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HYPERFUNCTIONS AS BOUNDARY VALUES OF GENERALIZED AXIALLY SYMMETRIC POTENTIALS

BY

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1. Introduction

In the classical Dirichlet problem on the unit disk, one starts with a given function $f(\theta)$ on the unit circle and seeks a harmonic function $f(r, \theta)$ in the open unit disk that converges to $f(\theta)$ as r goes to 1. The solution to this problem is very well known. However, the solution to the converse problem i.e. finding a boundary function to which a given harmonic function in the interior of the disk converges, was found relatively recently by Gelfand [1], Johnson [4], Köthe [5] and Sato [6]. It turns out that this solution always exists in the space of hyperfunctions on the unit circle and is unique.

In [4] Johnson gives a characterization for the solutions of Laplace equation in the unit disk, namely, f is harmonic on the unit disk if and only if there is a sequence $\{g_n\}$ of continuous functions on the unit circle such that

$$\lim_{n\to\infty}(n!\|g_n\|_\infty)^{1/n} = 0$$

and

$$f(r, \theta) = \sum_{n=0}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} p_r^{(n)}(\theta - t)g_n(t)\,dt$$

where $\|g_n\|_\infty = \sup_{0\le t<2\pi}|g_n(t)|$ and $P_r^{(n)}(\theta)$ is the nth derivative of Poisson kernel for the unit disk. In addition to that, he shows that the space $\mathscr{H}^*$ of hyperfunctions on the unit circle is isomorphic to the space $\mathscr{F}$ of harmonic functions on the unit disk. The correspondence $f \leftrightarrow \tilde{f}$ where $f \in \mathscr{F}$ and $\tilde{f} \in \mathscr{H}^*$ is given by $f_r(\theta) = P_r * \tilde{f}$ where $*$ stands for the convolution, and $f_r(\theta) \to \tilde{f}$ in $\mathscr{H}^*$ as $r \to 1$.

More recently, Staples and Kelingos [7] have characterized all solutions of a perturbed Laplace equation in the unit disk by identifying their generalized boundary values.

In this paper we prove similar results for the regular solutions to the partial differential equation of Generalized Axially Symmetric Potentials (GASP):

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{2\mu}{y}\frac{\partial\phi}{\partial y} = 0 \quad \text{where } \mu > 0.$$

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One is led naturally to consider this type of differential equations if one considers those solutions of the n-dimensional Laplace equation

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \cdots + \frac{\partial^2 \phi}{\partial x_n^2} = 0, \quad n > 2,$$

which depend solely on the variables $x = x_1$, $y = (x_2^2 + \cdots + x_n^2)^{1/2}$. In this case $2\mu = n - 2$.

Throughout this report, a hyperfunction means a hyperfunction on $[-1, 1]$ or equivalently a hyperfunction on the unit circle with the points $(1, \theta)$ and $(1, -\theta)$ identified and L denotes the Gegenbauer differential operator

$$L = (1 - x^2)\frac{d^2}{dx^2} - (2\mu + 1)x\frac{d}{dx} - \mu^2.$$

Hence, $LC_k^\mu(x) = -(k + \mu)^2 C_k^\mu(x)$ where $C_k^\mu(x)$ is the Gegenbauer polynomial of degree k. L_θ denotes the same operator with $\cos\theta$ as independent variable and $\mathscr{H}$ denotes the linear space of analytic functions on $[-1, 1]$ provided with the topology introduced in [5]. The strong dual $\mathscr{H}^*$ of $\mathscr{H}$ is the space of hyperfunctions.

In Section 2 we give a characterization for the GASP functions analogous to the one given by Johnson for harmonic functions. In Section 3 the boundary values of GASP functions and their relationship with the space of hyperfunctions are investigated. Unlike Johnson, we reserve the term "generalized function" for "Schwartz distribution", hence every generalized function is a hyperfunction.

2. A characterization of GASP functions

In order to prove the characterization theorem we need the following lemma whose main idea goes back to Johnson.

LEMMA 2.1. *Let $\{a_k\}_{k=1}^\infty$ be a sequence of complex numbers satisfying*

$$\lim_{k\to\infty} |a_k|^{1/k} \le 1.$$

Then, there are sequences $\{a_{k,n}\}$ and a finite-valued function $B(\varepsilon)$ such that $a_k = \sum_{n=0}^k a_{k,n}$ and

$$|a_{k,n}| \le \frac{B(\varepsilon)\varepsilon^{2n+2[\mu]+2}}{(2n + 2[\mu] + 2)!} k^{2n} \quad \text{for } 0 \le n \le k,\ k = 1, 2, 3, \ldots,$$

all $\varepsilon > 0$ and fixed $\mu > 0$, where $[\mu]$ is the greatest integer less or equal to μ.

