily close to this point of convergence, and so arbitrarily close together, and we can do the same, simultaneously, with b_n and b_{n+m}. But

$$a_n a_{n+m} \geq a_{n-1} a_{n+m-1} \geq \cdots \geq a a_m \quad \text{and} \quad b_n b_{n+m} \geq b b_m.$$

Thus we can find a_m and b_m arbitrarily close to a and b respectively. Since $a_m b_m \geq a_1 b_1 \geq ab$, $a_1 b_1 = ab$ and f is an isometry. Furthermore, since a_m can be found close to a, the range is dense. But X is compact, so f is onto.

COROLLARY 6.3. *If $X \subset E^n$ is convex with the standard metric and $f: X \to X$ is onto and a local dilation then it is a global dilation. If X is also compact, then f is an isometry.*

Proof. For such an $\langle X, \rho\rangle$, $\rho = \rho'$, so $f: \langle X, \rho'\rangle \to \langle X, \rho'\rangle$ is still continuous and we can apply Theorem 6.2. If X is compact we can apply the previously mentioned fact in Kaplansky [5, p. 104]. ∎

The next theorem generalizes Theorem 4.3.

THEOREM 6.4. *If $\langle X, \rho\rangle$ is complete and a manifold without boundary and $\rho' < \infty$ then all local dilations of $\langle X, \rho\rangle$ to itself are onto.*

Proof. Let $h: \langle X, \rho\rangle \to \langle X, \rho\rangle$ be a local dilation, Then by Invariance of Domain, since h is an embedding, $h(X)$ is open in $\langle X, \rho\rangle$. Suppose h is not onto. Let $x \in h(X)$ and $y \in X - h(X)$. Since $\rho' < \infty$, there is a finitely long path α from x to y. Let β be a half-open subpath from x to, but not including, the first point of α not in $h(X)$. Let $\gamma = h^{-1}(\beta)$. Then by the Path Length Theorem (2.2), $l(\gamma) \leq l(\beta) \leq l(\alpha) < \infty$. Let $T_1 = \{t_i^1\}_{i=1}^{m_1}$ be in order such that $\sum \rho(\gamma(t_i^1), \gamma(t_{i-1}^1))$ is within 1 of $l(\gamma)$.

For each $n > 1$, let $T_n = \{t_i^n\}_{i=1}^{m_n} \supset T_{n-1}$ be in order with $\sum \rho(\gamma(t_i^n), \gamma(t_{i=1}^n))$ within 2^{-n} of $l(\gamma)$. Then $\gamma(t_{m_n})$ is within 2^{-n} of $\gamma(t)$ for $t_{m_n} \leq t$ and t in the domain of γ. So $\{\gamma(t_{m_n})\}$ is Cauchy in $\langle X, \rho\rangle$ and hence converges to some z in X. Complete the proof as in Theorem 4.2. ∎

We note that none of the hypotheses can be dropped. To see that completeness and no boundary are required, consider an open ray and a closed ray respectively, with the usual metric and h a translation. Having ρ' finite insures that X is connected—consider $X = \{(x, y) \mid y \text{ is a positive integer}\} \subset E^2$ and $h(x, y) = (x, y + 1)$.

The following example shows that $\langle X, \rho\rangle$ can be very "nice" but $\langle X, \rho'\rangle$ very "bad". Such behavior can be avoided by assuming $\rho' < \infty$.

Example 6.5. $\langle X, \rho\rangle \approx E^1$ and $\langle X, \rho'\rangle$ is discrete. Let X consist of the reals with $\rho(x, y) = \sqrt{|x - y|}$. Then $\langle X, \rho\rangle$ is a metric space homeomorphic to E^1. But if a and b are at distance c in $\langle X, \rho\rangle$, then $\rho'(a, b) \geq \sqrt{c}n$ for every positive n. ∎

COROLLARY 6.6. *If $\langle X, \rho\rangle$ and $\langle X, \rho'\rangle$ are compact manifolds without boundary and $\rho' < \infty$ then all local dilations of $\langle X, \rho\rangle$ to itself are isometries on $\langle X, \rho'\rangle$.*

Proof. By Theorem 6.4 a local dilation will be onto. By Theorem 6.2 it will be a global dilation of $\langle X, \rho'\rangle$. And by Kaplansky [5, p. 104] it will be an isometry on $\langle X, \rho'\rangle$. ∎

Note that $\langle X, \rho \rangle$ can be a manifold and $\langle X, \rho' \rangle$ fail to be a manifold. Consider the "snowflake" curve, an infinitely long loop (or a double cone on the snowflake curve).

COROLLARY 6.7. *If $\langle X, \rho \rangle$ is a compact Riemannian manifold (connected and without boundary) then any local dilation is an isometry.*

Proof. For a Riemannian manifold, $\rho = \rho'$. ▮

We now have an easy proof of Theorem 5.1: A local dilation of a standard sphere to itself is an isometry. Although $\rho \neq \rho'$, $\rho(a, b) = \rho(c, d)$ if and only if $\rho'(a, b) = \rho'(c, d)$. So an isometry on $\langle X, \rho' \rangle$ is an isometry on $\langle X, \rho \rangle$. Now apply Corollary 6.6.

Conjecture 6.8. Suppose $S \subset E^{n+1}$ is a homeomorph of the unit n-sphere and S together with its interior, is convex. Then any local dilation of S is a global dilation (and hence an isometry). ▮

Note that Example 5.3 shows the necessity of some assumption such as convexity—piecewise linear doesn't suffice.

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