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EXTREME INVARIANT EXTENSIONS OF PROBABILITY MEASURES AND PROBABILITY CONTENTS

BY

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1. Introduction

Let G be a semigroup which acts from the left on a set X and let $\mathscr{A}$ and $\mathscr{B}$ be invariant σ-algebras on X with $\mathscr{B} \subset \mathscr{A}$. In this paper we characterize the extreme points of the convex set of all invariant probability measures on $\mathscr{A}$ which extend a given probability measure P on $\mathscr{B}$ and we give an extremal integral representation in this set. This problem has been investigated by Farrell [8] and by several other authors for $\mathscr{B} = \{\emptyset, X\}$ and by Plachky [17] for $G = \{\mathrm{id}_X\}$.

Starting with a known characterization by an approximation property [14] we clarify its relation to the notion of pairwise sufficient σ-subalgebras of $\mathscr{A}$. For a wide class of measurable spaces $(X, \mathscr{A})$ and semigroups G the extreme invariant extensions of P turn out to be those invariant extensions whose conditional probabilities with respect to the σ-algebra of P-almost invariant $\mathscr{B}$-measurable sets are multiplicative modulo an averaging process. As an application of a Choquet type theorem of v. Weizsäcker and Winkler [20] we obtain an extremal integral representation in the set of invariant extensions of P.

Finally, given invariant algebras $\mathscr{A}$ and $\mathscr{B}$ with $\mathscr{B} \subset \mathscr{A}$ we derive characterizations of the extreme points of the convex set of all invariant probability contents on $\mathscr{A}$ which extend a given probability content on $\mathscr{B}$.

2. Preliminaries

Let X be a set, let G be a semigroup which acts from the left on X, and let $\mathscr{A}$ be an invariant algebra on X, i.e.

$$g^{-1}A = \{x \in X: gx \in A\} \in \mathscr{A} \quad \text{for all } g \in G,\ A \in \mathscr{A}.$$

An additive set function $\mu: \mathscr{A} \to \mathbf{R}$ is called invariant if $\mu(g^{-1}A) = \mu(A)$ for all $g \in G$, $A \in \mathscr{A}$. By $ba(\mathscr{A})$ we denote the space of all bounded, (finitely) additive real set functions on $\mathscr{A}$ and by $ba(\mathscr{A})_G$ we denote the subspace of all invariant elements. Then $ba(\mathscr{A})_G$ is an order complete Banach sublattice of $ba(\mathscr{A})$. We may identify $ba(\mathscr{A})$ with the topological dual $B(\mathscr{A})'$ of $B(\mathscr{A})$, where $B(\mathscr{A})$ denotes the closed linear hull of the set $\{1_A: A \in \mathscr{A}\}$ in the Banach lattice $B(X)$

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of all bounded real functions on X. An additive real set function with values in $[0, 1]$ is called probability content; $nba(\mathscr{A})$ is the set of all probability contents on $\mathscr{A}$.

Given an invariant subalgebra $\mathscr{B}$ of $\mathscr{A}$ and $P \in nba(\mathscr{B})_G$ we set

$$F(P)_G = \{Q \in nba(\mathscr{A})_G \colon Q \mid \mathscr{B} = P\}.$$

Obviously $F(P)_G$ is a convex set. Let F_G denote the space of all set functions $\mu \in ba(\mathscr{B})_G$ such that $F(|\mu|)_G \neq \emptyset$. Furthermore, let

$$\mathscr{A}_G = \{A \in \mathscr{A} \colon g^{-1}A = A \quad \text{for all } g \in G\},$$

$$\mathscr{A}(I)_G = \{A \in \mathscr{A} \colon Q(g^{-1}A \,\Delta\, A) = 0 \text{ for all } g \in G,\ Q \in I\} \quad \text{for} \quad I \subset nba(\mathscr{A})_G,$$

and

$$\mathscr{A}(Q)_G = \mathscr{A}(\{Q\})_G \quad \text{for } Q \in nba(\mathscr{A})_G.$$

When dealing with σ-additive set functions we shall always assume that $\mathscr{A}$ and $\mathscr{B}$ are invariant σ-algebras. Let $ca(\mathscr{A})$ denote the space of all σ-additive real set functions on $\mathscr{A}$. Then $ca(\mathscr{A})_G$ is an order complete Banach sublattice of $ca(\mathscr{A})$. By $nca(\mathscr{A})$ we denote the set of all probability measures on $\mathscr{A}$. Given $P \in nca(\mathscr{B})_G$ we set

$$E(P)_G = \{Q \in nca(\mathscr{A})_G \colon Q \mid \mathscr{B} = P\}.$$

Let E_G denote the space of all set functions $\mu \in ca(\mathscr{B})_G$ such that $E(|\mu|)_G \neq \emptyset$. For the following information see [14].

PROPOSITION 2.1. (a) *F_G (resp. E_G) is a band in $ba(\mathscr{B})_G$ (resp. $ca(\mathscr{B})_G$).*
(b) *Suppose $\mu_0 \leq \mu \in F_G$ (resp. E_G) for $\mu_0 \in ba_+(\mathscr{B})_G$ (resp. $ca_+(\mathscr{B})_G$). Then for each $\nu \in F(\mu)_G$ (resp. $E(\mu)_G$) there exists $\nu_0 \in F(\mu_0)_G$ (resp. $E(\mu_0)_G$) that satisfies $\nu_0 \leq \nu$.*

Given a convex set K we shall denote by ex K the set of all extreme points of K.

A semigroup G is called left amenable (LA) if there exists a left invariant mean on G, i.e. a positive linear form m on $B(G)$ satisfying $m(1_G) = 1$ (or equivalently, a probability content on $\mathfrak{P}(G)$) which is invariant under the left translation operators. By interchanging "right" and "left" we obtain the definition of right amenable (RA) semigroups. A semigroup G is called extremely left amenable (ELA) if there exists a left invariant mean on G which is multiplicative.

3. Extreme invariant extensions of measures

Throughout this section $\mathscr{B}$ and $\mathscr{A}$ are invariant σ-algebras on X with $\mathscr{B} \subset \mathscr{A}$ and P is an invariant probability measure on $\mathscr{B}$.

The proof of our first observation is also suitable for probability contents.

