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LIE ALGEBRAS WITH THE SAME MODULES

BY

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Let L be a (finite-dimensional) complex Lie algebra and let $\mathcal{M}$ be the category of finite-dimensional complex L-modules. $\mathcal{M}$ is an abelian subcategory, closed under tensor product, of the category of finite-dimensional complex vector spaces. When $L = [L, L]$, the category $\mathcal{M}$ determines L [2, Theorem 6.1, p. 62], but in general this is not the case. It is natural to inquire, then, how much of the structure of L can be recovered from $\mathcal{M}$. This paper answers that question in the following form: if L_i, $i = 1, 2$ are complex Lie algebras, theorem 12 below describes how L_1 and L_2 are related if they have isomorphic module categories.

If G is the simply connected complex Lie group with Lie algebra L, then the categories of finite-dimensional L-modules and G-modules are the same. The category of G-modules is the same as the category of rational finite-dimensional modules for a pro-affine algebraic group $A(G)$. We are thus interested in how closely the structure of $A(G)$ determines G. (When G is solvable as well as simply connected, this latter problem was solved in [10].)

We show that $A(G)$ is a semi-direct product of a maximal normal affine subgroup and a subgroup which is an inverse limit of tori, that the normal affine is unique and the other subgroup is unique up to conjugacy by an element of the unipotent radical of $A(G)$. Thus the centralizer in this subgroup of the unipotent radical is uniquely defined. The quotient of $A(G)$ by this centralizer is then seen to be affine, and to determine $A(G)$. The problem then becomes that of determining as much of the structure of G as possible from this quotient. This problem in turn is reduced to the same problem for the radical of G, which is solved by the methods of [10]: in fact, much of the work here can be regarded as a revision of [10] keeping track of Levi factors.

Throughout, all Lie algebras, algebraic groups, and vectors spaces are over C.

By an *analytic group* G, we mean a connected complex Lie group. $R(G)$ denotes the Hopf algebra of representative functions on G [4, p. 496] (the complex algebras generated by the matrix coordinate functions of the finite dimensional analytic linear representations of G). Let $A(G)$ be the pro-affine

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algebraic group with coordinate ring $R(G)$ [7, Theorem 2.1, p. 1131]. There is a canonical homomorphism $s: G \to A(G)$ such that if $\bar{G}$ is a linear complex algebraic group and $f: G \to \bar{G}$ is an analytic homomorphism, there is a unique algebraic morphism $\bar{f}: A(G) \to \bar{G}$ such that $\bar{f}s = f$. Thus finite dimensional analytic G-modules are rational $A(G)$-modules and conversely, so $A(G)$, as a pro-affine algebraic group, determines the category of finite-dimensional analytic G-modules. This category, as a tensored abelian category of finite-dimensional vector spaces, can be used directly to define $A(G)$ and $R(G)$ [8, Prop. 2.3], so analytic groups G_1 and G_2 will have isomorphic categories of finite-dimensional analytic modules if and only if $A(G_1)$ and $A(G_2)$ are isomorphic pro-affine algebraic groups.

The structure of $A(G)$ is determined in [5]: $s(G)$ is normal in $A(G)$, and $A(G)$ is a semi-direct product $s(G) \cdot T$, where T is a pro-affine algebraic group which is an inverse limit of tori (i.e., *a pro-torus*) whose character group $X(T)$ is isomorphic to the rational vector space Hom (G, C) [5, Theorem 6.1, p. 127]. G has a faithful representation if and only if s is injective [4, Theorem 7.1, p. 522]. A *nucleus* K of G is a closed simply connected solvable normal subgroup such that G/K is reductive. G has a nucleus if and only if G has a faithful representation; if G has a nucleus K and P is a maximal reductive subgroup of G then G is the semi-direct product $K \cdot P$ [5, p. 113].

Now assume G has a faithful representation, let K be a nucleus of G and let P be a maximal reductive subgroup. Then the pro-torus T above can be chosen so that $(P, T) = \{e\}$. Moreover, the unipotent radical U of $A(G)$ is affine with dim $(U) =$ dim (K) [5, Corollary 5.1, p. 124].

In the above decomposition $A(G) = s(G) \cdot T$, $s(G)$ is not an algebraic subgroup of $A(G)$ in general: in fact, it is Zariski-dense. Nonetheless, we can deduce an algebraic semi-direct product decomposition from it.