Proof. Let $d_\nu = \sup_{k\ge\nu} |a_k|^{1/k}$, $\nu = 1, 2, \ldots$ Then d_ν is a monotone decreasing sequence with limit less or equal to one. Therefore, we can write $d_\nu \le 1 + \varepsilon_\nu$ where $\varepsilon_\nu \downarrow 0$. Hence, for $k \ge 1$

$$|a_k| \le d_k^k \le (1 + \varepsilon_k)^k = \sum_{n=0}^{k} \binom{k}{n} \varepsilon_k^n \le 2 \sum_{n=0}^{[k/2]} \binom{k}{2n} \varepsilon_k^{2n} \tag{1}$$

Inequality (1) follows from the inequality

$$(1-\varepsilon)^k = \sum_{n=0}^{[k/2]} \binom{k}{2n} \varepsilon^{2n} - \sum_{n=0}^{[k-1/2]} \binom{k}{2n+1} \varepsilon^{2n+1} > 0 \quad \text{for } 0 \le \varepsilon < 1.$$

Because of (1) we can write, for k even, $a_k = a_{k,0} + \cdots + a_{k,k/2}$ with

$$|a_{k,n}| \le 2\binom{k}{2n}\varepsilon_k^{2n} \le 2\frac{k^{2n}}{2n!}\varepsilon_k^{2n}, \quad n = 0, 1, \ldots, k/2. \tag{2}$$

For the sake of symmetry, we add $k/2$ terms (possibly zeros) without violating inequality (2) so that $a_k = a_{k,0} + a_{k,1} + \cdots + a_{k,k}$. Clearly this decomposition is not unique. The case k is odd is treated exactly the same way except the number of terms added is $(k+1)/2$. Therefore, we have

$$|a_{k,n}| \le 2\frac{k^{2n}}{2n!}\varepsilon_k^{2n}, \quad 0 \le n \le k \text{ and } k = 1, 2, 3, \ldots, \tag{3}$$

and for the sake of completeness we set $a_0 = a_{0,0}$.

For $\varepsilon > 0$, we choose $k(\varepsilon)$ such that $\varepsilon_k < \varepsilon/e$ for all $k \ge k(\varepsilon)$. Define A by

$$A = \max\left(\frac{(2n+1)(2n+2)\cdots(2n+2[\mu]+2)}{e^{2n}}\right).$$

A is finite and depends only on μ because the function

$$f_p(x) = \frac{(2x+1)(2x+2)\cdots(2x+p)}{e^{2x}}, \ p \text{ a positive integer},$$

is bounded in $[0, \infty)$ and goes to zero as $x \to \infty$.

Thus, $(2n+1)(2n+2) \cdots (2n+2[\mu]+2)e^{-2n} \le A$ for all n, and consequently we have

$$\varepsilon_k^{2n} \le \varepsilon^{2n}e^{-2n} \le \frac{A\varepsilon^{2n}}{(2n+1)\cdots(2n+2[\mu]+2)},$$

for all n and all $k \ge k(\varepsilon)$. But there are only finitely many k's and n's such that $0 \le n \le k < k(\varepsilon)$, hence we let

$$B_1(\varepsilon) = \sup_{\substack{0 \le n \le k, \\ 1 \le k < k(\varepsilon)}} \frac{(2n+1)\cdots(2n+2[\mu]+2)}{\varepsilon^{2[\mu]+2}}\left(\frac{\varepsilon_k}{\varepsilon}\right)^{2n} < \infty,$$

$$B_2(\varepsilon) = \frac{A}{\varepsilon^{2[\mu]+2}}$$

and

$$B(\varepsilon) = 2\max(B_1(\varepsilon), B_2(\varepsilon)). \tag{4}$$

Using (3) and (4), we immediately obtain

$$|a_{k,n}| \leq \frac{B(\varepsilon)\varepsilon^{2n+2[\mu]+2}}{(2n+2[\mu]+2)!} k^{2n} \quad \text{for } 0 \leq n \leq k, \text{ and } k = 1, 2, \ldots . \quad \text{Q.E.D.}$$