PROPOSITION 3.1. *Assume* ex $E(P)_G \neq \emptyset$. *Then the following statements are equivalent:*

(a) ex $E(P)_G = E(P)_G \cap$ ex $nca(\mathscr{A})_G$.

(b) *P is an extreme point of* $nca(\mathscr{B})_G$.

(c) *The cone* $\mathbf{R}_+ \cdot E(P)_G$ *is hereditary to the left in the cone* $ca_+(\mathscr{A})_G$, *i.e.* $\nu \in ca_+(\mathscr{A})_G$, $\mu \in \mathbf{R}_+ \cdot E(P)_G$, $\nu \leq \mu$ *imply* $\nu \in \mathbf{R}_+ \cdot E(P)_G$.

(d) $E(P)_G$ *is a face of* $nca(\mathscr{A})_G$, *i.e.* $Q \in E(P)_G$, Q_1, $Q_2 \in nca(\mathscr{A})_G$, $Q = (Q_1 + Q_2)/2$ *imply* $Q_1, Q_2 \in E(P)_G$.

Proof. (a)$\Rightarrow$(b) If $P = (P_1 + P_2)/2$ with P_1, $P_2 \in nca(\mathscr{B})_G$ and $Q \in$ ex $E(P)_G$, then according to Proposition 2.1 there exists a measure $Q_1 \in E(P_1)_G$ such that $Q_1 \leq 2Q$. Defining $Q_2 = 2Q - Q_1$ we have $Q_2 \in E(P_2)_G$ and $Q = (Q_1 + Q_2)/2$. Since, by (*a*), $Q \in$ ex $nca(\mathscr{A})_G$, it follows that $Q_1 = Q_2$ and hence, $P_1 = P_2$.

(b)$\Rightarrow$(c) Assume $\nu \in ca_+(\mathscr{A})_G$, $\mu \in \mathbf{R}_+ \cdot E(P)_G$ with $\nu \leq \mu$ and $0 \neq \nu \neq \mu$. Let $\alpha = \nu(X)$ and $\beta = \mu(X)$. Defining invariant probability measures on $\mathscr{A}$ by

$$Q = \beta^{-1}\mu, \quad Q_1 = \alpha^{-1}\nu \quad \text{and} \quad Q_2 = \beta(\beta - \alpha)^{-1}(Q - \beta^{-1}\nu)$$

we obtain $Q \in E(P)_G$ and $Q = \alpha\beta^{-1}Q_1 + (1 - \alpha\beta^{-1})Q_2$. According to (b) this implies $Q_1, Q_2 \in E(P)_G$; hence $\nu \in \mathbf{R}_+ \cdot E(P)_G$.

(c)$\Rightarrow$(d)$\Rightarrow$(a) is obvious.

The following generalization of the characterization of extreme invariant probability measures as ergodic measures is due to the author [14, Theorem 7].

THEOREM 3.2. *Let* $Q \in E(P)_G$. *Then Q is an extreme point of* $E(P)_G$ *if and only if for each* $A \in \mathscr{A}(Q)_G$ *there exists* $B \in \mathscr{B}$ *with* $Q(A \,\Delta\, B) = 0$.

In a situation treated by Bierlein [3] (without invariance considerations) we can conclude the existence of extreme points.

COROLLARY 3.3. *Let* $\mathscr{A}$ *be the σ-algebra generated by* $\mathscr{B} \cup \{A_n : n \in \mathbf{N}\}$, *where* A_n, $n \in \mathbf{N}$ *are disjoint invariant subsets of X. Then* ex $E(P)_G \neq \emptyset$.

Proof. We may assume $\bigcup_{n=1}^{\infty} A_n = X$. Obviously

$$\mathscr{A} = \{\bigcup_{n=1}^{\infty} (A_n \cap B_n) : B_n \in \mathscr{B} \text{ for all } n \in \mathbf{N}\}$$

is invariant. For the $\mathscr{B}$-measurable kernel C_n(resp. hull D_n) of A_n we have C_n, $D_n \in \mathscr{B}(P)_G$. Defining $\delta_n = 1_{D_n} - 1_{C_n}$,

$$\Lambda = \Big\{\lambda = (\lambda_n)_{n \in \mathbf{N}} : \lambda_n \text{ is a real } \mathscr{B}(P)_G\text{-measurable function on } X \text{ such that } 0 \leq \lambda_n \leq 1 \text{ and } \sum_{n=1}^{\infty} (1_{C_n} + \lambda_n \delta_n) = 1 \ P\text{-a.e.}\Big\}$$

and

$$Q^{\lambda}\left(\bigcup_{n=1}^{\infty}(A_n \cap B_n)\right) = \sum_{n=1}^{\infty}\int_{B_n}(1_{C_n} + \lambda_n \delta_n)\, dP \quad \text{for } \lambda \in \Lambda,$$

we have according to Bierlein [3, Satz 2*A*] and some simple invariance considerations $E(P)_G = \{Q^{\lambda}: \lambda \in \Lambda\}$.

We will show

$$\text{ex } E(P)_G = \{Q^{\lambda}: \lambda \in \Lambda_0\},$$

where

$\Lambda_0 =$

$\{\lambda = (\lambda_n)_{n \in \mathbf{N}}: \lambda \in \Lambda, \lambda_n \mid D_n \backslash C_n$ is an indicator function P-a.e. for all $n \in \mathbf{N}\}$.

If $Q^{\lambda} \in \text{ex } E(P)_G$, then according to Theorem 3.2 there exist $B_n \in \mathscr{B}$ with $Q^{\lambda}(A_n \,\Delta\, B_n) = 0$ for all $n \in \mathbf{N}$. Since

$$Q^{\lambda}(A_n \,\Delta\, B_n) = \int_{B_c^n}(1_{C_n} + \lambda_n \delta_n)\, dP + \int_{B_n}[1 - (1_{C_n} + \lambda_n \delta_n)]\, dP$$

we obtain $\lambda_n | D_n \backslash C_n = 1_{B_n} | D_n \backslash C_n$ P-a.e.; hence $\lambda = (\lambda_n)_{n \in \mathbf{N}} \in \Lambda_0$. If, conversely, $\lambda = (\lambda_n)_{n \in \mathbf{N}} \in \Lambda_0$, then by definition of Λ_0 there exist $B_n \in \mathscr{B}$, $B_n \subset D_n \backslash C_n$ such that

$$\lambda_n | D_n \backslash C_n = 1_{B_n} | D_n \backslash C_n \quad P\text{-a.e. for all } n \in \mathbf{N}.$$

Setting $B_n' = B_n \cup C_n$ we obtain $Q^{\lambda}(A_n \,\Delta\, B_n') = 0$ for all $n \in \mathbf{N}$. According to Theorem 3.2 this yields $Q^{\lambda} \in \text{ex } E(P)_G$. From [3, Satz 2*B*] and obvious invariance considerations it now follows that ex $E(P)_G \neq \emptyset$.

