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EQUIVARIANT BUNDLES

BY

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We develop a theory of equivariant bundles, i.e., bundles with a compact Lie group G of automorphisms. Equivariant vector bundles were discussed by Wasserman [9] and Atiyah-Segal [8]; Bierstone [1] considered smooth equivariant bundles; and in [6] we sketched the general theory of equivariant bundles with a finite group G of automorphisms. However, general equivariant bundles are needed in equivariant smoothing theory [5]; and unfortunately, none of the above expositions generalizes without important modifications. As in [6], we generalize the Dold numerable bundle theory [3] to the equivariant case.

1. Numerable G-bundles

Let $p\colon E \to X$ be a locally trivial bundle with fibre F and structure group A. We call p a *G-bundle*, or more precisely a G-A bundle if E and X are G-spaces, p is a G-map, and G acts on E through A-bundle maps. Two G-A bundles over X are called *G-A equivalent* if they are A-equivalent via a G-equivariant map.

Example 1. A G-vector bundle [8] of dimension n is a G-L_n bundle, L_n the group of linear isomorphisms of R^n.

If $p\colon E \to X$ is a G-A bundle, the action of G induces an action of G on the associated principal A-bundle P, again through A-bundle maps. That is, G acts on the left and A acts on the right of P and these actions commute. Conversely, if $p\colon P \to X$ is a principal G-A bundle and A acts on the left of F, then $E = P \times_A F$ is a G-A bundle with fibre F. Two G-A bundles with fibre F are G-A equivalent if and only if their associated principal G-A bundles are G-A equivalent.

In order to prove a covering homotopy property or to produce a classifying space for ordinary bundles, the local triviality condition is essential. Bierstone [1] pointed out that for equivariant bundles one needs a G-local triviality condition for the same purpose. Before defining this condition we recall the local structure of a completely regular G-space X (see [2]): For any $x \in X$ there is a G_x-invariant subspace V_x containing x, called a *slice through* x, such that

$$\mu\colon G \times_{G_x} V_x \to X, \quad \mu[g, v] = gv$$

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[1] By abuse of notation, we shall write $E_0 = E/X$.

is a homeomorphism onto an open neighborhood of the orbit Gx. The G-invariant neighborhood GV_x is called a *tube* about Gx. For an arbitrary G-space X we define an *H-slice* V to be an H-invariant subspace V of X, H some closed subgroup of G, such that

$$\mu: G \times_H V \to X, \quad \mu[g, v] = gv,$$

is a homeomorphism onto an open set. (Note that a slice through x is a G_x slice which contains x.) Palais [7] shows that, if $f: Y \to X$ is a G-map, then $f^{-1}(V)$ is an H-slice in Y.

We now describe the appropriate generalization of product bundle: Let H be a closed subgroup of G and $\rho: H \to A$ a homomorphism. For any H-space V, we denote by $\varepsilon^\rho(V)$ the G-A bundle over $G \times_H V$ with fibre F given by

$$p: G \times_H (V \times F) \to G \times_H V, \quad p[g, (v, y)] = [g, v],$$

where H acts on F via the homomorphism ρ and the left action of A on F.

DEFINITION. A G-A bundle $p: E \to X$ with fibre F is called *G-A locally trivial* (or simply *G-locally trivial* if A is fixed) if there is an open cover $\{GV_\alpha\}_{\alpha \in I}$ of X, where V_α is an H_α slice, such that $E \mid GV_\alpha$ is G-A equivalent to $\varepsilon^{\rho_\alpha}(V_\alpha)$ for some homomorphism $\rho_\alpha: H_\alpha \to A$ (under the identification $\mu: G \times_{H_\alpha} V_\alpha \to GV_\alpha$).

If $p: E \to X$ is a G-A bundle and $f: Y \to X$ is a G-map, the induced bundle $f^*(p): f^*E \to Y$ is a G-A bundle. Further, if p is G-A locally trivial, then $f^*(p)$ is G-A locally trivial. Also note that a G-A bundle is G-A locally trivial if and only if its associated principal bundle is G-A locally trivial.

Bierstone [1] gave a somewhat different definition of G locally trivial:

Bierstone's Condition. For each $x \in X$ there is a G_x invariant neighborhood U_x such that $p^{-1}(U_x)$ is G_x-A equivalent to $U_x \times F$ with G_x action

$$h(u, g) = (hu, \rho_x(h)y),$$

where $u \in U_x$, $h \in G_x$, $y \in F$ and $\rho_x: G_x \to A$ is a homomorphism.

LEMMA 1.1. *A G-A locally trivial bundle satisfies Bierstone's condition.*

Proof. Suppose $p: E \to X$ is G-A locally trivial. For any $x \in X$ there is an H-slice V such that $x \in GV$ and $p^{-1}(GV)$ is G-A equivalent to

$$\varepsilon^\rho(V) = G \times_H (V \times F)$$

over the homeomorphism of GV with $G \times_H V$. If $x = gv$, $v \in V$, then $G_v \subset H$ and $G_x = gG_v g^{-1}$. If D is a small normal disc to H through $1 \in G$ with respect to a biinvariant metric on G, then D is invariant under conjugation by any $h \in H$ and $\mu: D \times H \to DH$ is a diffeomorphism. Thus $U_x = gDV$ is invariant under G_x and $p^{-1}(gDV)$ is G_x-A equivalent to $gDV \times F$.