let $H(t, x, r)$ denote the kernel of Abel summability of Gegenbauer expansions:

$$\begin{aligned} H(t, x, r) &= \sum_{n=0}^{\infty} \frac{1}{h_n^\mu} (1-t^2)^{\mu-1/2} C_n^\mu(t) C_n^\mu(x) r^n \\ &= \frac{\mu}{\pi} \int_{-1}^{1} (1-t^2)^{\mu-1/2} (1-u^2)^{\mu-1} \frac{(1-r^2)}{(1-2r\cos\gamma + r^2)^{\mu+1}}\, du \end{aligned} \tag{5}$$

where $\cos\gamma = xt + u\sqrt{(1-t^2)(1-x^2)}$, $-1 \leq t, x \leq 1$, $0 \leq r < 1$, and

$$\int_{-1}^{1} (1-t^2)^{\mu-1/2} C_n^\mu(t) C_m^\mu(t)\, dt = h_n^\mu \delta_{nm}.$$

For short, when $x = \cos\theta$, we write $H(t, \theta, r)$ instead of $H(t, \cos\theta, r)$.

THEOREM 2.2. *A function $f(r, \theta)$ is a GASP function regular in the open unit disk $\{z\colon |z| < 1\}$ if and only if there exists a sequence of continuous functions $\{g_n\}$ on the interval $[-1, 1]$ such that*

$$\lim_{n\to\infty} (2n!\|g_n\|_\infty)^{1/n} = 0$$

and

$$f(r, \theta) = \sum_{n=0}^{\infty} L_\theta^n \int_{-1}^{1} g_n(t) H(t, \theta, r)\, dt, \quad 0 < |\theta| < \pi.$$

Proof. Sufficiency. Let $\{g_n\}$ be given as in the hypothesis of the theorem. Then, for $n = 0, 1, 2, \ldots$, we have

$$g_n(t) \sim \sum_{k=0}^{\infty} a_{k,n} C_k^\mu(t);$$

Set $f_n(r, \theta) = L_\theta^n \int_{-1}^{1} g_n(t) H(t, \theta, r)\, dt$. Then it can be easily verified that

$$\int_{-1}^{1} g_n(t) H(t, \theta, r)\, dt = \sum_{k=0}^{\infty} a_{k,n} C_k^\mu(\cos\theta) r^k, \quad 0 \leq r < 1$$

and hence

$$f_n(r, \theta) = (-1)^n \sum_{k=0}^{\infty} (k+\mu)^{2n} a_{k,n} C_k^\mu(\cos\theta) r^k, \quad 0 < |\theta| < \pi.$$

To show that $\sum_{n=0}^{\infty} f_n(r, \theta)$ converges for $0 \le r < 1$, we invoke Gilbert-Bergman integral operator (cf. Gilbert [2]) which maps the holomorphic function given by the series

$$\sum_{k=0}^{\infty} \frac{\Gamma(k+2\mu)}{k!\,\Gamma(2\mu)} a_k \sigma^k, \quad |\sigma| < 1, \sigma = x + iy \cos \alpha,$$

into a GASP function element given by

$$\sum_{k=0}^{\infty} a_k r^k C_k^{\mu}(\cos \theta), \quad 0 \le r < 1.$$

Thus,

$$\begin{aligned} f_n(r, \theta) &= (-1)^n \sum_{k=0}^{\infty} (k+\mu)^{2n} a_{k,n} C_k^{\mu}(\cos \theta) r^k \\ &= i\alpha_\mu \int_0^{\pi} F_n(x + iy \cos \alpha)(2i)^{2\mu-1}(\sin \alpha)^{2\mu-1}\, d\alpha \end{aligned}$$

where

$$F_n(\sigma) = (-1)^n \sum_{k=0}^{\infty} \frac{\Gamma(k+2\mu)}{k!\,\Gamma(2\mu)} (k+\mu)^{2n} a_{k,n} \sigma^k, \tag{6}$$

$$x = r\cos\theta, \quad y = r \sin \theta, \quad \alpha_\mu = \frac{4\Gamma(2\mu)}{(4i)^{2\mu}[\Gamma(\mu)]^2}.$$

