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LAPLACIANS AND RIEMANNIAN SUBMERSIONS WITH TOTALLY GEODESIC FIBRES

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0. Introduction

0.0 The Laplace-Beltrami operator on a compact Riemannian manifold (M, g) can be viewed as a natural generalization of the ordinary Laplacian on a bounded domain in $\mathbf{R}^n$. In particular, most properties of eigenvalues and eigenfunctions of the latter with either Dirichlet or Neuman boundary values carry out to the spectral data of the former.

As an example, the famous Faber-Krahn inequality (see [10], page 188) which bounds from below the product of the area of a domain in $\mathbf{R}^2$ by its first eigenvalue finds its counterpart in J. Hersch's theorem (cf. [14]) which gives an upper bound of the product of the first nonzero eigenvalue of any Riemannian metric on the 2-sphere S^2 by the Riemannian area. The Faber-Krahn inequality generalizes to domains in $\mathbf{R}^n$. In [2], M. Berger shows that such an extension does not exist for the n-sphere S^n, if one insists on the upper bound being sharp for the standard metric. This still leaves some hope for the existence of an upper bound.

Analogously, Courant's nodal line theorem (see [10], page 452) according to which the ith eigenfunction has at most i nodal domains (i.e., connected components of the complement of its zero set) is valid for a general Riemannian manifold. Such a result suggests that eigenfunctions corresponding to high eigenvalues should be complicated. In particular they were expected to have more than two nodal domains.

In the same vein, it is reasonable to believe that the more symmetries a Riemannian metric has, the more multiple its eigenvalues are. In particular, the standard metric on the sphere was expected to be the metric with the largest multiplicity of its first nonzero eigenvalue.

All these guesses turn out to be wrong for appropriate choices of (M, g). The right choices all belong to the class of manifolds to which this article is devoted, namely, *Riemannian submersions with totally geodesic fibres*. These metrics are the next simple metrics after Riemannian products. This probably indicates that, in a sense, Riemannian metrics on compact manifolds form a wider family than domains in $\mathbf{R}^n$.

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Before giving the details, let us mention how we came to this study.

0.1 In [24], *H. Urakawa* studies the spectrum of left-invariant metrics on the 3-sphere S^3. He *shows* in particular *that the first eigenvalue of the Laplace operator of some* U_2*-invariant metric on* S^3 *has multiplicity* 7, whereas for the standard metric this multiplicity is only 4. This surprising result of H. Urakawa was our starting point.

Namely we noticed that the U_2-invariant metrics on S^3 are precisely the so-called *Berger metrics* obtained by varying the lengths of the circles of the Hopf fibration from S^3 to S^2. Since this fibration is the prototype of a Riemannian submersion with totally geodesics fibres, we tried to understand H. Urakawa's result from this point of view. (Another generalization is due to S. Tanno; see [23].)

0.2 In this article we study the Laplace operator acting on functions defined on the total space of a Riemannian submersion. We concentrate on the subfamily of submersions with totally geodesic fibres. This includes most of the classical examples (Hopf fibrations, natural bundles on a Riemannian manifold). *For this family* an interesting phenomenon takes place: *the Laplace operator commutes with* the operator deduced from it by restricting the functions to the fibres, *the* so-called *vertical Laplacian* (this is our Theorem 1.5).

This property explains H. Urakawa's result and has many applications of which we now sketch a few.

0.3 For example we can decompose the eigenvalues of the Laplace operator on the total space in such a way that we can compute them very easily in the *canonical variation* of the metric on the total space (this variation is obtained by changing the relative sizes of the base and of the fibres). This extends the results on S^3 (see our Proposition 7.2). In this way we get many situations where there is no upper bound of the first eigenvalue in terms of the volume (cf. Proposition 7.7). We also obtain non product manifolds (like spheres) on which eigenfunctions corresponding to eigenvalues far away in the spectrum have only two nodal domains (cf. Proposition 7.2) or else whose spectrum can coincide as far as we like with the spectrum of a manifold of lower dimension.

0.4 Surprisingly enough the general setting that we come up with seems to be of some interest in quantum physics.[3] The connection is the following. Configuration spaces of physical systems are sometimes set into correspondence by non-bijective maps. The problem is then to give the "selection rules" which tell you which part of the spectrum of one system persists in the spectrum of the other. This is detailed in Section 4.