COROLLARY 1.2. *A G-A locally trivial bundle is locally trivial as an A-bundle.*

LEMMA 1.3. *If X is completely regular, a G-A bundle over X which satisfies Bierstone's condition is G-A locally trivial.*

Proof. Suppose U_x is a G_x invariant neighborhood of x such that $p^{-1}(U_x)$ is G_x-A equivalent to $U_x \times F$. Then U_x contains a slice V_x through x and there is a G_x-A equivalence ϕ_0: $V_x \times F \to p^{-1}(V_x)$. The map

$$\phi\colon G \times_H (V_x \times F) \to p^{-1}(GV_x), \quad \phi[g, (v, y)] = g\phi_0(v, y)$$

is then a G-A equivalence over the homeomorphism $G \times_H V_x \to GV_x$, $H = G_x$. Since X is covered by $\{GV_x\}_{x \in X}$, p is G-A locally trivial.

Of course, a G-A bundle satisfies Bierstone's condition if and only if the associated principal bundle does. For principal G-A bundles we have:

LEMMA 1.4. *A principal G-A bundle satisfies Bierstone's condition if and only if for each $x \in X$, $p\colon E \to X$ has a local G_x section where G_x acts on E by $z \to gz\rho_x(g)^{-1}$. ($\rho_x\colon G_x \to A$ is the homomorphism induced by the action of G_x on the fibre over x. This is unique up to conjugation by $a \in A$.)*

Proof. (a) If U_x exists with $p^{-1}(U_x)$ G_x-A equivalent to $U_x \times A$ with action

$$g(u, a) = (gu, \rho_x(g)a),$$

then $s\colon U_x \to U_x \times A$, $s(u) = (u, 1)$ satisfies $s(gu) = gs(u)\rho_x(g)^{-1}$.

(b) If U_x exists with $s\colon U_x \to p^{-1}(U_x)$ satisfying

$$s(gu) = gs(u)\rho_x(g)^{-1}, \quad g \in G_x,$$

define $\phi\colon U_x \times A \to p^{-1}(U_x)$ by $\phi(u, a) = s(u)a$. Then ϕ is a G_x-A equivalence where G_x acts by $g(u, a) = (gu, \rho_x(g)a)$.

COROLLARY 1.5. *A G-A bundle with A a compact Lie group and base space X completely regular is G-A locally trivial.*

Proof. It is sufficient to prove this for a principal bundle $p\colon E \to X$. For any $x \in X$, let U_x be a G_x-invariant neighborhood such that $p^{-1}(U_x)$ is A equivalent to $U_x \times A$. If $z_0 \in p^{-1}(x)$, then $gz_0 = z_0\rho_x(g)$, $g \in G_x$, $\rho_x\colon G_x \to A$ a homomorphism. Consider the action of $G_x \times A$ on $p^{-1}(U_x)$, $(g, a)z = gza^{-1}$. The isotropy group at z_0 is $\{(g, \rho_x(g))\}_{g \in G_x}$, and the orbit through z_0 is $p^{-1}(x)$. Let V_{z_0} be a slice. (Note that U_x is completely regular and hence $p^{-1}(U_x) = U_x \times A$ is completely regular.) Thus a tube about $p^{-1}(x)$ is $(G_x \times A) \times_{G_x} V_{z_0}$, where G_x acts on V_{z_0} by $v \to gv\rho_x(g)^{-1}$ and on $(G_x \times A)$ by right translation on G_x and right translation via ρ_x on A. Thus,

$$(G_x \times A) \times_{G_x} V_{z_0} \cong A \times V_{z_0}$$

by

$$[(g, a), v] \leftrightarrow (a\rho_x(g)^{-1}, gv\rho_x(g)^{-1}).$$

Note that left multiplication by a^{-1} in $A \times V_{z_0}$ corresponds to right multiplication by a in $p^{-1}(U_x)$. Thus $p \mid V_{z_0}$ is a G_x homeomorphism with respect to the action $v \to gv\rho_x(g)^{-1}$, and $(p \mid V_{z_0})^{-1}$ is the desired local G_x cross-section. Hence p satisfies the Bierstone condition and since X is completely regular, p is G-A locally trivial.

A locally trivial bundle $p: E \to X$ is *smooth* if p is a smooth map of smooth manifolds, the fibre F is a smooth manifold and the local trivializations $U \times F \to p^{-1}(U)$ are diffeomorphisms.

If A is a Lie group which acts smoothly and effectively on F, then any locally trivial A-bundle with fibre F over a smooth manifold admits the structure of a smooth bundle such that the smooth local trivializations are A-admissible. The smooth bundle structure is unique up to A-isotopy. A smooth bundle with a smooth trivializing A-atlas will be called a *smooth A-bundle*. Any smooth A-bundle is associated to a smooth principal A-bundle.

A *smooth equivariant bundle* is a smooth bundle on which G acts smoothly by bundle maps. A *smooth G-A bundle* is a smooth A-bundle on which G acts smoothly by A-bundle maps. Bierstone [1] shows that the action of G on the associated principal smooth A-bundle is smooth; so that every smooth G-A bundle has an associated principal smooth G-A bundle.