Clearly, $|\sigma| = |x + iy \cos \alpha| < r$ and

$$|f_n(r, \theta)| \le \max_{0 \le \alpha \le \pi} |F_n(\sigma)|. \tag{7}$$

Now

$$a_{k,n} = \frac{1}{h_k^{\mu}} \int_{-1}^{1} g_n(t)(1-t^2)^{\mu-1/2} C_k^{\mu}(t)\, dt;$$

therefore

$$|a_{k,n}| \le \frac{\|g_n\|_\infty}{h_k^{\mu}} \int_{-1}^{1} (1-t^2)^{\mu-1/2} |C_k^{\mu}(t)|\, dt.$$

Using the estimates (cf. [8])

$$C_k^{\mu}(\cos\theta) = \begin{cases} \theta^{-\mu} O(k^{\mu-1}), & c/k \le \theta \le \pi/2, \\ O(k^{2\mu-1}),) & 0 \le \theta \le c/k \end{cases}$$

we obtain

$$\int_0^1 (1-t^2)^{\mu-1/2} |C_k^{\mu}(t)|\, dt = O(k^{\mu-1})$$

from which we get

$$|a_{k,n}| \leq C \frac{\|g_n\|_\infty}{h_k^\mu} k^{\mu-1} = \|g_n\|_\infty O(k^{-\mu+1}) \tag{8}$$

since

$$h_k^\mu \sim \frac{2^{1-2\mu} k^{2\mu-2} \pi}{[\Gamma(\mu)]^2}.$$

Upon using the inequality $(k+\mu)^{2n}/k^{2n} \leq (1+\mu)^{2n}$; $k = 1, 2, \ldots$ and substituting (8) in (6) we obtain

$$\begin{aligned} |F_n(\sigma)| &\leq C \sum_{k=0}^{\infty} k^{2\mu-1} k^{2n} \|g_n\|_\infty k^{-\mu+1} |\sigma^k| \\ &\leq C \|g_n\|_\infty \sum_{k=0}^{\infty} k^{2n+\mu} |\sigma^k| \\ &\leq C \|g_n\|_\infty \sum_{k=0}^{\infty} k^{2n+[\mu+1]} |\sigma^k| \\ &\leq C \|g_n\|_\infty \{(2n+[\mu+1])!\} a^{2n+[\mu+1]+1}, \quad 0 \leq |\sigma| \leq r < 1, \end{aligned}$$

for a suitable constant $a > 0$ depending on r (see Johnson [4, Proposition 1]).

Therefore, the series $\sum_{n=0}^{\infty} |f_n(r, \theta)|$ is majorized by the series $\sum_{n=0}^{\infty} (2n + [\mu+1])! \|g_n\|_\infty a^{2n+[\mu+1]+1}$ which converges since

$$\begin{aligned} &\overline{\lim_{n\to\infty}} \{(2n+[\mu+1])! \|g_n\|_\infty a^{2n+[\mu+1]+1}\}^{1/n} \\ &\quad = \overline{\lim_{n\to\infty}} \{a^{2n+[\mu+1]+1}\}^{1/n} \{(2n+[\mu+1])! \|g_n\|_\infty\}^{1/n} \\ &\quad = A \overline{\lim_{n\to\infty}} \{2n! \|g_n\|_\infty (2n)^{[\mu+1]}\}^{1/n} \\ &\quad = A \overline{\lim_{n\to\infty}} \{2n! \|g_n\|_\infty\}^{1/n} = 0 \quad \text{by hypothesis.} \end{aligned}$$

Necessity. Let $f(r, \theta)$ be a GASP function in $\{z : |z| < 1\}$. Then

$$f(r, \theta) = \sum_{k=0}^{\infty} a_k r^k C_k^\mu(\cos\theta), \quad |r| < 1, \ \overline{\lim_{k\to\infty}} |a_k|^{1/k} \leq 1.$$

Let $\{a_{k,n}\}$ be the decomposition sequence of $\{a_k\}$ given by Lemma (2.1) and let

$$g_n(r, \theta) = \sum_{k=n-[\mu]-1}^{\infty} (-1)^n \frac{a_{k,n-[\mu]-1}}{(k+\mu)^{2n}} C_k^\mu(\cos\theta) r^k, \quad n \geq [\mu] + 1.$$

