ILLINOIS JOURNAL OF MATHEMATICS
Volume 25, Number 4, Winter 1981

# A CHARACTER TABLE BOUND FOR THE SCHUR INDEX

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## Introduction

Although there can be no universal formula giving the Schur index of an irreducible character of a finite group as a function of the character values (see [5]), various estimates for Schur indices have been obtained. Most recent work on the Schur index is based on the Brauer-Witt reduction, which relates the Schur index of an irreducible character of a group $G$ to Schur indices of irreducible characters of certain hyperelementary sections of $G$. Our approach has nothing to do with the Brauer-Witt reduction, but unlike other results, excepting perhaps [7], it gives a useful bound for Schur indices from the character table alone.

## 1. Main theorem and corollaries

We let $G$ be a finite group, $p$ a prime number, and $x$ a $p'$-element of $G$. We define the rational $p'$-section $S_x$ as the set $\{g \in G \mid \langle g_{p'} \rangle \sim_G \langle x \rangle\}$, where $\sim_G$ denotes $G$-conjugacy. The characteristic function of $S_x$ is denoted by $1_{S_x}$. For an irreducible character $\chi$ of $G$ we abbreviate the inner product $(\chi, 1_{S_x})_G$ by $S_x(\chi)$, and denote by $m(\chi)$ the rational Schur index of $\chi$. Finally, if $r$ is a rational number, $r_p$ will denote the $p$-part of $r$. We can now state the main theorem.

THEOREM. *For any $p'$-element $x$ in $G$, $1_{S_x}$ is a $p$-integral linear combination of permutation characters. Consequently $S_x(\chi)$ is a $p$-integral rational number for all $\chi \in \operatorname{Irr}(G)$, and $m(\chi)_p$ divides $S_x(\chi)_p$.*

The second sentence follows from the first by a standard property of Schur indices [4, Corollary 10.2(c)], so we do get a bound for $m(\chi)_p$.

We therefore concentrate on the proof of the first assertion. To this end we introduce the Burnside ring $\Omega(G)$, which may be defined as the Grothendieck ring of the category of finite $G$-sets. Thus $\Omega(G)$ consists of all formal integral linear combinations of transitive $G$-sets, with multiplication given by decomposing the cartesian product of 2 transitive $G$-sets into its transitive orbits. If we let $L^*(G)$ be a set of representatives of the conjugate classes of subgroups of $G$ and denote by $u_H$ the transitive $G$-set of left cosets of $H$, then $\{u_H \mid H \in L^*(G)\}$ is

Received January 11, 1980.

the natural basis of $\Omega(G)$. Furthermore there is a ring homomorphism Char from $\Omega(G)$ to the ring of generalized characters of $G$ which sends $u_H$ to the permutation character $1_H^G$.

We next compare the primitive idempotents in two coefficient ring extensions of $\Omega(G)$, and their images under Char. These images will be class functions on $G$, but not usually generalized characters.

In the Burnside algebra $\mathbf{Q} \otimes \Omega(G)$, the primitive idempotents $e_H$ again correspond to the elements of $L^*(G)$. It is easy to see that Char $(e_H)$ is 0 when $H$ is non-cyclic and Char $(e_H)$ is the characteristic function of the set of conjugates of generators of $H$ when $H$ is cyclic. See [8] for these and other facts about the Burnside algebra.

We next consider the ring $\Omega(G)_p = \mathbf{Z}_p \otimes \Omega(G)$, where $\mathbf{Z}_p$ denotes the integers localized at $p$. We call subgroups $H$ and $K$ of $G$ $p$-equivalent if $\mathbf{O}^p(H) \sim_G \mathbf{O}^p(K)$, where $\mathbf{O}^p$ denotes the smallest normal subgroup of $p$-power index. The next lemma describes the primitive idempotents of $\Omega(G)_p$.