COROLLARY 1.6 (Bierstone). *A smooth G-A bundle satisfies Bierstone's condition (and hence is G-A locally trivial).*

Proof. It is sufficient to consider a smooth principal G-A bundle $p: E \to X$. For any $x \in X$, let U_x be a G_x invariant neighborhood such that $p^{-1}(U_x)$ is smoothly A-equivalent to $U_x \times A$. If $z_0 \in p^{-1}(x)$, then let $\rho_x: G_x \to A$ be the homomorphism such that $gz_0 = z_0\rho_x(g)$. Choose a G_x invariant Riemannian metric on $p^{-1}(U_x)$ under the action $z \to gz\rho_x(g)^{-1}$. If D is a sufficiently small normal disc to $p^{-1}(x)$ through z_0, then $p \mid D$ is a G_x diffeomorphism and $(p \mid D)^{-1}$ is the desired local G_x cross-section. Hence p satisfies the Bierstone condition. Since a manifold is completely regular, p is G-A locally trivial.

Remark. Since $(p \mid D)^{-1}$ is a smooth G_x section, p satisfies a smooth Bierstone condition; i.e., the G_x-A equivalence of $p^{-1}(U_x)$ with $U_x \times F$ is a diffeomorphism.

LEMMA 1.7. *A principal G-A bundle $p: P \to X$ reduces to a G-B bundle, B a closed subgroup of A such that A/B has local cross-section in A, if and only if the associated bundle P/B with fibre A/B has an equivariant cross-section.*

Proof. Let $\lambda: P \to P/B$ be the quotient map. If $s: X \to P/B$ is an equivariant section, then $Q = \lambda^{-1}s(X)$ is a G-B invariant subspace of P. That $p \mid Q = q: Q \to X$ is a locally trivial B-bundle follows from the local A-triviality of P and the condition on A/B. But P is G-A equivalent to $Q \times_B A$.

Conversely, if P is G-A equivalent to $Q \times_B A$ for some G-B bundle Q, then P/B contains Q/B and the induced projection $\bar{p}\colon P/B \to X$ when restricted to Q/B is a G-equivalence and $s = (\bar{p} \mid Q/B)^{-1}$ is the G cross-section.

DEFINITION. If B is a closed subgroup of A, A/B will be called *A-B locally trivial* if for each compact Lie group $H \subset A$, $\lambda\colon A \to A/B$ is an H-B locally trivial bundle.

Example. If A is any Lie group, then $\lambda\colon A \to A/B$ is a smooth H-B bundle for any compact Lie group $H \subset A$, and hence H-B locally trivial (1.6).

The following is obvious:

LEMMA 1.8. *Let $\rho\colon B \to A$ be a homomorphism. If $q\colon Q \to X$ is a G-B locally trivial bundle, then the associated G-A bundle $P = Q \times_B A$ over X is G-A locally trivial.*

We also have the following partial converse:

PROPOSITION 1.9. *Let $p\colon P \to X$ be a (principal) G-A locally trivial bundle and suppose p reduces to a G-B bundle $q\colon Q \to X$, B a closed subgroup of A such that A/B is A-B locally trivial. Then q is G-B locally trivial.*

Proof. Locally P is equivalent to $G \times_H (V \times A)$ for some H-slice V in X and homomorphism $\rho\colon H \to A$. The reduction of P to a G-B bundle gives an equivariant section of P/B and hence a G-section of $G \times_H (V \times A/B)$. This last is equivalent to an H-map $f\colon V \to A/B$ with H acting on A/B via ρ. By assumption $A \to A/B$ is $\rho(H)$-B and hence H-B locally trivial. Let W be a K slice in A/B such that the preimage of $H \times_K W$ in A is $H \times_K (W \times B)$, K acting on B via some homomorphism $\sigma\colon K \to B$. Then, over

$$G \times_H (H \times_K f^{-1}(W)) = G \times_K f^{-1}(W),$$

Q is equivalent to $G \times_K (f^{-1}(W) \times B)$ since, with s the above section,

$$q^{-1}f^{-1}(W) = \lambda^{-1} s f^{-1}(W) = \{(v, w, b) \in f^{-1}(W) \times W \times B \mid f(v) = w\},$$

where $\lambda\colon P \to P/B$ is the quotient map and K acts on B via σ. Thus q is G-B locally trivial.

PROPOSITION 1.10. (Wasserman-Segal). *Any G-L_n bundle over a completely regular X is G-L_n locally trivial.*

Proof. It is sufficient to prove Bierstone's condition for the associated G-vector bundle $p\colon E \to X$. Let $x \in X$ and let

$$\bar{\phi}\colon \bar{U}_x \times R^n \to p^{-1}(\bar{U}_x)$$

be a local trivialization. We can assume $\bar{U}_x$ is G_x-invariant. Define

$$\rho_x\colon G_x \to L_n$$

by

$$\rho_x(h)y = \bar{\phi}_x^{-1}h\bar{\phi}_x(y), \quad h \in G_x,\ y \in R^n.$$

Now let ϕ: $\bar{U}_x \times R^n \to p^{-1}(\bar{U}_x)$ be the map obtained by averaging over G_x; i.e.,

$$\phi_u y = \bar{\phi}_u \int_{G_x} \bar{\phi}_u^{-1}h^{-1}\bar{\phi}_{hu}\rho_x(h(y))\,dh, \quad u \in \bar{U}_x.$$

Then ϕ_u is linear, $\phi_x = \bar{\phi}_x$ is an isomorphism and $h\phi(u, y) = \phi(hu, \rho_x(h)y)$, $h \in G_x$. Now for u in some smaller G_x invariant neighborhood $U_x \subset \bar{U}_x$, ϕ_u will still be an isomorphism. Thus $\phi | U_x \times R^n$: $U_x \times R^n \to p^{-1}(U_x)$ is a G_x-L_n equivalence and p is G-L_n locally trivial.