Each $g_n(r, \theta)$ defines a continuous function on the closed unit disk $\{z: |z| \leq 1\}$ since (except for a trivial modification when $n = [\mu] + 1$) we have

$$\begin{aligned} |g_n(r, \theta)| &\leq \sum_{k=n-[\mu]-1}^{\infty} \frac{|a_{k,n-[\mu]-1}|}{(k+\mu)^{2n}} |C_k^\mu(\cos\theta)| \\ &\leq \sum_{k=n-[\mu]-1}^{\infty} \frac{B(\varepsilon)\varepsilon^{2n} k^{2n-2[\mu]-2} k^{2\mu-1}}{2n!(k+\mu)^{2n}} \\ &\leq \frac{B(\varepsilon)\varepsilon^{2n}}{2n!} \sum_{k=n-[\mu]-1}^{\infty} \frac{k^{2n}}{(k+\mu)^{2n}} \frac{k^{2(\mu-[\mu])}}{k^3}, \quad 0 \leq r \leq 1. \end{aligned} \tag{9}$$

The last series converges since $2(\mu - [\mu]) < 2$ and hence the series defining $g_n(r, \theta)$ converges uniformly on $[|z| \leq 1]$. We denote by $g_n(t)$, for $-1 \leq t \leq 1$, the function $g_n(t) = g_n(1, \theta)$, $t = \cos\theta$ and $n \geq [\mu] + 1$.

The sequence $\{g_n\}$ satisfies the hypothesis of the theorem since

$$\int_{-1}^{1} g_n(t) H(t, \theta, r)\, dt = g_n(r, \theta)$$

and by relation (9),

$$\overline{\lim_{n\to\infty}} (2n! \|g_n\|_\infty)^{1/n} \leq \varepsilon^2 \quad \text{for all } \varepsilon > 0.$$

Finally,

$$L_\theta^n g_n(r, \theta) = \sum_{k=n-[\mu]-1}^{\infty} a_{k,n-[\mu]-1} C_k^\mu(\cos\theta) r^k, \quad 0 < |\theta| < \pi,$$

and

$$\sum_{n=[\mu]+1}^{\infty} L_\theta^n g_n(r, \theta) = \sum_{n=[\mu]+1}^{\infty} \sum_{k=n-[\mu]-1}^{\infty} a_{k,n-[\mu]-1} C_k^\mu(\cos\theta) r^k,$$

$$\sum_{k=0}^{\infty} a_k C_k^\mu(\cos\theta) r^k = f(r, \theta).$$

Q.E.D.

3. Hyperfunctions as boundary values of GASP functions

Throughout the rest of this section, μ will be restricted such that $\mu - \frac{1}{2}$ is a non-negative integer, hence $(1 - t^2)^{\mu - 1/2}$ is a holomorphic function in some neighborhood of $[-1, 1]$. It should be pointed out that under this restriction f is a hyperfunction on $[-1, 1]$ if and only if $(1 - t^2)^{\mu - 1/2} f$ is. The next three theorems have counterparts in [4] but the proofs require slightly different techniques.

THEOREM 3.1. *Let $\{a_k\}$ be a sequence of complex numbers, then the series $\sum_{k=0}^{\infty} a_k C_k^\mu$ converges to a hyperfunction f on $[-1, 1]$ if and only if* $\overline{\lim}_{k\to\infty} |a_k|^{1/k} \leq 1$.

Proof. Suppose that the series $\sum_{k=0}^{\infty} a_k C_k^\mu$ converges to a hyperfunction f. Then for any $\phi(z) \in \mathscr{H}$ (the space of analytic functions on $[-1, 1]$ we have

$$\phi(z) = \sum_{k=0}^{\infty} b_k C_k^\mu(z) \quad \text{with } \overline{\lim_{k\to\infty}} |b_k|^{1/k} < 1 \tag{10}$$

(cf. Szegö [8]) and the series converges to ϕ in the sense of $\mathscr{H}$.

Let $\phi_N(z) = \sum_{k=0}^{N} b_k C_k^\mu(z)$; then

$$(f, (1-t^2)^{\mu-1/2}\phi_N(z)) \to (f, (1-t^2)^{\mu-1/2}\phi(z)) \quad \text{as } N \to \infty.$$

That is

$$\sum_{k=0}^{N} \bar{b}_k a_k h_k^\mu \to \sum_{k=0}^{\infty} \bar{b}_k a_k h_k^\mu \quad \text{as } N \to \infty \tag{11}$$