The following slightly more general result is an immediate consequence of Corollary 3.3 and arguments of Ascherl and Lehn [1].

COROLLARY 3.4. *Let* $\mathscr{A}$ *be the* σ*-algebra generated by* $\mathscr{B} \cup \{A_t: t \in T\}$, *where* T *is any indexing set and* A_t, $t \in T$ *are disjoint invariant subsets of* X. *Then* ex $E(P)_G \neq \emptyset$.

If G is ELA, then $\mathscr{A}(Q)_G = \mathscr{A}$ for any invariant probability content Q on $\mathscr{A}$; see Granirer [10, p. 58]. Hence, one obtains from Theorem 3.2:

COROLLARY 3.5. *Assume that* G *is ELA and* $Q \in E(P)_G$. *Then* Q *is an extreme point of* $E(P)_G$ *if and only if for each* $A \in \mathscr{A}$ *there exists* $B \in \mathscr{B}$ *with* $Q(A \,\Delta\, B) = 0$.

We will weaken the approximation assertion of Theorem 3.2 using the notion of pairwise sufficient σ-subalgebras. The latter were studied in this setting by Plachky [17] for $G = \{\text{id}_X\}$, while Farrell [8] used the notion of sufficient σ-subalgebras in the case $\mathscr{B} = \{\emptyset, X\}$.

Given a σ-subalgebra $\mathscr{C}$ of $\mathscr{A}$ we set

$H(\mathscr{C}) = \{Q \in E(P)_G$: for each $A \in \mathscr{C}$ there exists $B \in \mathscr{B}$ with $Q(A \,\Delta\, B) = 0\}$.

THEOREM 3.6. *For a σ-subalgebra $\mathscr{C}$ of $\mathscr{A}(E(P)_G)_G$ consider the following statements:*

(a) ex $E(P)_G = H(\mathscr{C})$.

(b) *$\mathscr{C}$ is pairwise sufficient for $H(\mathscr{C})$.*

(c) *$\mathscr{C}$ is a determining σ-algebra for $H(\mathscr{C})$, i.e. $Q_1, Q_2 \in H(\mathscr{C})$, $Q_1 | \mathscr{C} = Q_2 | \mathscr{C}$ imply $Q_1 = Q_2$.*

Then (b) $\Rightarrow$ (c) $\Rightarrow$ (a) *and if $\mathscr{B}(P)_G \subset \mathscr{C}$, then all statements are equivalent.*

Proof. (b) $\Rightarrow$ (c) Let Q_1, $Q_2 \in H(\mathscr{C})$ such that $Q_1 | \mathscr{C} = Q_2 | \mathscr{C}$. Then for $A \in \mathscr{A}$ we obtain

$$Q_1(A) = \int E(1_A | \mathscr{C})\, dQ_1 = \int E(1_A | \mathscr{C})\, dQ_2 = Q_2(A),$$

where $E(1_A | \mathscr{C})$ denotes a simultaneous version of the $\mathscr{C}$-conditional probabilities of A with respect to Q_1 and Q_2.

(c) $\Rightarrow$ (a) According to Theorem 3.2 clearly ex $E(P)_G \subset H(\mathscr{C})$ holds. If conversely $Q \in H(\mathscr{C})$, $Q = (Q_1 + Q_2)/2$ with Q_1, $Q_2 \in E(P)_G$, then Q_1, $Q_2 \in H(\mathscr{C})$ and $Q_1 | \mathscr{C} = Q_2 | \mathscr{C}$. By hypothesis this implies $Q_1 = Q_2$ and hence, $Q \in$ ex $E(P)_G$.

(a) $\Rightarrow$ (b) Let us now assume that $\mathscr{B}(P)_G \subset \mathscr{C}$. Let Q_1, $Q_2 \in H(\mathscr{C})$ and $Q = (Q_1 + Q_2)/2$. Define measures Q_i' on $\mathscr{A}$ by

$$Q_i'(A) = \int_A \frac{d(Q_i | \mathscr{C})}{d(Q | \mathscr{C})}\, dQ, \quad i = 1, 2.$$

Then Q_1' and Q_2' are invariant probability measures on $\mathscr{A}$ such that $Q_i' | \mathscr{C} = Q_i | \mathscr{C}$, $i = 1, 2$. Since $\mathscr{B}(P)_G \subset \mathscr{C}$, this implies Q_1', $Q_2' \in E(P)_G$ [16, 10.2] and hence, Q_1', $Q_2' \in H(\mathscr{C})$ and moreover $(Q_i' + Q_i)/2 \in H(\mathscr{C})$, $i = 1, 2$. According to (a) we obtain $Q_i' = Q_i$, and it follows that

$$\frac{dQ_i}{dQ} = \frac{d(Q_i | \mathscr{C})}{d(Q | \mathscr{C})}\ Q\text{-a.e.} \quad \text{for } i = 1, 2.$$

This implies that $\mathscr{C}$ is sufficient for $\{Q_1, Q_2\}$ (cf. [22, Satz 3.21]).

The following example shows that in general ex $E(P)_G = H(\mathscr{C})$ for a σ-algebra $\mathscr{C}$ with $\mathscr{B}(P)_G \subset \mathscr{C} \subset \mathscr{A}(E(P)_G)_G$ does not imply that $\mathscr{C}$ is pairwise sufficient for $E(P)_G$, even in the case $\mathscr{B} = \{\emptyset, X\}$.