By (1.9) and (1.10) we have:

PROPOSITION 1.11. *If A is a closed subgroup of some general linear group L_n, then any G-A bundle over a completely regular space X is G-A locally trivial.*

DEFINITION. If X is a G-space, an open cover $\{U_\alpha\}_{\alpha \in I}$ will be called an *open G-cover* if each U_α is G-invariant. An open G-cover will be called *numerable* if there is a subordinate partition of unity $\{\lambda_\alpha\}_{\alpha \in I}$ such that each λ_α is G-invariant.

LEMMA 1.12. *If X is a paracompact G-space, then every open G cover has a numerable refinement.*

Proof. If X is paracompact so is $X | G$. Further, the quotient map q: $X \to X | G$ is open. Hence, if $\{U_\alpha\}$ is an open G cover of X, $\{qU_\alpha\}$ is an open cover of $X|G$ and has a locally finite open refinement $\{\bar{V}_\beta\}$. Then $\{V_\beta\}$, $V_\beta = q^{-1}\bar{V}_\beta$, is an open G cover refining $\{U_\alpha\}$. If $\{\bar{\lambda}_\beta\}$ is a partition of unity subordinate to $\{\bar{V}_\beta\}$, $\lambda_\beta = \bar{\lambda}_\beta \circ q$ is G-invariant and subordinate to $\{V_\beta\}$. Hence $\{V_\beta\}$ is numerable.

DEFINITION. A G-A bundle p: $E \to X$ will be called *numerable* if X has a trivializing numerable G-cover $\{GV_\alpha\}_{\alpha \in I}$. That is, $G \times_{H_\alpha} V_\alpha \to GV_\alpha$ is a G-homeomorphism onto an open set and μ is covered by a G-A bundle equivalence ϕ: $G \times_{H_\alpha} (V_\alpha \times F) \to p^{-1}(GV_\alpha)$ where H_α acts on F via a homomorphism ρ_α: $H_\alpha \to A$.

COROLLARY 1.13. *Every G-A locally trivial bundle over a paracompact X is numerable.*

2. Classification of equivariant bundles

We follow the methods of Dold [3]:

An equivariant map p: $E \to X$ of G-spaces has the *equivariant section extension property* (ESEP) if for any closed invariant subspace $X_0 \subset X$ with invariant halo W_0 and equivariant section s_0: $W_0 \to p^{-1}(W_0)$ there is an equivariant section s: $X \to E$ such that $s | X_0 = s_0 | X_0$. (Note: W_0 is an invariant halo of X_0

if there exists an invariant map $v: X \to [0, 1]$ such that $v^{-1}(1) = X_0$ and $v^{-1}(0, 1] = W_0$.)

LEMMA 2.1. *Let V and F be H-spaces and give $V \times F$ the diagonal H-action. Let $q: V \times F \to V$ be projection onto the first factor. If F is H-contractible, q satisfies the ESEP.*

Proof. Equivariant sections of q are equivalent to equivariant maps of V into F. Let V_0 be a closed invariant subspace of V with invariant halo W_0 and equivariant map $f_0: W_0 \to F$. We need to show there is an equivariant map $f: V \to F$ such that $f \mid V_0 = f_0 \mid V_0$.

Let $v: V \to [0, 1]$ be an invariant map such that $v^{-1}(1) = V_0$ and $v^{-1}(0, 1] = W_0$. Let $c: F \times I \to F$ be an H-contraction of F to $y_0 \in F$; i.e., $c(y, 0) = y$ and $c(y, 1) = y_0$. Define f by

$$f(x) = c(f_0(x), 1 - v(x)) \quad \text{for } x \in v^{-1}(0, 1]$$

and $f(x) = y_0$, elsewhere. Then f is a continuous equivariant function and $f = f_0$ on $v^{-1}(1) = V_0$.

COROLLARY 2.2. *Let $p: G \times_H (V \times F) \to G \times_H V$ be projection. Then p satisfies the ESEP (with respect to G).*

Proof. If X_0 is a closed invariant subspace of $G \times_H V$, then $X_0 = G \times_H V_0$ where $V_0 = X_0 \cap V$ is an H-invariant subspace of V. If W_0 is a G-invariant halo of X_0, $W_0 = G \times_H U_0$ where $U_0 = W_0 \cap V$ is an H-invariant halo of V_0 in V. Now G-sections of p are determined by H-sections of $q = p \mid V \times F$ and conversely. Hence (2.2) follows from (2.1).

Just as in the G-trivial case, one easily gets from the local ESEP the global result:

PROPOSITION 2.3. *Let $p: E \to X$ be an equivariant bundle. If for some numerable open G-cover $\{U_\alpha\}_{\alpha \in I}$, $E \mid U_\alpha$ has the ESEP, then E has the ESEP.*

COROLLARY 2.4. *Let $p: E \to X$ be a numerable G-A bundle with fibre F. If F is equivariantly contractible for compact Lie groups in A, then p has the ESEP and any two equivariant sections which agree over a halo of a closed invariant subspace $X_0 \subset X$, are equivariantly homotopic* rel X_0.