PROPOSITION 1. *Let G be an analytic group with a faithful representation, let P be a maximal reductive subgroup of G, and let T be a pro-torus in $A(G)$ such that $A(G) = s(G) \cdot T$ (semi-direct product) and $(T, s(P)) = e$. Let U be the unipotent radical of $A(G)$. Then $Us(P)$ is a normal affine subgroup of $A(G)$ with $A(G) = (Us(P)) \cdot T$ (semi-direct product) and UT is a normal pro-affine subgroup of $A(G)$ with $A(G) = (UT) \cdot s(P)$ (semi-direct product), and $s^{-1}(UT \cap s(G))$ is a nucleus of G.*

Proof. We regard G as a subgroup of $A(G)$, so s is the identity. Then P is Zariski-closed in $A(G)$, so its image in $A(G)/U$ is closed, and hence the inverse image UP is closed in $A(G)$. Similarly, UT is closed in $A(G)$. Moreover, UP is affine: for there is a finite-dimensional representation ρ of $A(G)$ faithful on both U and P, and $\rho(UP) = \rho(U) \cdot \rho(P)$ is a semi-direct product since $\rho(U)$ is unipotent and $\rho(P)$ is reductive, so ρ is faithful on UP.

Next, we see that UT is normal: if $\rho: A(G) \to GL(V)$ is a finite-dimensional representation with V semi-simple as an $A(G)$-module, then for any nucleus K

of G, $\rho(K)$ is normal in $\rho(G)$ and hence in its Zariski-closure $\rho(A(G)) = \bar{G}$. Now PT maps onto $\bar{G}/\rho(K)$, and T is normal in PT, so $\rho(K)\rho(T) = L$ is normal in $\rho(G)$. Then V is semi-simple as an L-module, and L is solvable, so $(\rho(K), \rho(T))$ acts trivially on V. It follows that (K, T) acts trivally on every semi-simple $A(G)$ module, so $(K, T) \subseteq U$. Thus UT is normalized by K. Since UT is also normalized by P, UT is normalized by G, and the Zariski-density of G in $A(G)$ implies UT is normal.

Now we consider $UT \cap G$. Every x in $A(G)$ can be uniquely written as $x = g(x)t(x)$, where $g(x) \in G$ and $t(x) \in T$, so $g(x) = xt(x)^{-1}$. If $u \in U$, then $g(u) \in UT \cap G$. If $y = ut$ is in $G \cap UT$ with $u \in U$ and $t \in T$, then $yt^{-1} = u$ so $g(u) = y$. It follows that g is a bijection. If we regard $A(G)$ as the inverse limit of complex analytic groups, then in the inverse limit topology g is a homeomorphism. In particular, $UT \cap G$ is a closed, simply connected, solvable normal subgroup of G of dimension equal to dim (U). This implies $UT \cap G$ is a nucleus of G, so $UT \cap P = (UT \cap G) \cap P$ is trivial, and also $G = (UT \cap G) \cdot P$, so $A(G) = G \cdot T = (UT) \cdot P$.

Finally, we now have $A(G) = UT \cdot P = UP \cdot T$ so $A(G)$ is also the semi-direct product of UP and T.

We will use Proposition 1 to show that the subgroup $Us(P)$ of $A(G)$ is characteristic and that, if P is semi-simple, that T is determined up to conjugacy, so $Us(P)$ and T depend only on $A(G)$ and not on G.

We will need a property of pro-tori. Let T be a pro-torus whose character group $X(T)$ is divisible (i.e., is a rational vector space), let S be a pro-affine subgroup of T and let S_0 be the connected component of the identity in S. Then there is a pro-affine subgroup T_0 of T with $T = T_0 \times S_0$, $X(S_0)$ and $X(T_0)$ are rational vector spaces with $X(T) = X(T_0) \oplus X(S_0)$, and S/S_0 is pro-finite. These facts are special cases of [11, Theorem 3.2, p. 24].

PROPOSITION 2. *Let G be an analytic group with a faithful representation, let P be a maximal reductive subgroup of G and let T be a pro-torus in $A(G)$ such that $A(G) = s(G) \cdot T$ (semi-direct product) and $(T, s(P)) = e$. Let U be the unipotent radical of $A(G)$. Then:*

(i) *$H = Us(P)$ is the (unique) maximal normal connected affine algebraic subgroup of $A(G)$, and any maximal reductive subgroup of H is conjugate to $s(P)$ by an element of $(s(G), s(G)) \cap U$.*

(ii) *If H/U is semi-simple and S is a pro-torus in $A(G)$ such that $A(G) = H \cdot S$ (semi-direct product) and S centralizes a maximal reductive subgroup of H, then S is conjugate to T by an element of $(s(G), s(G)) \cap U$.*