Example 3.7. Let $X = (-1, 0) \cup (0, 1)$, $\mathscr{A}$ the Borel σ-algebra on X, and $\mathscr{B} = \{\emptyset, X\}$. For $x \in (0, 1)$ define bijective mappings g_x: $X \to X$ by

$$g_x = \mathrm{id}_X 1_{\{-x, x\}^c} + x 1_{\{-x\}} - x 1_{\{x\}}$$

and let G denote the group generated by $\{g_x: x \in (0, 1)\}$. Then

$$\mathscr{A}_G = \mathscr{A}(nca(\mathscr{A})_G)_G = \{A \in \mathscr{A}: A = -A\}.$$

Clearly we have

$$\{(\delta_x + \delta_{-x})/2: x \in (0, 1)\} \subset \text{ ex } nca(\mathscr{A})_G \subset H(\mathscr{A}_G),$$

where δ_x denotes the unit mass on $\mathscr{A}$ placed at the point x. To see that

$$H(\mathscr{A}_G) \subset \{(\delta_x + \delta_{-x})/2: x \in (0, 1)\}$$

let $Q \in H(\mathscr{A}_G)$ and denote the $\{0, 1\}$-valued measure $Q|\mathscr{A}_G$ by Q_0. Since Q is inner regular with respect to compact sets it is easily verified that Q_0 is inner regular, i.e.

$$Q_0(A) = \sup\{Q_0(K): K \subset A, K \text{ compact}, K \in \mathscr{A}_G\} \quad \text{for all } A \in \mathscr{A}_G.$$

This implies that $Q_0(\{x, -x\}) = 1$ for some $x \in X$. In view of the invariance of Q we obtain $Q = (\delta_x + \delta_{-x})/2$. But according to Luschgy [13, Example 2], $\mathscr{A}_G$ is not pairwise sufficient for $nca(\mathscr{A})_G$.

. . . .

Next we will show that for certain LA semigroups G the extreme points of $E(P)_G$ are those invariant extensions of P whose $\mathscr{B}(P)_G$-conditional probabilities are multiplicative modulo an averaging process with respect to any left invariant mean on G.

Let us introduce some more notations. For a subset H of G, set

$$g^{-1}H = \{h \in G: gh \in H\}.$$

A measurable space $(X, \mathscr{A})$ is called Blackwell space if $\mathscr{A}$ is countably generated and the range of every real measurable function on X is Souslin (i.e. a continuous image of a polish space). If $(X, \mathscr{A})$ is a Blackwell space, then there exists a regular $\mathscr{B}(P)_G$-conditional probability of any probability measure Q on $\mathscr{A}$ denoted by R_Q. Furthermore, let Q_* denote inner measure of Q.

THEOREM 3.8. *Suppose that $(X, \mathscr{A})$ is a Blackwell space, that G is LA, and that the following conditions are satisfied:*

(i) *There exists a left invariant σ-algebra $\mathscr{G}$ on G such that the action $G \times X \to X$, $(g, x) \mapsto gx$ is $(\mathscr{G} \otimes \mathscr{A}, \mathscr{A})$-measurable and a non-zero, σ-finite quasi-left invariant measure ω on $\mathscr{G}$, i.e. $\omega(H) = 0$ implies $\omega^g(H) = \omega(g^{-1}H) = 0$ for all $g \in G$, $H \in \mathscr{G}$.*

(ii) *There exists a countably generated σ-subalgebra $\mathscr{C}$ of $\mathscr{A}(nca(\mathscr{A})_G)_G$ that is pairwise sufficient for nca $(\mathscr{A})_G$.*

Then for $Q \in E(P)_G$ the following assertions are equivalent:

(a) *Q is an extreme point of $E(P)_G$.*

(b) *For each left invariant mean m on G we have*

$$Q_*(\{x \in X: m(R_Q(x, g^{-1}A \cap B)) = R_Q(x, A)R_Q(x, B)\}) = 1$$

for all $A, B \in \mathscr{A}$.

(c) *For some mean m on G we have*

$$Q_*(\{x \in X: m(R_Q(x, g^{-1}A \cap B)) = R_Q(x, A)R_Q(x, B)\}) = 1$$

for all $A, B \in \mathscr{A}$.

Proof. We may assume that ω is a probability measure. For shorter notation set $R = R_Q$. We will show that $R(x, \cdot)$ is an invariant probability measure on $\mathscr{A}$ for all x in some P-null set. It is easily seen that the function

$$G \times X \to [0, 1], \quad (g, x) \mapsto R(x, g^{-1}A)$$

is $\mathscr{G} \otimes \mathscr{B}(P)_G$-measurable for all $A \in \mathscr{A}$. Furthermore,

$$R(\cdot, g^{-1}A) = R(\cdot, A) \quad P\text{-a.e.}$$

is valid for all $g \in G$, $A \in \mathscr{A}$. For $A \in \mathscr{A}$ set

$$S(A) = \{(g, x) \in G \times X: R(x, g^{-1}A) \neq R(x, A)\};$$

then $S(A) \in \mathscr{G} \otimes \mathscr{B}(P)_G$ and $P(S(A)_g) = 0$ for all $g \in G$. Setting

$$N(A) = \{x \in X: \omega(S(A)_x) > 0\}$$

we obtain according to Fubini's theorem $N(A) \in \mathscr{B}(P)_G$ and $P(N(A)) = 0$. Define

$$R_0(x, A) = \int R(x, g^{-1}A)\, d\omega(g) \quad \text{for } x \in X,\ A \in \mathscr{A}.$$

Then $R_0(\cdot, A)$ is $\mathscr{B}(P)_G$-measurable for all $A \in \mathscr{A}$ and $R_0(x, \cdot) \in nca(\mathscr{A})$ for all $x \in X$. Let $\mathscr{A}_0$ be a countable algebra that generates $\mathscr{A}$ and set $N_0 = \bigcup_{A \in \mathscr{A}_0} N(A)$. Then $P(N_0) = 0$ and since

$$N(A)^c \subset \{x \in X: R_0(x, A) = R(x, g^{-1}A)\ \omega\text{-a.e.}\}$$

we obtain

$$R_0(x, h^{-1}A) = \int R(x, (hg)^{-1}A)\, d\omega(g)$$

$$= \int R(x, g^{-1}A)\, d\omega^h(g) = \int R_0(x, A)\, d\omega^h(g)$$

$$= R_0(x, A) \quad \text{for all } x \in N_0^c,\ A \in \mathscr{A}_0,\ h \in G.$$

This implies $R_0(x, \cdot) \in nca(\mathscr{A})_G$ for all $x \in N_0^c$. Finally, since

$$N(A)^c \subset \{x \in X: R(x, A) = R_0(x, A)\}$$

we have $R(x, A) = R_0(x, A)$ for all $x \in N_0^c$, $A \in \mathscr{A}_0$. This yields $R(x, \cdot) = R_0(x, \cdot)$ for all $x \in N_0^c$ and hence, $R(x, \cdot) \in nca(\mathscr{A})_G$ for all $x \in N_0^c$.