Suppose $\overline{\lim}_{k\to\infty} |a_k|^{1/k} = \rho > 1$; then for all $\varepsilon > 0$ such that $1 < \rho - 2\varepsilon$, there exists a subsequence $\{a_{k_v}\}$ so that $(\rho - \varepsilon) < |a_{k_v}|^{1/k_v}$ for $v = 0, 1, 2, \ldots$. Since (11) is true for all $\{b_k\}_{k=0}^{\infty}$ satisfying (10), then we can choose $\phi(z) \in \mathscr{H}$ for which

$$\frac{1}{\rho - 2\varepsilon} < \overline{\lim_{k\to\infty}} |b_k|^{1/k} < 1.$$

By passing to a subsequence $\{b_{k_v}\}$, we obtain

$$|\bar{b}_{k_v} a_{k_v}|^{1/k_v} > \frac{\rho - \varepsilon}{\rho - 2\varepsilon} > 1.$$

Therefore, since $\lim_{k\to\infty} |h_k^\mu|^{1/k} = 1$, the series in (11) diverges which is a contradiction. Thus $\overline{\lim}_{k\to\infty} |a_k|^{1/k} = \rho \leq 1$.

Conversely, let $\{a_k\}_{k=0}^{\infty}$ be a sequence such that $\overline{\lim}_{k\to\infty} |a_k|^{1/k} \leq 1$. Set

$$f_n = \sum_{k=0}^{n} a_k C_k^\mu.$$

We want to show that $\{f_n\}_{n=0}^{\infty}$ converges in $\mathscr{H}^*$ as $n \to \infty$. Since both $\mathscr{H}$ and $\mathscr{H}^*$ are Fréchet-Montel spaces and every Montel space is reflexive, hence weak and strong sequential convergence coincide in $\mathscr{H}^*$, it suffices to show that $\lim_{n\to\infty} (f_n, \phi)$ exists for all $\phi \in \mathscr{H}$. First, we show that

$$\lim_{n\to\infty} ((1-t^2)^{\mu-1/2} f_n, \phi) \tag{12}$$

exists, from which we deduce that $\lim_{n\to\infty} f_n$ is a hyperfunction on $(-1, 1)$. It is evident that (12) holds since for all $\phi(z) \in \mathscr{H}$, we have

$$\phi(z) = \sum_{k=0}^{\infty} b_k C_k^\mu(z) \text{ with } \overline{\lim_{k\to\infty}} |b_k|^{1/k} = \alpha < 1,$$

and also $\lim_{k\to\infty} |h_k^\mu|^{1/k} = 1$ and consequently

$$\lim_{n\to\infty}((1-t^2)^{\mu-1/2}f_n, \phi) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \bar{b}_k h_k^\mu = \sum_{k=0}^{\infty} a_k \bar{b}_k h_k^\mu.$$

But the last series converges since $\lim_{k\to\infty} |\bar{b}_k a_k h_k^\mu|^{1/k} < 1$. Q.E.D.

Using the characterization given above for $\mathscr{H}$ and $\mathscr{H}^*$, we can define a direct product $\otimes$: $\mathscr{H}^* \times \mathscr{H}^* \to \mathscr{H}^*$ as follows:

$$(f \otimes g)(x) = \sum_{k=0}^{\infty} a_k b_k C_k^\mu(x)$$

where $f, g \in \mathscr{H}^*$ and $f = \sum_{k=0}^{\infty} a_k C_k^\mu$, $g = \sum_{k=0}^{\infty} b_k C_k^\mu$.

This operation is well defined since $\overline{\lim}_{k\to\infty} |a_k b_k|^{1/k} \leq 1$ if

$$\overline{\lim_{k\to\infty}} |a_k|^{1/k} \leq 1 \quad \text{and} \quad \overline{\lim_{k\to\infty}} |b_k|^{1/k} \leq 1.$$

This direct product plays a role similar to that of convolution for hyperfunctions on the unit circle.

It is easy to see that $(\mathscr{H}^*, +, \otimes)$ is a commutative ring with identity $e(x) = \sum_{k=0}^{\infty} C_k^\mu(x)$. In fact, $(\mathscr{H}, +, \otimes)$ is an ideal of $(\mathscr{H}^*, +, \otimes)$. We will denote by $e_r(x)$ that element of $\mathscr{H}$ given by

$$e_r(x) = \sum_{k=0}^{\infty} r^k C_k^\mu(x) = \frac{1}{(1 - 2xr + r^2)^\mu}, \quad 0 \leq r < 1.$$

By the same argument we used in Theorem (3.1) we can easily show that $e_r(x) \to e(x)$ in the sense of $\mathscr{H}^*$ as $r \to 1$. With no difficulty, one can show that the representation of Dirac delta function is given by

$$\delta(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\mu)_m}{h_{2m}^\mu m!} C_{2m}^\mu(x) = H(0, x, 1)$$

where $H(t, x, r)$ is the kernel given by equation (5). We should keep in mind that for trigonometric series the Poisson kernel

$$P(t, x, r) = \sum_{k=-\infty}^{\infty} r^{|k|} \exp ik(x - t)$$

has the property that $P(0, x, 1)$ is the identity for the operation of convolution.