Proof. (i) By Proposition 1, H is a normal, connected affine algebraic subgroup of $A(G)$ and $T \to A(G)/H$ is an isomorphism. Thus $A(G)/H$ is a pro-torus with $X(A(G)/H)$ divisible, so it has no connected affine subgroups other than

$\{e\}$, and hence H contains every connected affine subgroup of $A(G)$. If Q is a maximal reductive subgroup of H, Q is conjugate to $s(P)$ by an element of

$$(H, U) \subseteq (A(G), A(G), A(G)) = (s(G), s(G)).$$

(ii) Let Q be a maximal reductive subgroup of H centralized by S. By part (i) we may assume $Q = s(P)$, after conjugation. Since US is normalized by Q, US is normal in $A(G)$, and by arguments similar to Proposition 1, US is a connected pro-affine subgroup of $A(G)$. Since $A(G) = H \cdot S = (U \cdot Q) \cdot S = (US) \cdot Q$, and Q is semi-simple, US is the pro-radical of $A(G)$ (i.e., the inverse limit of the radicals of any surjective inverse system of affine algebraic groups with limit $A(G)$). By Proposition 1, $A(G) = (U \cdot T) \cdot Q$ also, so UT is also the pro-radical and $US = UT$. Now S and T are both maximal reductive subgroups of US, so by [7, Theorem 3.3, p. 1138] they are conjugate by an element of the closure of $(U, US) \subseteq (s(G), s(G)) \cap U$.

Among the consequences of Proposition 2, we want to note in particular that part (i) implies that the isomorphism type of the maximal reductive subgroup P of G is determined by $A(G)$. We also want to note that part (ii), along with Proposition 1, implies that the nucleus s^{-1} $(UT \cap s(G))$ is uniquely determined. This merely reflects the fact that if H/U is semi-simple then a maximal reductive subgroup of G is semi-simple, so any nucleus of G is actually the radical of G. Hence part (ii) only could apply to groups with simply connected radical.

Proposition 2 suggests that we make the following definitions:

DEFINITION 3. Let G be an analytic group with a faithful representation. The subgroup H of Proposition 2(i) is the *maximal affine* of $A(G)$, and the quotient H/U is the *reductive type* of $A(G)$. A pro-torus T in $A(G)$ such that $A(G) = H \cdot T$ (semi-direct product) and T centralizes a maximal reductive subgroup of H is called a *complementary pro-torus* of $A(G)$. If T is a complementary pro-torus in $A(G)$ and S is the centralizer of U in T, and if the reductive type of $A(G)$ is semi-simple, then $A(G)/S$ is called the *bottom group* of $A(G)$.

By Proposition 2(ii), the bottom group of $A(G)$ is independent of the choice of the pro-torus T. We will show that the bottom group of $A(G)$ is affine and can be characterized intrinsically, and that when the reductive type of $A(G)$ is simply connected, the bottom group determines $A(G)$ (whence its name).

LEMMA 4. *Let G be an analytic group with a faithful representation, let H be the maximal affine of $A(G)$ and suppose the reductive type of $A(G)$ is semi-simple. Let $\bar{A}$ be the bottom of $A(G)$ and let p: $A(G) \to \bar{A}$ be the canonical surjection. Then $\bar{A}$ is an affine algebraic group whose radical R is a semi-direct factor, and R contains no central semi-simple elements. Moreover, $H \cap \operatorname{Ker}(p) = \{e\}$. If A' is an affine algebraic group whose radical R' is a semi-direct factor such that R' contains no central semi-simple elements, and if f: $A(G) \to A'$ is a surjective homomorphism with $H \cap \operatorname{Ker}(f) = \{e\}$, then A' is isomorphic to $\bar{A}$.*

Proof. Let T be a complementary pro-torus of $A(G)$. Let $\rho: T \to \text{Aut}(U)$ represent T as inner automorphisms. Let $S = \text{Ker}(\rho)$ and let $\bar{T} = \rho(T)$. Since $\text{Aut}(U) = \text{Aut}(\text{Lie}(U))$ is an affine algebraic group, $\bar{T}$ is a finite dimensional torus. Let $\bar{R}$ be the semi-direct product $U \cdot \bar{T}$. Any semi-simple central element of $\bar{R}$ lies in $\bar{T}$ and hence comes from an element of T centralizing U. By construction, any such element of $\bar{T}$ is trivial. Let P be a maximal reductive subgroup of H centralized by T. P also acts on U, commuting with the action of $\bar{T}$, so we can form the semi-direct product $\bar{R} \cdot P$. Since $A(G) = UT \cdot P$, we have an induced onto map $A(G) \to \bar{R} \cdot P$ whose kernel is S. Thus $\bar{A} = A(G)/S$ has the required properties.