(a) $\Rightarrow$ (b) Note first that $R(x, \cdot) \in \text{ex}\, nca(\mathscr{A})_G$ for all x in some Q-null set. In fact, if $A \in \mathscr{C}$, then according to Theorem 3.2 there exists $B \in \mathscr{B}$ with $Q(A \,\Delta\, B) = \int R(x, A \,\Delta\, B)\, dQ = 0$, which yields $R(\cdot, A) = 1_A Q$-a.e. Since $\mathscr{C}$ is countably generated, this implies $R(x, \cdot)|\mathscr{C} = \delta_x|\mathscr{C}$ for all x in some Q-null set N_1, thus according to Theorem 3.6 $R(x, \cdot) \in \text{ex } nca(\mathscr{A})_G$ for all $x \in N^c$, where $N = N_0 \cup N_1$, $Q(N) = 0$.

Now let m be a left invariant mean on G and $x \in N^c$. For $A, B \in \mathscr{A}$ define

$$U(x, A) = m(R(x, g^{-1}A \cap B)) - R(x, A)R(x, B).$$

It is easily seen that $R(x, \cdot) \pm U(x, \cdot)$ are invariant probability contents on $\mathscr{A}$ and $R(x, \cdot) \pm U(x, \cdot) \le 2R(x, \cdot)$ ensures that $R(x, \cdot) \pm U(x, \cdot)$ are σ-additive. This implies $U(x, \cdot) = 0$.

(b) $\Rightarrow$ (c) Clear.

(c) $\Rightarrow$ (a) Let $C \in \mathscr{C}$. Since $\mathscr{C} \subset \mathscr{A}(nca(\mathscr{A})_G)_G$, we have, by (c),

$$R(\cdot, C \cap A) = R(\cdot, C)R(\cdot, A) \ Q\text{-a.e.} \quad \text{for all } A \in \mathscr{A}.$$

This implies $Q(C \cap A) = \int R(x, C)1_A \, dQ$ for all $A \in \mathscr{A}$ and hence, $R(\cdot, C) = 1_C$ Q-a.e. Setting $B = \{R(\cdot, C) = 1\}$ we obtain $B \in \mathscr{B}(P)_G$ and $Q(C \,\Delta\, B) = 0$. Thus $Q \in \text{ex } E(P)_G$ follows from Theorem 3.6.

Remarks. (1) If G is a locally compact second countable Hausdorff group or an Abelian locally compact second countable Hausdorff semigroup which admits a non-zero sub-invariant measure ω with $\omega(K) < \infty$ for all compact subsets K of G such that the action of G on X is measurable with respect to the Borel σ-algebra on G, then condition (i) and (ii) are satisfied; see Farrell [8, Theorem 3].

(2) It is not without interest to observe that in general the extreme points of $E(P)_G$ are not pairwise orthogonal, in contrast to the case $\mathscr{B} = \{\emptyset, X\}$; compare Blum and Hanson [4] Corollary 2, for $\mathscr{B} = \{\emptyset, X\}$.

4. Extremal integral representations

In this section X is a topological Hausdorff space with its Borel σ-algebra $\mathscr{A}$ and $\mathscr{B}$ is a σ-subalgebra of $\mathscr{A}$. G is a semigroup which acts from the left on X such that $\mathscr{B}$ and $\mathscr{A}$ are invariant σ-algebras and P is an invariant probability measure on $\mathscr{B}$. Let $nca(\mathscr{A}, r)$ denote the set of all Borel probability measures on X which are inner regular with respect to compact sets. We set $E(P, r)_G = E(P)_G \cap nca(\mathscr{A}, r)$; then $\text{ex } E(P, r)_G = \text{ex } E(P)_G \cap nca(\mathscr{A}, r)_G$. As an application of an integral representation theorem of v. Weizsäcker and Winkler [20] for convex non-compact sets of inner regular measures we will give an integral representation for every $Q \in E(P, r)_G$.

Let τ denote the narrow topology on $nca(\mathscr{A}, r)$. Then τ is Hausdorff (cf. [19, p. 371]). For a family $\mathscr{F}$ of bounded real Borel functions on X let $\sigma(\mathscr{F})$ be the initial topology on $nca(\mathscr{A}, r)$ of the functions $Q \mapsto \int f \, dQ$, $f \in \mathscr{F}$; by $\tau(\mathscr{F})$ we denote the topology generated by τ and $\sigma(\mathscr{F})$. The Hausdorff topology $\tau(\mathscr{F})$ is called admissible if $\mathscr{F}$ is countable. Let $\sum(\text{ex } E(P, r)_G)$ denote the σ-algebra on $\text{ex } E(P, r)_G$ generated by the functions $Q \mapsto Q(A)$, $A \in \mathscr{A}$.

PROPOSITION 4.1. *Suppose that at least one of the conditions* (i)–(iv) *and one of the conditions* (v)–(viii) *is satisfied:*

(i) $\mathscr{B}$ *is countably generated.*

(ii) $\mathscr{B}$ *is generated by a family of bounded real continuous functions on* X.

(iii) $\mathscr{B} = \phi^{-1}(\mathscr{C})$ *for some continuous mapping* $\phi: X \to Y$, *where* Y *is a topological Hausdorff space with its Borel* σ-*algebra* $\mathscr{C}$.

(iv) P *is inner regular, i.e.*

$$P(B) = \sup\{P(K): K \subset B,\ K \text{ compact},\ K \in \mathscr{B}\} \quad \textit{for all } B \in \mathscr{B}.$$

(v) G *acts continuously on* X.