LEMMA. *For $H \in L^*(G)$ let $\tilde{e}_H = \sum_K e_K$, where $K$ ranges over all subgroups in $L^*(G)$ which are $p$-equivalent to $H$ and the sum is taken in $\mathbf{Q} \otimes \Omega(G)$. Then the $\tilde{e}_H$ are the primitive idempotents of $\Omega(G)_p$.*

*Proof.* This is a somewhat disguised version of the main results of [2], and a fuller discussion may be found in [3]. We sketch the proof for the reader's convenience.

The prime ideals of $\Omega(G)_p$ are of two types. There are minimal prime ideals

$$\mathfrak{p}(H, 0) \underset{\text{def}}{=} \{x \in \Omega(G)_p \mid \langle e_H, x \rangle = 0\},$$

where $\langle\ ,\ \rangle$ denotes the natural bilinear form on $\mathbf{Q} \otimes \Omega(G)$ determined by the primitive idempotents of $\mathbf{Q} \otimes \Omega(G)$, and there are maximal ideals

$$\mathfrak{p}(H, p) \underset{\text{def}}{=} \{x \in \Omega(G)_p \mid \langle e_H, x \rangle \equiv 0 \bmod p\}.$$

The $p(H, 0)$ are distinct for distinct $H \in L^*(G)$ but $\mathfrak{p}(H, p) = \mathfrak{p}(K, p)$ if and only if $H$ and $K$ are $p$-equivalent. Each $p$-equivalence class thereby determines a connected component of Spec $\Omega(G)_p$ consisting of one maximal ideal $\mathfrak{p}(H, p)$ and the minimal prime ideals $\mathfrak{p}(K, 0)$ for those $K$ in $L^*(G)$ which are $p$-equivalent to $H$. All the above $\mathfrak{p}(K, 0)$ are contained in $\mathfrak{p}(H, p)$.

On the other hand, for any commutative ring $R$ the connected components of Spec $R$ correspond to the primitive idempotents of $R$; the connected component of Spec $R$ corresponding to a primitive idempotent $e$ in $R$ consists of all prime ideals of $R$ which contain $1 - e$, so if $e$ is the primitive idempotent of $\Omega(G)_p$ corresponding to the connected component of Spec $\Omega(G)_p$ described in the previous paragraph, $1 - e$ is contained in $\mathfrak{p}(K, 0)$ if and only if $K$ is $p$-equivalent to $H$. The statement of the lemma follows.

To complete the proof of the theorem we consider the primitive idempotent $\tilde{e}_{\langle x\rangle}$ of $\Omega(G)_p$. By our earlier remarks on Char $(e_H)$ it follows that Char $(\tilde{e}_{\langle x\rangle}) = 1_{S_x}$. The theorem then follows from the fact that $\tilde{e}_{\langle x\rangle}$ is a $p$-integral combination of the $u_H$'s.

The theorem has several striking applications to Schur indices. We give two simple ones, the first of which has already appeared in [7].

COROLLARY 1 (L. Solomon). *Let* $\chi \in \operatorname{Irr}(G)$ *have p-defect* 0. *Then* $p \nmid m(\chi)$.

*Proof.* The $p$-elements of $G$ comprise a rational $p'$-section $S_1$. Since $\chi$ vanishes on non-identity $p$-elements, $S_1(\chi) = \chi(1)/|G|$ is not divisible by $p$.

The theory of blocks with cyclic defect group [1] can be used to get corollaries of the main theorem which are more widely applicable than Corollary 1. One considers elements $x$ of $G$ which are $p$-regular and $q$-singular for some prime $q \neq p$. Here one should keep in mind the case where $G$ is simple and $q$ is a large prime divisor of $|G|$. If $\chi$ is an irreducible character of $G$ whose $q$-defect group is cyclic and contains $x_q$, one can express $S_x(\chi)$ in terms of the irreducible $q$-Brauer characters of $C_G(x_q)$. The following corollary considers only the simplest case of this type, but one which occurs frequently.