From the decomposition $A(G) = UT \cdot P$ we see that $f(UT) = R'$ and that $P' = f(P)$ is a maximal reductive subgroup of A'. Moreover, $U' = f(U)$ is the unipotent radical of A'. Since R' has no central semi-simple elements, $f(S) = \{e\}$. Thus we have a surjection $\bar{A} = A(G)/S \to A'$. Let $\bar{R} = UT/S$ be the radical of $\bar{A}$. Then the map $\bar{R} \to R'$ is an isomorphism on unipotent radicals so its kernel is a normal closed subgroup of $\bar{T} = T/S$. Hence the kernel consists of central semi-simple elements of $\bar{R}$ and is thus trivial. Since both $p: P \to p(P)$ and $f: P \to P'$ are also isomorphisms, the map $\bar{A} \to A'$ is an isomorphism.

To use Lemma 4, we need to recall some facts about representations of analytic groups. Let G be an analytic group with a faithful representation, and let P be a maximal reductive subgroup of G. Then there is an algebraic group G', an analytic injective homomorphism with Zariski-dense image $f: G \to G'$ and a torus T in G' such that G' is the analytic semi-direct product of $f(G)$ and T, and $f(P)$ and T commute [9, Theorem 10, p. 880]. Following the terminology of [10], we call (G', f) a *split hull* of G. By [10, Lemma 3], we can assume that T contains no central elements of G', in which case we say (G', f) is a *reduced* split hull.

THEOREM 5. *Let G be an analytic group with simply connected radical and let (G', h) be a reduced split hull of G. Then the reductive type of $A(G)$ is semi-simple and G' is isomorphic to the bottom group of $A(G)$.*

Proof. For notational convenience we assume h is inclusion. Let P be a maximal reductive subgroup of G. Since the radical K of G is a nucleus, P is semi-simple and Proposition 2(i) implies $A(G)$ has semi-simple reductive type. Let U be the unipotent radical of G' and let T be a torus in G' with $G' = G \cdot T$ (semi-direct product) and $(T, P) = \{e\}$. Exactly as in the proof of Proposition 1, we see that $(T, K) \subseteq U$ so UT is normal in G', $UT \cap G' = K$ and $G' = (UT) \cdot P$ (semi-direct product), so UT is the radical of G'. Moreover, UT contains no central semi-simple elements. Since G is Zariski-dense in G', the inclusion induces a surjective homomorphism $f: A(G) \to G'$. We show that Lemma 4 applies to f: by [10, Theorem 1], $T_0 = f^{-1}(T)$ is a pro-torus in $A(G)$ such that $A(G)$ is the semi-direct product $s(G) \cdot T_0$ and $(T_0, s(P)) = \{e\}$. Let U_0 be the unipotent radical of $A(G)$ and let $P_0 = s(P)$. By Proposition 2(i), $H = U_0 P_0$ is the maximal affine of $A(G)$ and by Proposition 1, T_0 is a complementary pro-torus. Now Ker (f) is contained in T_0 and $T_0 \cap H = \{e\}$, so Ker $(f) \cap H = \{e\}$. By Lemma 4, we conclude that G' is isomorphic to the bottom group of $A(G)$.

Next we will see that when G is simply connected the bottom group of $A(G)$ determines $A(G)$.

THEOREM 6. *Let G be a simply connected analytic group. Then the bottom group of $A(G)$ determines $A(G)$.*