(vi) G *is countable and* $\mathscr{A}$ *is countably generated.*

(vii) G *has a countable dense subsemigroup with respect to the initial topology on* G *of the mappings* $g \mapsto gx$, $x \in X$, X *is metrizable, and* $\mathscr{A}$ *is countably generated.*

(viii) G *is a locally compact second countable Hausdorff group, the action* $G \times X \to X, (g, x) \mapsto gx$ *is* $(\mathscr{G} \otimes \mathscr{A}, \mathscr{A})$-*measurable* ($\mathscr{G}$ *is the Borel* σ-*algebra*), *and* $\mathscr{A}$ *is countably generated.*

Then for every $Q \in E(P, r)_G$ *there exists a probability measure* ρ *on* $\sum$ (ex $E(P, r)_G$) *such that*

$$Q(A) = \int Q'(A)\, d\rho(Q') \quad \textit{for all } A \in \mathscr{A}.$$

Proof. According to v. Weizsäcker and Winkler [20, Theorem 1] it suffices to prove that $E(P, r)_G$ is closed with respect to some admissible topology on $nca(\mathscr{A}, r)$. We show first that under any one of the conditions (i)–(iv) $E(P, r)$ is closed with respect to some admissible topology.

Assume (i). Set $\mathscr{F} = \{1_B: B \in \mathscr{B}_0\}$, where $\mathscr{B}_0$ is a countable algebra generating $\mathscr{B}$. Clearly $E(P, r)$ is $\tau(\mathscr{F})$-closed.

Assume (ii). Then $\mathscr{B}$ is generated by $\mathscr{E} \subset C_b(X)$. Let $\hat{\mathscr{E}}$ denote the smallest vector sublattice of $C_b(X)$ that contains $\mathscr{E}$ and 1_X. Since $\hat{\mathscr{E}} \subset B(\mathscr{B}) \cap C_b(X)$, $\mathscr{B}$ is generated by $\hat{\mathscr{E}}$. Now let $(Q_\alpha)_\alpha$ be a net in $E(P, r)$, $Q \in nca(\mathscr{A}, r)$ such that $\lim_\alpha Q_\alpha = Q$ with respect to τ. Then $\lim_\alpha \int f\, dQ_\alpha = \int f\, dQ$ for all $f \in C_b(X)$ which yields $\int f\, dP = \int f\, dQ$ for all $f \in \hat{\mathscr{E}}$. This implies $Q \in E(P, r)$ (cf. [2, 39.3]) and hence, $E(P, r)$ is τ-closed.

Assume (iii). Consider a net $(Q_\alpha)_\alpha$ in $E(P, r)$, $Q \in nca(\mathscr{A}, r)$ such that $\lim_\alpha Q_\alpha = Q$ with respect to τ. Then $\lim_\alpha Q_\alpha^\phi = Q^\phi$ narrowly in $nca(\mathscr{C}, r)$ (cf. [19, p. 372]). Since $Q_\alpha^\phi = P^\phi$ for all α, we obtain $Q^\phi = P^\phi$ and hence, $Q \in E(P, r)$. Thus $E(P, r)$ is τ-closed.

Assume (iv). Let $(Q_\alpha)_\alpha$ be a net in $E(P, r)$, $Q \in nca(\mathscr{A}, r)$ such that $\lim_\alpha Q_\alpha = Q$ with respect to τ. Then $\overline{\lim}_\alpha Q_\alpha(K) \leq Q(K)$ for every compact set K which yields $P(K) \leq Q(K)$ for all $K \in \mathscr{B}$, K compact. Inner regularity of P and Q implies $Q \in E(P, r)$ and hence, $E(P, r)$ is τ-closed.

To complete the proof we remark that under any one of the conditions (v)–(viii) $nca(\mathscr{A}, r)_G$ is closed with respect to some admissible topology. For (v) and (vi) compare v. Weizsäcker and Winkler [21, Proposition 8]. For (vii) (resp. (viii)) let G_0 be a countable dense subsemigroup (resp. subgroup) of G, $\mathscr{A}_0$ a countable algebra generating $\mathscr{A}$, and set

$$\mathscr{F} = \{1_{g^{-1}A}: A \in \mathscr{A}_0, g \in G_0\}.$$

Consider the limit Q of a net in $nca(\mathscr{A}, r)_G$ with respect to the admissible topology $\tau(\mathscr{F})$. It is obvious that Q is G_0-invariant and according to arguments of Farrell [8] (proofs of Corollary 3 and Corollary 4) Q is invariant. Hence, $nca(\mathscr{A}, r)_G$ is $\tau(\mathscr{F})$-closed.

Remarks. (1) For the existence of extreme points and hence, for the integral representation in $E(P, r)_G$ the assumptions for $\mathscr{B}$ and G cannot be dropped. For $\mathscr{B}$ see f.e. Plachky [18]. For G let $X = \mathbf{R}$ and consider the semigroup

$$G = \{g \in X^X: \#\{x \in X: gx \neq x\} < \infty\}$$

and $\mathscr{B} = \{\emptyset, X\}$. Then $nca(\mathscr{A}, r)_G = nca(\mathscr{A})_G = \{Q \in nca(\mathscr{A}): Q \text{ is non-atomic}\}$ but ex $nca(\mathscr{A})_G = \emptyset$.

(2) It is easily seen by simple examples that the above integral representation is in general not unique.

5. Extreme invariant extensions of contents

Throughout this section $\mathscr{B}$ and $\mathscr{A}$ are invariant algebras on X with $\mathscr{B} \subset \mathscr{A}$ and P is an invariant probability content on $\mathscr{B}$. Then the convex set $F(P)_G$ of all invariant probability contents on $\mathscr{A}$ which extend P is $\sigma(ba(\mathscr{A}), B(\mathscr{A}))$-compact. Hence, according to the theorem of Krein-Milman, ex $F(P)_G \neq \emptyset$ iff $F(P)_G \neq \emptyset$. Moreover, according to the theorem of Bishop-de Leeuw, for every $Q \in F(P)_G$ there exists a (non-unique) probability measure ρ on $\sum$ (ex $F(P)_G$) (cf. Section 4) such that $Q(A) = \int Q'(A)\, d\rho(Q')$ for all $A \in \mathscr{A}$.