Now we are able to show that the class of hyperfunction on $[-1, 1]$ can be considered as "boundary values" for GASP functions in the unit disk.

THEOREM 3.2. *A function f is a* GASP *on $\{z: |z| < 1\}$ if and only if there is a hyperfunction $\tilde{f}$ on $[-1, 1]$ such that $f(r, \theta) = (e_r \otimes \tilde{f})(\theta)$ $0 \leq r < 1$. Moreover, $f(r, \theta) \to \tilde{f}$ in $\mathscr{H}^*$ as $r \to 1$ and hence $\tilde{f}$ is uniquely determined on $(-1, 1)$.*

Proof. Let f be a GASP function. Then

$$f(r, \theta) = \sum_{k=0}^{\infty} a_k r^k C_k^\mu(\cos\theta) \quad \text{and} \quad \overline{\lim_{k\to\infty}} |a_k|^{1/k} \leq 1.$$

Define $\tilde{f} = \sum_{k=0}^{\infty} a_k C_k^\mu(\cos\theta)$. Then $\tilde{f} \in \mathscr{H}^*$ and

$$(e_r \otimes \tilde{f})(\theta) = \sum_{k=0}^{\infty} a_k r^k C_k^\mu(\cos\theta) = f(r, \theta).$$

That $f(r, \theta) \to \tilde{f}$ in $\mathscr{H}^*$ as $r \to 1$ was shown in the proof of Theorem (3.1). Conversely, if $\tilde{f} \in \mathscr{H}^*$, then

$$\tilde{f} = \sum_{k=0}^{\infty} a_k C_k^\mu \quad \text{with} \quad \overline{\lim_{k\to\infty}} |a_k|^{1/k} \le 1$$

and it is evident that $(\tilde{f} \otimes e_r)(\theta)$ is a GASP function and $(\tilde{f} \otimes e_r)(\theta) \to \tilde{f}$ in $\mathscr{H}^*$ as $r \to 1$.

Remarks. (i) By using standard arguments it can be shown that $\tilde{f} \in \mathscr{H}^*$ is a generalized function "Schwartz distribution" if and only if $\tilde{f} = \sum_{k=0}^{\infty} a_k C_k^\mu$ with $a_k = O(k^p)$ for some integer p.

(ii) In virtue of Theorem (2.2) and Theorem (3.2) one would expect to have a theorem similar to (2.2) for hyperfunctions. Indeed, this is the case and that is what we will show next.

THEOREM 3.3. *$\tilde{f}$ is a hyperfunction if and only if there exists a sequence of continuous functions $\{g_n\}$ on $[-1, 1]$ such that*

$$\overline{\lim_{n\to\infty}} [2n! \|g_n\|_\infty]^{1/n} = 0$$

and $\tilde{f} = \sum_{n=0}^{\infty} L^n g_n$ on $(-1, 1)$.

Proof. Observe that since g_n is continuous and $L^n g_n$ is just a finite linear combination of some derivatives of g_n up to order $2n$ and multiplication by C^∞ functions, then $L^n g_n$ is a generalized function and so is any finite sum of the series.

Now let $\tilde{f}$ be a hyperfunction, then define $f(r, \theta) = (e_r \otimes \tilde{f})(\theta)$. By Theorem (2.2) there exists a sequence of continuous functions $\{g_n\}$ on $[-1, 1]$ such that

$$\lim_{n\to\infty} [2n! \|g_n\|_\infty]^{1/n} = 0 \quad \text{and} \quad f(r, \theta) = \sum_{n=0}^{\infty} L_\theta^n g_n(r, \theta).$$

Then

$$\begin{aligned} f(r, \theta) &= \sum_{n=0}^{\infty} L_\theta^n[(e_r \otimes g_n)(\theta)] \\ &= \sum_{n=0}^{\infty} (e_r \otimes L_\theta^n g_n)(\theta) \\ &= \left(e_r \otimes \sum_{n=0}^{\infty} L_\theta^n g_n\right)(\theta) \\ &= (e_r \otimes \tilde{f})(\theta). \end{aligned}$$

Therefore, by the uniqueness part of Theorem (3.2), it follows that $\tilde{f} = \sum_{n=0}^{\infty} L^n g_n$ on $(-1, 1)$.