0.5 The paper is organized as follows: we prove our main observation in Section 1 where we present our notations and definitions. In Section 2 we

[3] We are indebted to M. Boiteux for pointing out the relevance of our study to this question.

discuss examples of Riemannian submersions. We specify the consequences for eigenvalues and eigenfunctions in Section 3. In Section 4, we show how our observation is related to some physical problems. We introduce the canonical variation of a Riemannian submersion in Section 5. We take up H. Urakawa's example in the Section 6 and generalize some of his considerations on universal inequalities between first eigenvalue and volume or between first eigenvalue and diameter in Section 7. We refine the analysis of the horizontal Laplace operator in Section 8: this leads us to introduce the notion of an infinitesimally transitive holonomy group.

0.6 We are indebted to Marcel Berger for insisting on the importance of eigenvalue properties to Riemannian geometry. We also thank Jerry L. Kazdan for constructive criticism of an earlier version of this paper.

1. The commutation theorem

1.1 Let (M, g) and (B, j) be two complete Riemannian manifolds of respective dimensions n and p. Let π: $M \to B$ be a submersion. The map π is said to be a *Riemannian submersion* from (M, g) to (B, j) if at each point m of M the restriction of $T_m \pi$ to the *horizontal space* $H_m M$ (i.e., the space orthogonal for g_m to the kernel $V_m M$ of $T_m \pi$; one usually calls $V_m M$ the *vertical space*) is an isometry from $(H_m M, g_m \restriction H_m M)$ to $(T_{\pi(m)} B, j_{\pi(m)})$. Since the metrics g and j are part of our data, later we shall refer to the Riemannian manifolds M and B.

We *denote by* Δ^M the Laplacian of M acting on functions.

1.2 Definition. The *vertical Laplacian* Δ_v is the second-order differential operator defined on a C^2 function f on M by

$$(\Delta_v f)(m) = (\Delta^{F_m}(f \restriction F_m))(m)$$

where $F_m = \pi^{-1}(\pi(m))$ is the fibre of π through m and Δ^{F_m} the Laplace operator of the metric induced by M on F_m.

1.3 Definition. The difference operator $\Delta_h = \Delta^M - \Delta_v$ is called *the horizontal Laplacian.*

1.4 Notice that both Δ_h and Δ_v are in general non-elliptic (unless B is a point or π a covering) since they take into account the behaviour of a function only in certain directions. In that respect one may find the term Laplacian misleading.

The main observation on which this paper is based is the following:

1.5 Theorem. *If the fibres of the Riemannian submersion* π: $M \to B$ *are totally geodesic, the operators* Δ^M, Δ_v *and* Δ_h *commute with each other.*

Proof. The proof is based on an appropriate local expression for the operators Δ_v, Δ^M and Δ_h and on a lemma of R. Hermann characterizing Riemannian submersions with totally geodesic fibres by the action of basic vector fields on the total space.

(We recall that a vector field X on M is called *basic* if X is the horizontal lift of a vector field $\overline{X}$ on B. In particular X is π-projectable.)

1.6 LEMMA. *A Riemannian submersion has totally geodesic fibres if and only if the vertical Laplacian commutes with any basic vector field (viewed as an operator on the space of functions on the total space).*

Proof. Let X be a basic vector field and $(\xi_t)_{t \in \mathbf{R}}$ its flow. In [13], it is shown that, if the fibres are totally geodesic, ξ_t maps the fibre F_m *isometrically* into the fibre $F_{\xi_t(m)}$. Therefore

$$\xi_t^*(\Delta^{F_{\xi_t(m)}}) = \Delta^{F_m}.$$

Since Δ_v coincides with Δ^{F_m} along F_m, we get $\xi_t^*(\Delta_v) = \Delta_v$, so that by differentiating at $t = 0$, $[X, \Delta_v] = 0$.

Conversely, if $[X, \Delta_v] = 0$, then X leaves the symbol of Δ_v invariant, i.e., the metric of F_m. So ξ_t is an isometry from F_m into $F_{\xi_t(m)}$ and the proof in [13] is in fact an equivalence, as shown by J. Vilms in [25]. ∎

1.7 *Proof of Theorem 1.5.* Let $(\overline{X}_i)_{i=1,\dots,p}$ be a (local) orthonormal moving frame on B. We denote by X_i the basic vector field associated with $\overline{X}_i$. Then $(X_i)_{i=1,\dots,p}$ is an orthonormal basis of H_mM for each point m in M. Let $(U_j)_{j=1,\dots,n-p}$ be vertical vector fields in M which form an orthonormal basis of VM around m. Then $\{X_i, U_j\}$ is an orthonormal moving frame on M, so that

$$\Delta^M = -\sum_{i=1}^{p} (X_i \circ X_i - D_{X_i}X_i) - \sum_{j=1}^{n-p} (U_j \circ U_j - D_{U_j}U_j)$$