Analogous to Proposition 3.1 we have:

PROPOSITION 5.1. *Assume $F(P)_G \neq \emptyset$. Then the following statements are equivalent:*

(a) ex $F(P)_G = F(P)_G \cap$ ex $nba(\mathscr{A})_G$.
(b) *P is an extreme point of $nba(\mathscr{B})_G$.*
(c) *The cone $\mathbf{R}_+ \cdot F(P)_G$ is hereditary to the left in the cone $ba_+(\mathscr{A})_G$.*
(d) *$F(P)_G$ is a face of $nba(\mathscr{A})_G$.*

We need the following information (cf. [7, IV.6.18, IV.9.10 and IV.9.11]). Let X' be the Stone representation space of $\mathscr{A}$ so that X may be identified with the $\sigma(ba(\mathscr{A}), B(\mathscr{A}))$-compact totally disconnected Hausdorff space of $\{0, 1\}$-valued probability contents on $\mathscr{A}$. Then the evaluation map $T: B(\mathscr{A}) \to C(X')$ is an isometric lattice isomorphism onto $C(X')$. Let $\mathscr{A}''$ be the algebra of clopen subsets of X'. Then the map $\phi: \mathscr{A} \to \mathscr{A}''$ defined by $\phi(A) = \{T(1_A) = 1\}$ is an isomorphism of $\mathscr{A}$ onto $\mathscr{A}''$. By $\mathscr{A}'$ we denote the σ-algebra generated by

$\mathscr{A}''$($\mathscr{A}'$ is the Baire σ-algebra). ϕ induces an isometric lattice isomorphism U of $ba(\mathscr{A})$ onto $ca(\mathscr{A}')$ determined by $U\mu(C) = \mu(\phi^{-1}C)$, $C \in \mathscr{A}''$. G acts continuously from the left on X', hence $\mathscr{A}''$ and $\mathscr{A}'$ are invariant. Since $g^{-1}\phi(A) = \phi(g^{-1}A)$ for all $g \in G$, $A \in \mathscr{A}$, we obtain $U(ba(\mathscr{A})_G) = ca(\mathscr{A}')_G$. Let $\mathscr{B}'$ be the invariant σ-subalgebra of $\mathscr{A}'$ generated by the invariant algebra $\mathscr{B}'' = \phi(\mathscr{B})$ and let $P' \in nca(\mathscr{B}')_G$ be the (uniquely determined) extension of P'' defined by $P''(C) = P(\phi^{-1}C)$, $C \in \mathscr{B}''$. Then $U(F(P)_G) = E(P')_G$.

From this facts and Theorem 3.2 we obtain the following generalization of results of Olshen [15] and Plachky [18].

THEOREM 5.2. *For $Q \in F(P)_G$ the following statements are equivalent*:

(a) *Q is an extreme point of $F(P)_G$.*

(b) *For each sequence $(A_n)_n$ in $\mathscr{A}$ such that*

$$\lim_{n,m\to\infty} Q(A_n \Delta A_m) = 0 \text{ and } \lim_{n\to\infty} Q(A_n \Delta g^{-1}A_n) = 0$$

for all $g \in G$ and for each $\varepsilon > 0$ there exists $B \in \mathscr{B}$ with $\inf_n Q(A_n \Delta B) < \varepsilon$.

COROLLARY 5.3. *Assume that G is ELA and $Q \in F(P)_G$. Then Q is an extreme point of $F(P)_G$ if and only if for each $A \in \mathscr{A}$ and $\varepsilon > 0$* there exists *$B \in \mathscr{B}$ with $Q(A \Delta B) < \varepsilon$.*

Remark. If G is LA, then according to a fixed point theorem of Day [5], $F(P)_G = \emptyset$ and hence, ex $F(P)_G \neq \emptyset$. If moreover G is ELA, then in view of Corollary 5.3 there exists $Q \in F(P)_G$ such that the closures of the set of all values of Q and P are equal. This is an invariant version of a result due to Sikorski (cf. [12]).

For the following theorem let us remark that G acts from the right on $B(\mathscr{A})$ by $G \times B(\mathscr{A}) \to B(\mathscr{A})$, $(g, f) \mapsto gf$ such that $gf(x) = f(gx)$, $x \in X$.

THEOREM 5.4. *Assume that G is RA and that at least one of the following conditions is satisfied:*

(i) *G admits a discrete right invariant mean.*

(ii) *G is LA, $\mathscr{A}$ is a σ*-algebra, *and the action of G on $B(\mathscr{A})$ is weakly almost periodic, i.e. for each $f \in B(\mathscr{A})$ the set $\{gf: g \in G\}$ is relatively weakly compact.*

Then $Q \in F(P)_G$ is an extreme point of $F(P)_G$ if and only if for each $A \in \mathscr{A}_G$ and $\varepsilon > 0$ there exists $B \in \mathscr{B}$ with $Q(A \Delta B) < \varepsilon$.

There is an extensive discussion by Granirer [9] of semigroups which satisfy (i). Theorem 5.4 is an immediate consequence of Theorem 5.2 and the following lemma. Let $D(\mathscr{A})$ denote the linear hull of the set $\{1_{g^{-1}A} - 1_A : A \in \mathscr{A}, g \in G\}$.

LEMMA 5.5. *In the situation of Theorem 5.4, $\mathscr{A}_G$ is a determining algebra for $nba(\mathscr{A})_G$.*

Proof. Assume (i). Then according to Granirer [9, Theorem 4.2 and Remark 5.4] there exists a discrete right invariant mean m on G such that $H = \{h \in G: m(h) > 0\}$ is finite $(m(h) = m(\{h\}))$. Observe that $Hg = H$ for all $g \in G$. To see this consider $(Hg)g^{-1} = \{k \in G: kg \in Hg\}$ and $\{hg\}g^{-1}$ for $h \in H$, $g \in G$. Then

$$1 = m(H) \leq m((Hg)g^{-1}) = m(Hg) \leq 1$$

and

$$0 < m(h) \leq m(\{hg\}g^{-1}) = m(hg).$$

We thus obtain $\mathscr{A}_G = \mathscr{A}_H$.