Conversely, let $f_N = \sum_{n=0}^{N} L^n g_n$, where $\{g_n\}$ satisfy the hypothesis of the theorem. Since $g_n(x)$ is continuous on $[-1, 1]$, then

$$g_n(x) \sim \sum_{k=0}^{\infty} a_{nk} C_k^{\mu}(x) \quad \text{and} \quad L^n g_n = \sum_{k=0}^{\infty} (-1)^n (k+\mu)^{2n} a_{nk} C_k^{\mu}(x)$$

where the series converges on $(-1, 1)$ in the sense of generalized functions. Hence

$$f_N = \sum_{k=0}^{\infty} b_{Nk} C_k^{\mu} \quad \text{where} \quad b_{Nk} = \sum_{n=0}^{N} (-1)^n (k+\mu)^{2n} a_{nk}.$$

Using the estimate given in the proof of Theorem (2.2) equation (8) for a_{nk}, we obtain

$$|b_{Nk}| \le C \sum_{n=0}^{N} (k+\mu)^{2n} \|g_n\|_{\infty} k^{-\mu}$$

(the case $k = 0$ is trivial). But since

$$\overline{\lim_{n \to 0}} \, (2n! \|g_n\|_{\infty})^{1/n} = 0,$$

then there is a finite valued decreasing function $B(\varepsilon)$ such that $2n! \|g_n\|_{\infty} \le B(\varepsilon)\varepsilon^{2n}$ for all n and all $\varepsilon > 0$. Hence

$$|b_{Nk}| \le \frac{C}{k^{\mu}} \sum_{n=0}^{N} \frac{(k+\mu)^{2n} \varepsilon^{2n} B(\varepsilon)}{2n!} \le \frac{C}{k^{\mu}} B(\varepsilon) \exp\, (k+\mu)\varepsilon$$

for all k, N and $\varepsilon > 0$. For each fixed k, the sequence $\{b_{Nk}\}_{N=0}^{\infty}$ has a limit since it is the sequence of partial sums of a dominated series. Taking the limit as $N \to \infty$ and setting $b_k = \lim_{N \to \infty} b_{Nk}$, we get $\tilde{f} = \sum_{k=0}^{\infty} b_k C_k^{\mu}$ which is in $\mathscr{H}^*$ since

$$\overline{\lim_{k \to \infty}} \, |b_k|^{1/k} \le \overline{\lim_{k \to \infty}} \left(\frac{1}{k^{\mu}} B(\varepsilon) \exp\, (k+\mu)\varepsilon \right)^{1/k} = \exp \varepsilon$$

for all $\varepsilon > 0$ and consequently

$$\overline{\lim_{k \to \infty}} \, |b_k|^{1/k} \le 1.$$

COROLLARY. *f is a generalized function on $[-1, 1]$ if and only if there is a finite number of continuous functions $\{g_0, \ldots, g_N\}$ on $[-1, 1]$ such that $f = \sum_{n=0}^{N} L^n g_n$ on $(-1, 1)$.*

REFERENCES

1. I. M. GEL'FAND and G. E. SHILOV, *Quelques applications de la theories des fonctions generalisees*, J. Math. Pures Appl., vol. 35 (1956), pp. 383–413.

2. R. P. GILBERT, *Function theoretic methods in partial differential equations*, Academic Press, New York, 1969.
3. G. H. HARDY, *Divergent series*, Oxford University Press, London 1963, pp. 171.
4. G. JOHNSON, *Harmonic functions on the unit disk I*, Illinois J. Math., vol. 12 (1968), pp. 366–385.
5. G. KÖTHE, *Topological vector spaces I*, Springer-Verlag, New York, 1969.
6. M. SATO, *Theory of hyperfunctions I and II*, J. Fac. Sci. Univ. Tokyo, vol. 8 (1959–1960), pp. 139–193, 287–437.
7. P. O. STAPLES and J. A. KELINGOS, *A characterization of solutions to a perturbed Laplace equation*, Illinois J. Math., vol. 22 (1978), pp. 208–216.
8. G. SZEGÖ, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Amer. Math. Soc., Providence, R.I., 1959.

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