Proof. The first statement is immediate from (2.2) and (2.3). The second follows by applying (2.3) to $E \times I$ over $X \times I$.

Propositions (2.5) and (2.6) below are simple extensions of results of Bierstone [1]:

PROPOSITION 2.5. *Let B be a closed subgroup of the Lie group A. If A/B is equivariantly contractible for compact subgroups of A, there is a bijection between equivalence classes of numerable G-A bundles and numerable G-B bundles.*

Proof. By (1.7) and (2.4) every G-A bundle comes from a G-B bundle. Reductions of equivalent G-A bundles give equivalent G-B bundles as can be seen by applying (1.7) and (2.4) to bundles over $X \times I$. Conversely, G-A bundles associated to equivalent G-B bundles are G-A equivalent. By the argument of (1.7) this sets up the desired bijection.

By (1.9), G-locally trivial bundles correspond to G-locally trivial bundles. Further, the G-A bundle associated to a numerable G-B bundle is clearly numerable. Conversely, using the fact that for any compact $H \subset G$ and $\rho\colon H \to A$, the bundle $A \to A/B$ is H-B numerable, we see that if $p\colon P \to X$ is G-A numerable any reduction Q of P to a G-B bundle will be G-B numerable by the method of constructing the G-B local trivialization in (1.9). (That is, as $H \times_K W$ runs over a numerable K-B trivializing cover of A/B,

$$G \times_K f^{-1}(W) = G \times_H (H \times_K f^{-1}(W))$$

runs over a numerable trivializing cover of $G \times_H V$.)

Remark. Instead of assuming A is a Lie group in (2.5), it is sufficient to assume $A \to A/B$ is H-B numerable for every compact Lie subgroup of A.

PROPOSITION 2.6. *If B is a maximal compact subgroup of a connected semi-simple Lie group A, then A/B is equivariantly contractible for compact subgroups of A. Hence by* (2.5) *there is a bijective correspondence between equivalence classes of numerable G-A bundles and numerable G-B bundles over a G-space X.*

Proof. We follow an argument of Bierstone [1]. For any A-invariant metric on A/B, A/B is a complete 1-connected manifold with negative curvature. Hence $\exp_{aB}\colon T_{aB}(A/B) \to A/B$ is a diffeomorphism for each $a \in A$. If $K \subset A$ is a compact subgroup, $K \subset aBa^{-1}$ for some a, since all maximal compact subgroups are conjugate. But then aB is a fixed point of A/B under K, $T_{aB}(A/B)$ is an Euclidean K- space and $\exp_{aB}$ is K-equivariant. Since a Euclidean K-space is K-contractible, so is A/B.

COROLLARY 2.7. *There is a bijective correspondence between equivalence classes of numerable G-L_n bundles and G-O_n bundles.*

Proof. L_n^+ is a connected semi-simple group and SO_n its maximal compact subgroup. Since $L_n/O_n = L_n^+/SO_n$ and O_n is a maximal compact subgroup of L_n and any two such are conjugate, the argument of (2.6) applies.

In order to prove the equivariant covering homotopy property (ECHP), just as in the non-equivariant case, one must prove that a numerable G-A bundle E over $X \times I$ is equivalent to $E_0 \times I$, $E_0 = E|(X \times (0))$.[1] Following Husemoller [4], it is sufficient to find a numerable open G-cover $\{W_\alpha\}$ of X such that $E|(W_\alpha \times I) \simeq (E|W_\alpha) \times I$. Bierstone [1] has proved this in the case that X is paracompact. In fact, in the paracompact case by (1.1) one can find for each $x \in X$, $t \in I$ a G_x invariant neighborhood U_x of x in X and an $\varepsilon > 0$ such that

$$E \,|\, (U_x \times [t - \varepsilon, t + \varepsilon])$$

is G_x equivalent to $(U_x \times F) \times [t - \varepsilon, t + \varepsilon]$. Since a finite number,

$$U_x^i \times [t_i - \varepsilon_i, t_i + \varepsilon_i], \quad i = 1, 2, \ldots, r,$$

cover $(x) \times I$, $U_x^0 = \bigcap_{i=1}^r U_x^i$ is a G_x invariant neighborhood of x in X such that $E \,|\, (U_x^0 \times I)$ is G_x equivalent to $(E \,|\, U_x^0) \times I$. If V_x is a slice through x in U_x^0, then $E \,|\, (GV_x \times I) \simeq (E \,|\, GV_x) \times I$. Taking $\{W_\alpha\}$ to be a numerable open G refinement of $\{GV_x\}$ we have $E \,|\, (W_\alpha \times I) = (E \,|\, W_\alpha) \times I$.