(where D is the Levi-Civita connection of M). B. O'Neill's calculations in [22] show that $D_{U_j}U_j$ is vertical if the fibres are totally geodesic. Also the Levi-Civita connection of a fibre F_m coincides with D restricted to vertical vector fields. Since $D_{X_i}X_i$ is a horizontal vector field, we obtain

$$\Delta_v = -\sum_{j=1}^{n-p} (U_j \circ U_j - D_{U_j}U_j) \tag{1.8}$$

and

$$\Delta_h = -\sum_{i=1}^{p} (X_i \circ X_i - D_{X_i}X_i). \tag{1.9}$$

Moreover $D_{X_i}X_i$ is the basic vector field associated with $\overline{D}_{\overline{X}_i}\overline{X}_i$ where $\overline{D}$ is the Levi-Civita connection of B (see [22]). It follows then from Lemma 2.6 that X_i and $D_{X_i}X_i$ commute with Δ_v. Therefore Δ_h and Δ_v commute with each other. ∎

1.10 *Remarks.* (i) Another geometric condition on the fibres which is of interest is to suppose that the fibres of π: $M \to B$ are *minimal submanifolds* of M. This ensures for example that π intertwines Δ^M and Δ^B (see [26], where it is also shown that π is then a harmonic map). From a topological point of view, this means that the structure group of the bundle (which is a priori the group of diffeomorphisms of the fibre F) reduces to the group of volume preserving diffeomorphisms of F. Such a reduction is always possible since the volume elements on a manifold form a convex, hence contractible, set.

However, this condition is not sufficient to ensure the identity $[\Delta^M, \Delta_v] = 0$ as one easily checks on local examples.

(ii) In order that the fibration π: $M \to B$ be a Riemannian submersion with totally geodesic fibres, the structure group of the bundle must reduce to a finite dimensional Lie group G. This follows directly from R. Hermann's result used in the proof of Lemma 1.6 since G appears as a group of isometries of F. In particular if F is compact, G must be compact.

This is a strong restriction on the bundle π: $M \to B$, but many of the interesting geometric bundles have this property (the Hopf fibrations, the natural bundles deduced from the tangent bundle of a Riemannian manifold with the Sasaki metric; for more examples, see the next section).

The group G appears also as the *holonomy group* of the horizontal distribution given by the Riemannian metric on the total space. It is usually under this name that we shall later refer to G.

(iii) It is important to notice that when F is one-dimensional, the fibres are minimal if and only if they are totally geodesic. Hence the special role played by real line bundles or circle bundles.

1.11 When the fibres are totally geodesic, because of R. Hermann's result, one can say that the fibre F is a Riemannian manifold. We insist here that the metric k on F is given up to a diffeomorphism since there is no preferred way to map a fibre to another one.

1.12 Theorem 1.5 does not generalize for the Hodge-de Rham Laplacians defined on i-forms $(0 < i < n - p)$. Already, to require that π intertwines the Hodge-de Rham Laplacians of M and B forces π: $M \to B$ to have integrable horizontal distribution as is shown in [11].

2. Some examples

2.1 We saw in 1.10 that the total space M and the base B of a fibration π: $M \to B$ can be given Riemannian metrics so that π is a Riemannian submersion with totally geodesic fibres only if the structure group of the bundle reduces to a finite-dimensional Lie group.

Conversely, if π: $M \to B$ is a G-bundle (where G is a Lie group) with fibre F, there exist adapted metrics on M. More precisely, given *a Riemannian metric j* *on B, a G-invariant Riemannian metric k on F and a G-connection θ for π*, there exists a unique Riemannian metric g on M such that π is a Riemannian submersion with totally geodesic fibres isometric to (F, k) and such that the horizontal distribution associated with θ is the orthogonal complement of the vertical distribution (see [25], page 78, for a proof).

2.2 The following describes an interesting family of Riemannian submersions with totally geodesic fibres with compact total space.