Proof. Let U be the unipotent radical of $A(G)$, let T be a complementary pro-torus, let S be the centralizer of U in T and let S_0 be the connected component of the identity in S. Let $\bar{A}$ be the bottom group of $A(G)$ and let $p\colon A(G) \to \bar{A}$ be the canonical surjection. As we noted above, there is a pro-torus T_1 in T with $T = S_0 \times T_1$ (since $X(T) = \mathrm{Hom}\ (G, C)$ is a rational vector space) and S/S_0 is pro-finite. Since $p(T) = p(T_1)$ is finite dimensional and the connected component of Ker $(p \mid T_1)$ is in S_0, $\dim_Q (X(T_1))$ is finite. Since $X(T) = X(S_0) \oplus X(T_1)$, and $\dim_Q (X(T_1))$ is 0 or uncountable, $\dim_Q (X(T)) = \dim_Q (X(S_0))$. Let H be the maximal affine of $A(G)$. Then S_0 centralizes H so $A(G) = (H \cdot T_1) \times S_0$ and $A' = A(G)/S_0$ is isomorphic to $H \cdot T_1$. Moreover, the map $p'\colon A' \to \bar{A}$ has pro-finite kernel S/S_0. We claim that $p'\colon A' \to \bar{A}$ is the universal pro-finite extension [3, p. 410] of $\bar{A}$. We need to show that if $f\colon A \to A'$ is a morphism of connected pro-affine algebraic groups with finite kernel then f is an isomorphism. Let P be a maximal reductive subgroup of G. Then $H = Us(P)$ is simply connected. Let H' and T' be the connected components of the identity of $f^{-1}(H)$ and $f^{-1}(T_1)$ in A (here we are regarding A' as $H \cdot T_1$). Then $A = H'T'$ and T' is a pro-torus. Let $\Gamma = \mathrm{Ker}\ (f)$. The surjection $T' \to T_1$ induces an injection $X(T_1) \to X(T')$ which splits since $X(T_1)$ is a rational vector space, and we have $X(T') = X(T_1) \oplus X(\Gamma)$. But $X(T')$ is torsion free and $X(\Gamma)$ is finite, so $\Gamma = \{e\}$. Thus we can recover A' from $\bar{A}$. To obtain $A(G)$, we need only show that the dimension over Q of $X(S_0)$ can be obtained from $\bar{A}$, since $A(G) = A' \times S_0$. As we saw above, this dimension is the same as that of $X(T) = \mathrm{Hom}\ (G, C)$. The analytic homomorphisms from G to C are the algebraic homomorphisms from $A(G)$ to C, and these all vanish on T and hence S, so the dimension in question is that of $\mathrm{Hom}\ (\bar{A}, C)$ (algebraic homomorphisms).

COROLLARY 7. *Let G_1 and G_2 be simply connected analytic groups. Then $A(G_1)$ is isomorphic to $A(G_2)$ if and only if G_1 and G_2 have reduced split hulls isomorphic as algebraic groups.*

Proof. Combine Theorems 5 and 6.

Now let G_1 and G_2 be simply connected analytic groups, and suppose they have reduced split hulls isomorphic as algebraic groups. We can assume the algebraic groups are actually the same, that is, there is an algebraic group G and analytic embeddings f_i: $G_i \to G$ such that (G, f_i) is a reduced split hull of G_i. Let T_i be the torus in G and P_i the maximal reductive subgroup of G_i such that $G = f_i(G_i) \cdot T_i$ and $(f_i(P_i), T_i) = \{e\}$.

The choice of T_i and P_i is not unique; we want to show that we can make these choices such that $T_1 = T_2$ and $f_1(P_1) = f_2(P_2)$. Let U be the unipotent radical of G and let R be the radical. Then we know $UT_i = R$ so there is g in (R, R) with $gT_2g^{-1} = T_1$. Let $P'_i = f_i^{-1}(gf_2(P_2)g^{-1})$. Clearly, P'_2 is a maximal reductive subgroup of G_2 with $(f_2(P'_2), T_1) = \{e\}$. Since $f_2(P_2)$ is semi-simple,

$$f_2(P_2) = (f_2(P_2), f_2(P_2)),$$

so $gf_2(P_2)g^{-1}$ is contained in $(G, G) = (f_1(G_1), f_1(G_1))$, and it follows that P'_1 is also a maximal reductive subgroup of G_1. Moreover, $G = f_i(G_i) \cdot T_1$. Let $T = T_1$ and $P = gf_2(P_2)g^{-1}$. Replace G_i by its image $f_i(G_i)$ so f_i becomes inclusion. Then G_i is a Zariski-dense analytic subgroup of G containing P as a maximal reductive subgroup, $(P, T) = \{e\}$, $G = G_i \cdot T$ (semi-direct product), $R = UT$ is the radical of G and $G = R \cdot P$ (semi-direct product). We also know that $R_i = R \cap G_i$ is the radical (and a nucleus) of G_i, so $G_i = R_i \cdot P$ (semi-direct product). Since $G = G_i \cdot T = (R_i \cdot P) \cdot T$ and P normalizes $R_i \cdot T$, $G = (R_i \cdot T) \cdot P$. It follows that $R_i \cdot T = R$. By definition, R contains no central semi-simple elements. We claim that R is a reduced split hull of R_i: we need only see that R_i is Zariski-dense in R. If $\bar{R}_i$ is the Zariski-closure of R_i in G, then $\bar{R}_i \subseteq R$ and $\bar{R}_i \cdot P$ contains G and is Zariski-closed in G, so $\bar{R}_i \cdot P = G$ and hence $\bar{R}_i = R$.