COROLLARY 2. *Let $q$ be a prime which divides $|G|$ to the first power and suppose that a $q$-Sylow of $G$ is self-centralizing. Let $\chi$ be an irreducible character of $G$ such that $q \nmid \chi(1)$. Then $m(\chi) = 1$ if $\chi$ is exceptional, and $m(\chi)$ divides the number of conjugate classes of elements of order $q$ in $G$ if $\chi$ is non-exceptional.*

*Proof.* Let $x$ be an element of order $q$ in $G$, and let $p$ be a prime different from $q$. Then the conjugates of $\langle x\rangle - \{1\}$ form a rational $p'$-section $S_x$. Let $x, x^a, x^b, \ldots,$ be a full set of non-conjugate powers of $x$, and let $N = N_G\langle x\rangle$.

If $\chi$ is exceptional, there is a non-principal character $\lambda$ of $\langle x\rangle$ so that

$$S_x(\chi) = \pm(1/q)(\lambda^N(x) + \lambda^N(x^a) + \lambda^N(x^b) + \cdots) = \pm 1/q.$$

Therefore $p \nmid m(\chi)$ for any prime $p$ different from $q$. Since $q \nmid \chi(1)$, it follows that $m(\chi) = 1$.

If $\chi$ is non-exceptional, then $S_x(\chi) = \pm(1/q)(\varepsilon + \cdots + \varepsilon)$, where $\varepsilon = \pm 1$ and one $\varepsilon$ appears for each of $x, x^a, x^b \ldots$. The result follows.

## 2. Examples

We shall apply the main theorem to $PSL(3, 3)$ and $M_{11}$. These are the two simple groups having an involution with centralizer isomorphic to $GL(2, 3)$. In both examples we take $p = 2$ and use rational $2'$-sections to estimate the 2-parts of Schur indices. In both examples it is clear that no odd prime can divide any Schur indices; the groups do not have the appropriate hyperelementary sections. We shall omit trivial calculations and merely state the results. The character tables can be found, for example, in [4] and [9].

*Example 1.* $G = PSL(3, 3)$.

First note that $PSL(3, 3)$, $SL(3, 3)$ and $PGL(3, 3)$ are all isomorphic. In the notation of Steinberg [9], $G$ has 12 conjugate classes, one each of type $A_1$, $A_2$, $A_3$, $A_4$ and $A_5$, three of type $B_1$, and four of type $C_1$. There are four rational 2′-sections, one consisting of the two classes of type $A_2$ and $A_5$, the second consisting of the single class of type $A_3$, the third consisting of the four classes of type $C_1$, and the fourth (the 2-elements) consisting of the remaining classes. There are eight characters of even degree: four exceptional characters of degree 16 in the principal 13-block, three characters of degree 26, and an irreducible character $\chi_{12}^{(2)}$ of degree 12. All but the last of these can be shown to have odd Schur index by considering the rational 2′-section consisting of the single class of type $A_3$. No rational 2′-section eliminates $\chi_{12}^{(2)}$, but $1_G + \chi_{12}^{(2)}$ is the character of the doubly transitive action of $G$ on the 13-point projective space over $GF(3)$, so $\chi_{12}^{(2)}$ has Schur index 1.

*Example 2.* $G = M_{11}$.

Here, Corollary 2 shows that a character with Schur index greater than 1 must have degree divisible by 10. There are three such characters, all of degree exactly 10. Our method eliminates one (the permutation character of $M_{11}$ on 11 points), but fails to eliminate the other two, which are algebraically conjugate. One can restrict the latter characters to $GL(2, 3) \leq M_{11}$ and find that the restrictions are multiplicity-free and contain characters of

$$GL(2, 3)/Z(GL(2, 3)) = S_4.$$

By a basic property of Schur indices [4, Lemma 10.4] this proves that the two algebraically conjugate characters of $M_{11}$ of degree 10 actually have Schur index 1.

Finally, it should be pointed out that there is an important case in which our method fails completely; namely when $p \mid |Z(G)|$. In this case all $S_x(\chi)$ are 0 unless $\chi$ contains the $p$-Sylow of $Z(G)$ in its kernel.

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