For a general X, one may again follow Husemoller to find a numerable open G-cover $\{W_\alpha\}$ of X such that

$$W_\alpha \times [(q - 1)/n(\alpha), q/n(\alpha)], \quad q = 1, 2, \ldots, n(\alpha),$$

refines the given trivializing numerable open G-cover $\{GV_\beta\}$ of $X \times I$. Since the restriction of a trivial G-A bundle to a G-invariant subspace is again trivial, the result for the general case will follow from the lemma below and its corollary applied to

$$E \,|\, (W_\alpha \times [(q - 1)/n(\alpha), q/n(\alpha)].$$

LEMMA 2.8. *Let X be a G-space and V an H-space and suppose we have a G-homeomorphism ϕ of $X \times I$ with $G \times_H V$. Let*

$$V_0 = \phi^{-1}(V) \cap (X \times (0)),$$

so that we may write $X = G \times_H V_0$. Then there is

(a) *an H-homeomorphism ψ of $V_0 \times I$ with V, and*

(b) *an H-map $\theta\colon V_0 \times I \to G$, where H acts on G via $g \to hgh^{-1}$, $g \in G$, $h \in H$,*

such that

$$\phi([g, v_0], t) = [g\theta(v_0, t), \psi(v_0, t)], \; [g, v_0] \in G \times_H V_0 = X.$$

COROLLARY 2.9. *If E is a trivial G-A bundle over $X \times I$, then*

$$E \simeq (E \,|\, X) \times I.$$

Proof of Corollary 2.9. By definition of G-trivial, there is an H-slice V in $X \times I$ for some closed $H \subset G$, such that $X \times I$ is G-homeomorphic with $G \times_H V$ and there is a G-A bundle equivalence of E with $G \times_H (V \times F)$ covering this G-homeomorphism ϕ. With $X = G \times_H V_0$ as above, $E \,|\, X$ is then equivalent to $G \times_H (V_0 \times F)$, and we need to define a G-A bundle equivalence

$$\hat{\phi}\colon G \times_H (V_0 \times F) \times I \to G \times_H (V \times F)$$

covering $\phi\colon (G \times_H V_0) \times I \to G \times_H V$.

Define $\hat{\phi}$ by

$$\hat{\phi}([g, (v_0, y)], t) = [g\theta(v_0, t), (\psi(v_0, t), y)].$$

Before proving the lemma we note the following: Let $q: G \to G/H$ be the quotient map. Let H act on G via $g \to hgh^{-1}$ and G/H by $gH \to hgH$; then q is an H-map. Indeed, q may be regarded as the associated bundle with fibre H to the principal H-$H \times H$ bundle

$$p: G \times H \to G/H, \quad p(g, h) = gH$$

(with left H-action given by $h_1(g, h) = (h_1 g, h_1 h)$ and right $H \times H$ action $(g, h)(h_1, h_2) = (gh_1, hh_2)$) where $H \times H$ acts on H by

$$(h_1, h_2)h = h_1 h h_2^{-1}.$$

The equivalence of $(G \times H) \times_{H \times H} H$ with G is given by

$$[(g, h_1), h_2] \to gh_2 h_1^{-1}.$$

Thus $q: G \to G/H$ is H-locally trivial and satisfies the ECHP for paracompact H-spaces; in particular, for the contraction of the space of paths P in G/H beginning at eH (with the C-O = metric topology). That is, if $\pi: P \to G/H$ is the endpoint map, there is an H-map $\hat{\pi}: P \to G$ such that $q\hat{\pi} = \pi$.

Proof of 2.8. Let $\lambda: G \times_H V \to G/H$ be projection. Then

$$\lambda\phi: (G \times_H V_0) \times I \to G/H$$

is a G-map such that $\lambda\phi(V_0) = eH$. Define an H-map $\phi_\#: V_0 \times I \to P$ by

$$\phi_\#(v_0, s)(t) = \lambda\phi(v_0, st), \quad s, t \in I.$$

Then $\pi\phi_\# = \lambda\phi$ and $\hat{\pi}\phi_\#: V_0 \times I \to G$ satisfies $q\hat{\pi}\phi_\# = \lambda\phi$. Let $\theta = \hat{\pi}\phi_\#$ and define ψ by $\psi(v_0, t) = \theta(v_0, t)^{-1}\phi(v_0, t)$. Then (1)

$$\psi(v_0, t) \in V,$$

since

$$\begin{aligned} \lambda\psi(v_0, t) &= \theta(v_0, t)^{-1}\lambda\phi(v_0, t) \\ &= \theta(v_0, t)^{-1}q\hat{\pi}\phi_\#(v_0, t) \\ &= \theta(v_0, t)^{-1}q\theta(v_0, t) \\ &= q(\theta(v_0, t)^{-1}\theta(v_0, t)) = eH; \end{aligned}$$

and (2)

$$\phi([g, v_0], t) = [g\theta(v_0, t), \psi(v_0, t)],$$

by equivariance. It remains to show that ψ is a homeomorphism:

(a) ψ is injective.

If $\psi(v_0, t) = \psi(v'_0, t')$, then

$$\theta(v_0, t)^{-1}\phi(v_0, t) = \theta(v'_0, t')^{-1}\phi(v'_0, t').$$

Taking ϕ^{-1} of both sides yields $\theta(v_0, t)^{-1}(v_0, t) = \theta(v'_0, t')^{-1}(v'_0, t')$. But then there is an $h \in H$ such that $\theta(v_0, t) = h\theta(v'_0, t')$ and $(v_0, t) = (hv'_0, t')$. Thus $t = t'$, $v_0 = hv'_0$ and

$$\theta(v_0, t) = h\theta(h^{-1}v_0, t) = hh^{-1}\theta(v_0, t)h = \theta(v_0, t)h.$$

Hence $h = e$ and $v_0 = v'_0$.