Let G be a compact Lie group, H and K two closed subgroups of G with $K \subset H$. The natural coset map π: $G/K \to G/H$ is a fibration with fibre H/K and structure group H. We now construct G-invariant metrics on these spaces as follows. Let $\mathfrak{G}$ be the Lie algebra of G, $\mathfrak{H}$ and $\mathfrak{K}$ the Lie subalgebras of $\mathfrak{G}$ corresponding to the subgroups H and K of G. We choose an H-invariant complement $\mathfrak{B}$ to $\mathfrak{H}$ in $\mathfrak{G}$, a K-invariant complement $\mathfrak{M}$ to $\mathfrak{K}$ in $\mathfrak{H}$. Then $\mathfrak{B} \oplus \mathfrak{M}$ is a K-invariant complement to $\mathfrak{K}$ in $\mathfrak{G}$.

A G-invariant metric j on G/H is given by an H-invariant scalar product ι on $\mathfrak{P}$. An H-invariant metric k on H/K is given by a K-invariant scalar product κ on $\mathfrak{M}$. Then $\gamma = \iota \oplus^{\perp} \kappa$ is a K-invariant scalar product on $\mathfrak{P} \oplus \mathfrak{M}$ which corresponds to a G-invariant metric on G/K. Then π: $(G/K, g) \to (G/H, j)$ is a Riemannian submersion with totally geodesic fibres $(H/K, k)$ (see [1]). Moreover, the O'Neill tensor A of this submersion is given, after suitable identifications, by a map from $\mathfrak{P} \times \mathfrak{P}$ to $\mathfrak{M}$ which is

$$A_X Y = \tfrac{1}{2}[X, Y]_{\mathfrak{M}}.$$

The holonomy group of the fibration here is the subgroup N of H generated by the intersection of $\mathfrak{H}$ with the ideal $\mathfrak{N}$ of $\mathfrak{G}$ spanned by iterated brackets of elements of $\mathfrak{P}$.

If $\mathfrak{N}$ does not contain all of $\mathfrak{M}$, then N is not transitive on H/K and there exists a non constant function on G/K such that $\Delta_h f = 0$.

2.3 The preceding family includes the Hopf fibrations $S^{2q+1} \to \mathbf{C}P^q$ by taking $G = SU(q+1)$, $H = SU(q)$ and $K = S(U(1) \times U(q))$ or $S^{4r+3} \to \mathbf{H}P^r$ (where $\mathbf{H}$ is the field of quaternions) by taking $G = Sp(r+1)$, $H = Sp(r)$ and $K = Sp(r) \cdot Sp(1)$. The spaces can be endowed with their canonical metrics but we will see in Section 5 that some other metrics belonging to the family turn out to be of interest. For the algebra **Caa** of Cayley numbers, $S^{15} \to \mathbf{Caa}P^1 = S^8$ is again of the preceding type (the fibres are 7-spheres), but $\mathbf{Caa}P^2$ is not the base space of a fibration of S^{23} by 7-spheres. (It must be written as $F_4/Spin\,9$ where F_4 is one of the exceptional Lie groups, see [7]).

2.4 Among the preceding family one also finds fibrations of flat tori over flat tori. Consider for example the flat torus $T^2_{\alpha,\beta} = \mathbf{R}^2/\alpha\mathbf{Z} + \beta\mathbf{Z}$ where $\alpha = (0, 1)$ and $\beta = (1, a)$, $0 \leq a < 1$. Then the map $(x, y) \mapsto x$ from $T^2_{\alpha,\beta}$ to

$S^1 = \mathbf{R}/\mathbf{Z}$ is a Riemannian submersion with totally geodesic fibres. Using the obvious coordinates (x, y) on T^2, one has

$$\Delta_v = -\frac{\partial^2}{\partial y^2}, \quad \Delta_h = -\frac{\partial^2}{\partial x^2}, \quad \Delta^M = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}.$$

This very simple example suggested to the authors by Y. Colin de Verdiere will show to be useful in the next section.

2.5 Other examples of Riemannian submersions with totally geodesic fibres include the fibrations associated with the tangent bundles when they are endowed with their Sasaki metrics. (This metric is obtained as described in 2.1 with $G = 0_n$ and the G-connection θ being the Levi-Civita connection. For a proof, see [5], page 46.)

3. On eigenvalues and eigenfunctions

3.1 Now *suppose M to be compact and connected.* Consider a Reimannian submersion $\pi\colon M \to B$ with totally geodesic fibres. We recall that π is then a G-bundle where G is a compact Lie group acting by isometries on the fibre F.