Thus the simply connected solvable analytic groups R_i have the same reduced split hull $R = R_i \cdot T$. In [10] we determined the relationship between Lie (R_1) and Lie (R_2) in such a case. We now recall the appropriate definition with an extension to cover the case at hand. We recall that if L is a Lie algebra, C is a nilpotent subalgebra, and a is a root of C on L, then $L_a(C)$ (or just L_a) is the corresponding root space.

DEFINITION 8. Let L_i, $i = 1, 2$ be solvable Lie algebras on which the semi-simple Lie algebra H acts as derivations. An *H-near isomorphism* f: $L_1 \to L_2$ with associated root bijection g is an H-module isomorphism of L_1 and L_2 with the following properties:

(1) $f([L_1, L_1]) = [L_2, L_2]$ and $f \mid [L_1, L_2]$ is a Lie algebra homomorphism.

(2) There are Cartan subalgebras C_i of L_i which are H-submodules such that $f(C_1) = C_2$ and $f \mid C_1$ is a Lie algebra homomorphism.

(3) If Φ_i denotes the set of non-zero roots of C_i on L_i, then there is a bijection g: $\Phi_1 \to \Phi_2$ such that

(a) $f(L_{1,a}) = L_{2,g(a)}$ for all a in Φ_1,

(b) $f((c - a(c))x) = (f(c) - g(a)f(c))f(x)$ for all x in $L_{1,a}$ and c in C_1, and

(c) g induces an isomorphism from the subgroup of C_1^* generated by Φ_1 to the subgroup of C_2^* generated by Φ_2.

PROPOSITION 9. *Let G_1 and G_2 be simply connected analytic subgroups of the algebraic group such that G is a reduced split hull of G_1 and G_2, and let P be a maximal reductive subgroup of G_1 and G_2. Then the radical of* Lie (G_1) *is* Lie (P)*-near isomorphic to the radical of* Lie (G_2).

Proof. Let T be a torus in G such that $G = G_i \cdot T$ and $(T, P) = \{e\}$, and let R_i, R be the radical of G_i, G respectively. We know that $R = R_i \cdot T$ and that R is a reduced split hull of R_i by the above discussion. Let $L_i =$ Lie (R_i) and let $L =$ Lie (R). Let $H =$ Lie (P). In [10, Theorem 21] we showed how to construct a near-isomorphism (for the trivial semi-simple algebra) f: $L_1 \to L_2$. We will show that this f is actually an H-near isomorphism.

Let $S =$ Lie (T) and $M =$ Lie (G). Since R_i is normal in G_i, it is normal in G and hence L_i is an ideal of M. The H-module action on L_i is then just Lie multiplication. Now $L = L_i \oplus S$, and f: $L_1 \to L_2$ is $L_1 \to L \to L_2$ (inclusion followed by projection). We check that f is an H-module homomorphism: the inclusion $L_1 \to L$ is an H-module homomorphism, and since $[H, S] = 0$, so is the projection $L \to L_2$. The Cartan subalgebras C_i are defined in the proof of [10, Theorem 21] as follows. Let C be the centralizer of S in L. Then $C_i = L_i \cap C$. Since H centralizes S, $[H, C] \subseteq C$, and hence C_i is an H-submodule of L_i. Thus f is an H-near isomorphism.

Proposition 9 has a converse. To state it simply, we make the following definition, so as to avoid having to single out the special reductive subgroup P.

DEFINITION 10. Let L_i, $i = 1, 2$ be Lie algebras. Then L_1 and L_2 are said to be *nearly isomorphic* if there is a semi-simple Lie algebra H and Lie algebra injections h_i: $H \to L_i$ such that $h_i(H)$ is a maximal semi-simple subalgebra of L_i, and such that the radical of L_1 is H-near isomorphic to the radical of L_2, with the H-module structures induced from the h_i.

THEOREM 11. *Let G_i, $i = 1, 2$, be simply connected analytic groups. Then the G_i have reduced split hulls isomorphic as algebraic groups if and only if* Lie (G_1) *is nearly isomorphic to* Lie (G_2).