(b) ψ is surjective.

If $v \in V$,

$$v = [g\theta(v_0, t), \psi(v_0, t)] \quad \text{for some } g, v_0, t.$$

But then $g\theta(v_0, t) = h \in H$ and $v = h\psi(v_0, t) = \psi(hv_0, t)$.

(c) ψ^{-1} is continuous:

Let λ': $(G \times_H V_0) \times I \to G/H$ be the projection. Then

$$\lambda'\phi^{-1}: V \to G/H.$$

If $v \in V$, there is a local cross-section s of q: $G \to G/H$ defined on a neighborhood of $\lambda'\phi^{-1}(v)$. For v' near v, let

$$\gamma(v') = s\lambda'\phi^{-1}(v').$$

Then $\tau(v') = \gamma(v')^{-1}\phi^{-1}(v') \in V_0 \times I$ and $\phi^{-1}(v') = [\gamma(v'), \tau(v')]$. Hence

$$v' = \phi \circ \phi^{-1}(v') = \phi[\gamma(v'), \tau(v')] = [\gamma(v')\theta(\tau(v')), \psi(\tau(v'))].$$

So $\gamma(v')\theta(\tau(v')) \in H$ and $v' = \psi(\gamma(v')\theta(\tau v')\tau(v'))$. Thus

$$\psi^{-1}(v') = \gamma(v')\theta(\tau v')\tau(v')$$

is defined and continuous for v' near v. Thus ψ^{-1} is continuous.

We have proved:

THEOREM 2.10. *A numerable G-A bundle E over $X \times I$ (trivial G-action on I) is G-A equivalent to $(E|X) \times I$, where by $E|X$ we mean $E|(X \times (0))$.*

COROLLARY 2.11. *Let $p: E \to Y$ be a numerable G-A bundle and let $f: X \times I \to Y$ be an equivariant homotopy between the G-maps $f_0, f_1: X \to Y$. Then $f_0^* E$ and $f_1^* E$ are G-A equivalent.*

COROLLARY 2.12. *A numerable G-A bundle satisfies the ECHP.*

Just as in the non-equivariant case, there are at least two possible definitions of G-A universal bundle (which turn out to be equivalent):

A *universal G-A* bundle is a numerable principal G-A bundle $\pi: E \to B$ such that for any G-space X, the equivalence classes of numerable G-A bundles over X are in bijective correspondence with $[X, B]_G$, the equivariant homotopy classes of equivariant maps of X into B; the correspondence being given by induced bundles.

A *strongly universal G-A* bundle is a numerable principal G-A bundle $\pi: E \to B$ which satisfies: Let $p: P \to X$ be a numerable principal G-A bundle and let $X_0 \subset X$ be a closed invariant subspace with invariant halo W_0. If $\phi_0: P\,|\,W_0 \to E$ is a G-A bundle map, then there is a G-A bundle map $\phi: P \to E$ such that ϕ agrees with ϕ_0 on $P\,|\,X_0$.

It is trivial to see that:

LEMMA 2.13. *Strongly universal implies universal.*

THEOREM 2.14. *A numerable principal G-A bundle $p: P \to X$ is strongly universal if and only if for each closed $H \subset G$ and homomorphism $\rho: H \to A$, P is H-contractible to a point under the action $z \to hz\rho(h)^{-1}$, $z \in P$, $h \in H$.*

Proof. (a) Suppose $p: P \to X$ is strongly universal.

First consider the G-A bundle $\pi: G \times_H A \to G/H$, H acting on A through $\rho: H \to A$. Since p is universal there is a G-A bundle map $\phi: G \times_H A \to P$. Let $e_1 \in G$, $e_2 \in A$ be the unit elements. Then

$$h\phi[e_1, e_2] = \phi[h, e_2] = \phi[e_1, \rho(h)] = (\phi[e_1, e_2]\rho(h).$$

Thus $\phi[e_1, e_2]$ is a fixed point of the H-action on P described in the statement of the theorem.

Now consider the G-A bundle $\pi: G \times_H (P \times A) \to G \times_H P$, H acting on P as described and on A through ρ. Define the G-A bundle maps

$$\lambda_i: G \times_H (P \times A) \to P, \quad i = 0, 1,$$

by

$$\lambda_0[g, z, a] = gza \quad \text{and} \quad \lambda_1[g, z, a] = gz_0 a,$$

z_0 any fixed point of P under the given action. Since P is strongly universal, there is a G-A homotopy $\lambda_t: G \times_H (P \times A) \to P$, $0 \le t \le 1$, between λ_0 and λ_1. In particular, $z = \lambda_0[e_1, z, e_2]$ is deformed to

$$\lambda_1[e_1, z, e_2] = z_0.$$

But

$$\lambda_t[e_1, hz\rho(h)^{-1}, e_2] = \lambda_t[h, z, \rho(h)^{-1}] = h(\lambda_t[e_1, z, e_2])\rho(h)^{-1}.$$

Hence, λ_t defines an H-contraction of P to z_0.

For the converse we shall need the following result.