As before X' denotes the Stone representation space of $\mathscr{A}$ and $\mathscr{A}'$ the σ-algebra generated by $\mathscr{A}'' = \phi(\mathscr{A})$. Let $(\mathscr{A}_H)'' = \phi(\mathscr{A}_H)$ and let $(\mathscr{A}_H)'$ be the σ-subalgebra of $\mathscr{A}'$ generated by $(\mathscr{A}_H)''$. Then $(\mathscr{A}_H)'' = (\mathscr{A}'')_H$ and moreover, $(\mathscr{A}_H)' = (\mathscr{A}')_H$. In fact,

$$\mathscr{M} = \{C \in \mathscr{A}': \bigcup_{h \in H} h^{-1}C \in (\mathscr{A}_H)'\}$$

is a monotone class containing $\mathscr{A}''$ and hence, $\mathscr{M} = \mathscr{A}'$. Now suppose that Q_1, $Q_2 \in nba(\mathscr{A})_G$ with $Q_1|\mathscr{A}_H = Q_2|\mathscr{A}_H$. Then $Q_1'(\mathscr{A}')_H = Q_2'(\mathscr{A}')_H$, where $Q_i' = UQ_i$, $i = 1, 2$. Since $(\mathscr{A}')_H$ is obviously sufficient for $nca(\mathscr{A}')_G$, we obtain $Q_1' = Q_2'$ (cf. [8, Theorem 1]) and hence, $Q_1 = Q_2$.

Assume (ii). Then according to a mean ergodic theorem of Dixmier [6] we have $B(\mathscr{A}) = B(\mathscr{A}_G) \oplus D(\mathscr{A})^-$, where $D(\mathscr{A})^-$ denotes the norm closure of $D(\mathscr{A})$. From this decomposition the assertion follows immediately.

For the reader's convenience we finally give some statements which are equivalent to the proposition $F(P)_G \neq \emptyset$. Let $L(\mathscr{A})$ denote the linear hull of the set $\{1_A: A \in \mathscr{A}\}$ and let $M(P, \mathscr{A})$ denote the linear hull of the set

$$\{1_{g^{-1}A} - 1_A + 1_B - P(B)1_B: A \in \mathscr{A}, B \in \mathscr{B}, g \in G\}.$$

Define $P_e: L(\mathscr{A}) \to \mathbf{R} \cup \{-\infty\}$ by

$$P_e(f) = \inf\{P(h): f \leq h + d, h \in L(\mathscr{B}), d \in D(\mathscr{A})\}.$$

PROPOSITION 5.6. *The following statements are equivalent:*

(a) $F(P)_G \neq \emptyset$.

(b) $P_e(f) > -\infty$ *for some* $f \in L(\mathscr{A})$.

(c) $P_e(f) > -\infty$ *for each* $f \in L(\mathscr{A})$.

(d) *Whenever* $n, m \geq 1$, $(B_1, \ldots, B_n) \in \mathscr{B}^n$, $(C_1, \ldots, C_m) \in \mathscr{B}^m$, *and* $d \in D(\mathscr{A})$ *are such that* $\sum_{i=1}^n 1_{B_i} \geq \sum_{j=1}^m 1_{C_j} + d$, *we have*

$$\sum_{i=1}^n P(B_i) \geq \sum_{j=1}^m P(C_j).$$

(e) *Whenever* $h_1, h_2 \in L(\mathscr{B})$ *and* $d \in D(\mathscr{A})$ *are such that* $h_1 \geq h_2 + d$, *we have* $P(h_1) \geq P(h_2)$.

(f) $\sup\{f(x): x \in X\} \geq 0$ *for each* $f \in M(P, \mathscr{A})$.

(g) $\inf\{\|f - 1_X\|: f \in M(P, \mathscr{A})\} = 1$.

(h) $1_X \notin M(P, \mathscr{A})^-$, *where* $M(P, \mathscr{A})^-$ *denotes the norm closure of* $M(P, \mathscr{A})$.

Proof. (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c). This follows from Klee [11].

(a)$\Rightarrow$(d) Obvious.

(d)$\Rightarrow$(e) Suppose that there exist h_1, $h_2 \in L(\mathscr{B})$, $d \in D(\mathscr{A})$ such that $h_1 \geq h_2 + d$ and $P(h_1) < P(h_2)$. We may assume that $h_i(X) \subset \mathbf{Q}$, $i = 1, 2$. Hence, there exists $n \in \mathbf{N}$ such that $nh_i(X) \subset \mathbf{Z}$, $i = 1, 2$. The above inequalities are valid for nh_1, nh_2 and nd. But this contradicts (d).

(e)$\Rightarrow$(f) For $f \in M(P, \mathscr{A})$ we can find h_1, $h_2 \in L(\mathscr{B})$, $d \in D(\mathscr{A})$ such that $f = d - h_1 + h_2$ and $P(h_1) = P(h_2)$. Setting

$$c = \sup\{f(x): x \in X\}$$

we obtain $h_1 + c1_X \geq h_2 + d$ and hence, by (e), $P(h_1) + c \geq P(h_2)$. This implies $c \geq 0$.

(f)$\Rightarrow$(g)$\Rightarrow$(h). Obvious.

(h)$\Rightarrow$(a) A routine separation argument shows that there exists $\mu \in ba(\mathscr{A})$ such that $\mu(f) = 0$ for all $f \in M(P, \mathscr{A})^-$ and $\mu(X) = 1$. Then μ is invariant and $\mu | \mathscr{B} = P$. Hence, μ^+ is invariant and $\mu^+ | \mathscr{B} \geq P$. The assertion follows from Proposition 2.1.

COROLLARY 5.7. *Let* $\mathscr{A}$ *be the algebra generated by*

$$\mathscr{B} \cup \{A \subset X: A \textit{ is invariant}\}.$$

Then $F(P)_G \neq \emptyset$.

Proof. Clearly $\mathscr{A}$ is invariant. Let $T = \{t: t$ is a finite set of invariant subsets of $X\}$ and let $\mathscr{A}_t$ be the invariant algebra generated by $\mathscr{B} \cup t$. From Loś and Marczewski [12] follows that for each $t \in T$ there exists an invariant probability content on $\mathscr{A}_t$ which extends P. Since $\mathscr{A} = \bigcup_{t \in T} \mathscr{A}_t$ and $D(\mathscr{A}) = \cup_{t \in T} D(\mathscr{A}_t)$, the assertion follows from Proposition 5.6.

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