3.2 We *denote by $L^2(M)$* the Hilbert space of (real-valued) L^2-functions on M (with respect to the canonical measure v_g associated with the metric g). Then Δ^M, Δ_v and Δ_h may be considered as self-adjoint unbounded operators on $L^2(M)$. One knows (see for example [3], page 142) that the spectrum of the elliptic operator Δ^M is discrete. We denote by $(\mu_i)_{i\in\mathbf{N}}$ the eigenvalues of Δ^M and by $(m_i)_{i\in\mathbf{N}}$ their (always finite) multiplicities. Then $\mu_0 = 0$, $m_0 = 1$, $\mu_i > 0$ for $i \geq 1$ and $\mu_i \to +\infty$ when $i \to +\infty$.

Also, $L^2(M)$ has a Hilbert basis consisting of eigenfunctions of Δ^M. The eigenfunctions associated with μ_0 are of course the constant functions.

3.3 Since, for any C^2 function f on M, the restriction of $\Delta_v f$ to F_m is by definition $\Delta^{F_m}(f \restriction F_m)$ and since all fibres are isometric, *the spectrum of Δ_v is also discrete.* Let $(\phi_i)_{i\in\mathbf{N}}$ be the eigenvalues of Δ_v (and also of Δ^F). Of course $\phi_0 = 0, \phi_i > 0$ if $i \geq 1$ and $\phi_i \to +\infty$ when $i \to +\infty$. However, multiplicities are not necessarily finite: for examples functions f such that $\Delta_v f = 0$ are only constant *along the fibres* and hence are functions of the form $\bar{f} \circ \pi$ where $\bar{f}$ is a function defined on B.

3.4 *Warning. The spectrum of Δ_h need not be discrete.* This appears already in the simple example that we introduced in 2.4 as we now explain. In this case the eigenvalues of Δ_v are the numbers $4\pi^2 q^2$ $(q \in \mathbf{N})$, those of Δ^M are $4\pi^2((p - aq)^2 + q^2)$ $(p, q \in \mathbf{Z})$. Therefore one sees that the eigenvalues of Δ_h are all numbers $4\pi^2(p - aq)^2$ $(p, q \in \mathbf{Z})$.

If a is irrational, 0 is obtained only for $p = q = 0$, the corresponding eigenfunctions being constant. But the spectrum of Δ_h does accumulate at 0.

3.5 When Δ^M and Δ_v commute, there exists a decomposition of $L^2(M)$ into joint eigenspaces for these two operators. More precisely, we have the following result.

3.6 THEOREM. *The Hilbert space $L^2(M)$ admits Hilbert basis consisting of simultaneous eigenfunctions for Δ^M and Δ_v.*

3.7 We set

$$\mathbf{H}^g(b, \phi) = \{f \mid f \in L^2(M), \Delta_h f = bf, \Delta_v f = \phi f\}.$$

We emphasize that we use letters of different alphabets to mean the following. *Greek letters are reserved for eigenvalues of Laplace-Beltrami operators* (μ_i for Δ^M, β_i for Δ^B, ϕ_i for Δ^F and therefore for Δ_v). We use *Latin letters* to denote *the eigenvalues of* Δ_h, since this operator is of a different nature.

If f is a function in $\mathbf{H}^g(b, \phi)$, then $\Delta^M f = (b + \phi)f$. Hence $\mathbf{H}^g(b, \phi)$ must be finite dimensional. If $\mathbf{H}^g(b, \phi) \neq 0$, then ϕ (resp. b) belongs to the spectrum of Δ_v (resp. of Δ_h). Since Δ_v and Δ_h are non-negative operators, we must have $\phi \geq 0$ and $b \geq 0$.

3.8 Notice that the eigenvalues of Δ^M *are not all possible sums* of one eigenvalue of Δ_h and one of Δ_v, if the bundle is not trivial. How to select the permitted combinations is one of the main problems and depends on the global geometry of the situation.

3.9 We emphasize that *the spectrum of Δ_h contains but does not coincide* in general *with the spectrum of the Laplace operator* Δ^B of the base manifold B. Indeed, if $\bar{f}$ is a function on B, then

$$(\Delta^B \bar{f}) \circ \pi = \Delta^M(\bar{f} \circ \pi) = \Delta_h(\bar{f} \circ \pi). \tag{3.10}$$

Notice that (3.10) already holds when the fibres are minimal as follows from [26].