LEMMA 2.15. *Let $p: P \to X$ and $p': P' \to X'$ be G-A bundles. There is a bijective correspondence between G-A bundle maps of P into P' and equivariant sections of the associated bundle $P \times_A P'$ to P with fibre P' and G-action $g[z, z'] = [gz, gz']$.*

The lemma follows trivially from the non-equivariant case.

(b) Suppose for each $H \subset G$ and $\rho: H \to A$, P is H-contractible.

Let $q: Q \to Y$ be any G-A bundle, Y_0 a closed invariant subspace of Y, and $\phi_0: Q | W_0 \to P$ be a G-A bundle map, W_0 an invariant halo of Y_0. Now ϕ_0 corresponds to an equivariant section s_0 of $Q \times_A P$ defined on $W_0 \subset Y$, and it is sufficient to define an equivariant section s of $Q \times_A P$ such that $s | Y_0 = s_0 | Y_0$. But this will follow if the ESEP is true locally; that is, over each trivializing open set $G \times_H V$ of a numerable cover for Y. But if

$$Q | G \times_H V = G \times_H (V \times A) \quad \text{for some } \rho: H \to A,$$

then

$$Q \times_A P | G \times_H V = G \times_H (V \times P),$$

H acting on P through $z \to z\rho(h)^{-1}$. But a G-section

$$s: G \times_H V \to G \times_H (V \times P)$$

with G acting on $[g, (y, z)]$ by $g_1[g, (y, z)] = [g_1 g,(y, g_1 z)]$ is equivalent to an H-section of $V \times P$ with H acting on P via $z \to hz\rho(h)^{-1}$. Since P is H-contractible, the local ESEP holds, and hence the desired global section exists and defines the desired bundle map ϕ with $\phi = \phi_0$ on $Q | Y_0$.

The above characterization of strongly universal bundle implies:

LEMMA 2.16. *If a strongly universal G-A bundle exists, then every universal G-A bundle is strongly universal.*

Existence of universal G-A bundles. There are as many ways to construct universal bundles in the equivariant as in the non-equivariant case. One may use Steenrod's approach via Stiefel manifolds (see [9]) or one may use the geometric bar construction (see [10]). Perhaps the most general is Milnor's infinite join construction first generalized to the equivariant case in [6]:

Choose a representative H from each conjugacy class of closed subgroups of G and a representative $\rho: H \to A$ from each A-equivalence class of homomorphisms. Let $\{\rho_\beta\}_{\beta \in I}$ be this set. Let $E_\beta = G \times_{H_\beta} A$, where H_β acts on A through ρ_β. Let $E = \coprod_{\beta \in I} E_\beta$. Then E is a numerable G-A bundle. Finally, let $P = \bigstar_{i=1}^{\infty} P_i$, where each $P_i = E$. Then, following Husemoller [4], see [6], it is easy to prove that P is a *strongly universal G-A bundle.* (We shall not repeat the proof here.)

Our final theorem gives information on the universal base space for G-A bundles. To state the result we need some notation. Let $H \subset G$ be a closed subgroup and $\rho\colon H \to A$ a homomorphism. Let A^ρ be the centralizer of $\rho(H)$ in A; i.e.,

$$A^\rho = \{a \in A \mid \rho(h)a\rho(h)^{-1} = a,\ h \in H\}.$$

Then A^ρ is a closed subgroup of A and we let BA^ρ denote its universal base space. (Note that if ρ' is A-equivalent to ρ, then $A^{\rho'}$ is conjugate to A^ρ and $BA^{\rho'}$ may be identified with BA^ρ.) Let R_H be a collection of homomorphisms of H in A containing exactly one representative from each A-equivalence class. Then from (2.14); we have:

THEOREM 2.17. *Let $\pi\colon E \to B$ be a universal G-A bundle and $H \subset G$ a closed subgroup. Then B^H, the fixed point set of B under H, is the disjoint union of the BA^ρ, $\rho \in R_H$. If $K \subset H$, $B^H \subset B^K$ corresponds to the maps $BA^\rho \to BA^{\rho|K}$ induced by $A^\rho \subset A^{\rho|K}$.*

Proof. Let

$$E = [z \in E \,|\, hz\rho(h)^{-1} = z,\ h \in H].$$

Then E^ρ is contractable by (2.14). If $z \in E^\rho$ and $a \in A$, then $za \in E^{\rho'}$, ρ' A-equivalent to ρ; and $za \in E^\rho$ if and only if $a \in A^\rho$. Thus

$$\pi(E^\rho) \cap \pi(E^{\rho'}) = \phi$$

if ρ' is not A equivalent to ρ. Finally, E^ρ is a locally trivial A^ρ bundle over an open subspace of B^H; since if $x \in \pi(E^\rho)$, $x \in B^H$, and if U_x is a G_x invariant neighborhood such that $\pi^{-1}(U_x)$ is G_x-A equivalent to $U_x \times A$ with G_x acting via $\sigma\colon G_x \to A$, we can assume $\sigma|H = \rho$. Thus $E^\rho \cap \pi^{-1}(U_x)$ is A^ρ equivalent to $U_x^H \times A^\rho$.

Added in Proof. The paper [11] (of which Bierstone was also unaware) was recently pointed out to the author. In it, tom Dieck gives essentially the same definition of numerable bundles and uses the join construction for universal equivariant bundles. He also proves the equivariant covering homotopy property for σ-compact spaces under the assumption that the structure group of the bundle is a compact Lie group.

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