Proof. "Only if" was done in Proposition 9 and the discussion preceding it. For "if', we let L_i be the radical of Lie (G_i) and let H be a semi-simple algebra as in definition 10 and f: $L_1 \to L_2$ an H-near isomorphism. We will regard H as a subalgebra of Lie (G_i) so that Lie $(G_i) = L_i \oplus H$, and let P be the simply connected, semi-simple subgroup of G_i with Lie $(P) = H$. Let R_i be the radical of G_i, so Lie $(R_i) = L_i$. In [10, Theorem 22] we showed how to construct isomorphic reduced split hulls $\bar{R}_i$ from R_i from the near isomorphism f. An examination of that construction will yield the desired (isomorphic) reduced split hulls of the G_i.

Let C_i be the Cartan subalgebra of L_i of Definition 8, part (2), and let Φ_i be as in part (3). If $a \in \Phi_i$, then since $[H, C_i] \subseteq C_i$, $[H, L_{i,a}] \subseteq L_{i,a}$. Thus the Lie subalgebra N_i of L_i generated by $\{L_{i,a} | a \in \Phi_i\}$ is an H-submodule. In the proof of [10, Theorem 22] (which precedes the theorem), we defined an action D of C_i on N_i such that $D(c)(x_a) = cx_a - a(c)x_a$ for $a \in \Phi_i$ and $x_a \in L_{i,a}$. Let ρ denote the H-action on N_i. Then an easy calculation shows that

$$\rho(h)D(c) - D(c)\rho(h) = D([h, c]) \quad \text{for } h \text{ in } H \text{ and } c \text{ in } C_i$$

so there is an action of $C_i + H$ on N_i compatible with D and ρ. This action of $C_i + H$ then extends to the nilpotent algebra $U(L_i, C_i)$ of [10]. Now let Λ_i be the subgroup of C_i^* generated by Φ_i, and let T_i be a torus with character group Λ_i. If a is a root of C_i on L_i, let λ_a be the corresponding character of T. In [10] we define an action α of T_i on $U(L_i, C_i)$ such that if $x \in L_{i,a}$ and $t \in T_i$, $\alpha(t)x = \lambda_a(t)x$. Thus if $h \in H$, $\alpha(t)(hx) = h(\alpha(t)x)$, so the actions of H (as a subalgebra of $C_i + H$) and T_i on $U(L_i, C_i)$ commute. Now let U_i be the unipotent algebraic group with Lie algebra $U(L_i, C_i)$. T_i acts on U_i, and in [10] we show that $U_i \cdot T_i$ (semi-direct product) is the split hull $\bar{R}_i$ of R_i to be produced. The action of H on $U(L_i, C_i)$ induces an action of P on U_i, commuting with the action of T_i, and hence an action of P on $\bar{R}_i$ commuting with the action of T_i. So we can form the semi-direct product $\bar{G}_i = \bar{R}_i \cdot P$. This is an algebraic group, since the action of P on U_i is algebraic.

$$\text{Lie}\ (\bar{G}_i) = (\text{Lie}\ (R_i) \oplus \text{Lie}\ (T_i)) \oplus \text{Lie}\ (P) = (L_i \oplus \text{Lie}\ (T_i)) \oplus H,$$

and the construction is such that Lie $(G_i) = L_i \oplus H \to$ Lie $(\bar{G}_i)$ is a Lie algebra homomorphism. Thus the injections $R_i \to \bar{R}_i \subseteq \bar{G}_i$ and $P \to \bar{G}_i$ extend to an analytic group injection $G_i \to \bar{G}_i$ such that $\bar{G}_i = G_i \cdot T_i$ (semi-direct product). Since R_i is Zariski-dense in $\bar{R}_i$, it follows that G_i is Zariski-dense in $\bar{G}_i$. By [10], f induces an algebraic group isomorphism $\bar{f}$: $\bar{R}_1 \to \bar{R}_2$, and since f is an H-module homomorphism we see that $\bar{f}$ is P-equivariant, and hence extends to an algebraic group isomorphism $\bar{G}_1 \to \bar{G}_2$. So G_1 and G_2 have isomorphic split hulls.

Theorem 11 and Corollary 7 combined describe when Lie algebras (or simply connected analytic groups) have isomorphic module categories. We summarize these facts in the following theorem.

THEOREM 12. *Let L_1 and L_2 be Lie algebras and let G_i be the simply connected analytic group with Lie algebra L_i. Then the following are equivalent:*

(1) *The categories* Mod (L_i) *of finite-dimensional modules for L_1 and L_2 are isomorphic as tensored abelian categories of vector spaces.*

(2) *The categories* Mod (G_i) *of finite-dimensional modules for G_1 and G_2 are isomorphic as tensored abelian categories of vector spaces.*

(3) *The algebras $R(G_1)$ and $R(G_2)$ of representative functions on G_1 and G_2 are isomorphic as Hopf algebras.*

(4) *The Hopf algebras $H(L_1)$ and $H(L_2)$ of L_1 and L_2* [1, 2.8.16, p. 99] *are isomorphic as Hopf algebras.*

(5) *The continuous duals of the universal enveloping algebras* $U(L_1)$ *and* $U(L_2)$ [1, 2.8.17, p. 100] *are isomorphic as Hopf algebras.*
(6) $A(G_1)$ *and* $A(G_2)$ *are isomorphic as pro-affine algebraic groups.*
(7) G_1 *and* G_2 *have reduced split hulls isomorphic as algebraic groups.*
(8) L_1 *and* L_2 *are nearly isomorphic.*

Proof. Since Mod (G_i) = Mod (L_i), (1) and (2) are equivalent. Since $R(G_i)$, as a Hopf algebra, determines and is determined by Mod (G_i)·[8, Prop. 2.3], (2) and (3) are equivalent. Since $H(L_i)$ and $R(G_i)$ are isomorphic Hopf algebras, (3) and (4) are equivalent. Since the continuous dual of $U(L_i)$ is $H(L_i)$ [1, 2.8.17, p. 100], (4) and (5) are equivalent. Since $A(G_i)$ determines and is determined by $R(G_i)$, (6) and (3) are equivalent. By Corollary 7, (6) and (7) are equivalent, and by Theorem 11, (7) and (8) are equivalent.

The equivalence of (1) and (8) can be regarded as an answer to how close the category Mod (L) comes to determining the Lie algebra L. In general, there may be many Lie algebras with the same module category—uncountably many, even, as the example of [7, p. 1150] shows. One could still raise the weaker question: given the category Mod (L), can we find some Lie algebra L' with Mod (L) = Mod (L')? If G is the simply connected analytic group with Lie algebra L, then knowledge of Mod (L) implies that $A(G)$ is known (although the subgroup $s(G)$ of $A(G)$ is not known) and hence that the bottom group $\bar{A}$ of $A(G)$ is known. To find an L', therefore, we need to find a simply connected analytic group G' in $\bar{A}$ which has $\bar{A}$ as a reduced split hull. Thus we want to recognize which algebraic groups are reduced split hulls of simply connected analytic groups. This is done in the following theorem.

THEOREM 13. *Let G be an algebraic group which is the semi-direct product of its radical R and a simply connected semi-simple subgroup P. Assume that P commutes with a maximal torus T of R, that R has no central semi-simple elements, and that if $T \neq \{e\}$ there is a non-trivial algebraic homomorphism $a: G \to C$. Then G is the reduced split hull of a simply connected analytic subgroup G'.*

Proof. If $T = \{e\}$ we can take $G' = G$, so we assume $T \neq \{e\}$ and hence the existence of $a: G \to C$ (non-trivial). Let U be the unipotent radical of G. Let $T = (C^*)^{(n)}$. Choose $\alpha_1, \ldots, \alpha_n \in C$ linearly independent over Q. Let $b: U \to C$ be the restriction of a to U, which is non-trivial. Map U to T by sending u to

$$(\exp(\alpha_1 b(u)), \ldots, \exp(\alpha_n b(n)).$$

The image is Zariski-dense; let K be its graph. We can regard K as a subset of UT. It is in fact a subgroup: if $t = \Pi \exp \alpha_i b(u))$ then

$$tu' = (tu't^{-1})t \quad \text{and} \quad b(u') = b(tu't^{-1})$$

sine a vanishes on t, so $tu' = (tu't^{-1})\Pi \exp(\alpha_i b(tu't^{-1}))$ from whence the closure of K under multiplication readily follows. Regarding K as a graph we see that it is closed and homeomorphic to U, hence simply connected. It is Zariski-dense in R since its Zariski-closure projects onto both U and T, and hence normal in R. Since P and T commute, and $b(pup^{-1}) = b(u)$ for $p \in P$ and $u \in U$, K is normalized by P. The subgroup $G' = KP$ of G is then a Zariski-dense simply connected analytic subgroup of G, with $G' \cap R = K$. Thus $G' \cap T = K \cap T = \{e\}$, and since $R = KT$, $G'T = G$ and this is a semi-direct product. Thus G is a (necessarily reduced) split hull of G'.

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