3.11 From now on, we *suppose that the fibres of π: $M \to B$ are connected.*

3.12 It follows from (3.10) that

$$\mathbf{H}^g(b, 0) = \{\bar{f} \circ \pi \mid \Delta^B \bar{f} = b\bar{f}\}.$$

In particular, $\mathbf{H}^g(b, 0) \neq 0$ only if $b = \beta_i$ where β_i is an eigenvalue of Δ^B. The map $\bar{f} \mapsto \bar{f} \circ \pi$ is an embedding of $L^2(B)$ into $L^2(M)$ as a direct factor. It is also easy to describe *the orthogonal projection from $L^2(M)$ onto the image of $L^2(B)$*. It is related to "integration along the fibres" in the following way: let us *denote by* $(\int_F f v_g)(q)$ the integral of $f \restriction F_q$ (where q is a point in B) for the canonical measure v_{g_q} associated with g_q.

3.13 PROPOSITION. *The following identities hold:*

$$\Delta^B \circ \int_F = \int_F \circ \Delta_h = \int_F \circ \Delta^M.$$

Proof. The second equality follows from the identity

$$\left(\int_F \Delta_v f v_g\right)(q) = \int_{F_q} (\Delta^{F_q}(f \restriction F_q)) v_{g_q} = 0.$$

For the first equality, one uses the expression (1.9) of Δ_h. The proposition then follows from the following lemma.

3.14 LEMMA. *If X is a basic vector field which projects down to the vector field $\overline{X}$ on B, then*

$$\overline{X} \cdot \left(\int_F f v_g\right) = \int_F (X \cdot f) v_g.$$

Proof. Since the flow of X maps a fibre isometrically to another fibre, X preserves v_g (this would also hold if the fibres were only minimal). ∎

3.15 It follows from Proposition 3.13 that if we set

$$f_F = \frac{\int_F f v_g}{\int_F v_g},$$

then the map $\bar{f} \mapsto \bar{f}_F$ is the orthogonal projection of $L^2(M)$ onto the image of $L^2(B)$.

4. Applications to quantum physics

4.1 Recently there has been a renewed interest in classical physics for *non-bijective canonical transformations* (see [19], [6]). This very general expression should not be taken literally, but more in the sense that certain interesting maps between configuration spaces turn out to be non-linear and non-bijective. From a mathematician's point of view these maps are in fact extremely nice (namely, coverings or Hopf fibrations in the examples that we detail later).

When going to the quantum level, one has to describe how the spectrum of the quantum operators are related. Once more the quantum operators are not the most general operators but very natural ones related to the Riemannian geometry of the situation (for example the Laplace operator of a Riemannian metric plus a potential for the energy).

4.2 As we recalled in 1.10(i), to ensure that the spectrum of the base Riemannian manifold B is contained in the spectrum of the total space M of the fibration it is enough that the fibres be minimal. Moreover, the inclusion of eigenfunctions is given (as explained at the end of Section 3) by composing functions on B with the projection map. The function on M obtained in this way are constant along the fibres, hence annihilated by the vertical Laplacian Δ_v. This condition is in turn sufficient if the fibres are compact and connected. But to give a nice description of this condition in a basis of pure states (i.e., eigenfunctions of Δ^M plus a potential depending only on the base), one needs that Δ_v and Δ^M commute. This is precisely the case that we came up with in Theorem 1.5. The holonomy group of the bundle is usually referred to in the physics literature as the *group of ambiguity*. We now come to some examples to see more concretely how the notions really appear.

4.3 The first instance which should be cited of non-bijective canonical transformation is the use of Euler angles to parametrize a rigid body in $\mathbf{R}^3$. Mathematically speaking, this amounts to considering S^3 as a two-fold cover of $\mathbf{R}P^3$, since S^3 identifies itself with the group SU_2 and $\mathbf{R}P^3$ with SO_3. Later, S^3 was recognized as having itself a physical meaning with the discovery of the spin of particles.

Another instance where a two-fold cover occurs is known as *Levi-Civita's regularization of the plane motion.* Let $z = x + iy$ be a complex parameter describing a body moving in a plane. In [17], Levi-Civita proposed to introduce a mathematical complex parameter $\zeta = \xi + i\eta$ mapped onto the physical z-plane by

$$z = \pi(\zeta) = \zeta^2 \qquad \text{(in other words } x = \xi^2 - \eta^2,\ y = 2\xi\eta).$$

This map π is a *two-sheeted ramified covering of* $\mathbf{C} = \mathbf{R}^2$ *over* $\mathbf{C} = \mathbf{R}^2$. The origin is a singular point of π. Under π, the distance to the origin is squared. The interest of the transformation lies in what it does to the equation of motion of a body governed by Kepler law. Indeed, under π, a conical section centered at the origin of the ζ-plane is transformed into a conical section of the z-plane having one focus at the origin, hence the regularization. By going to polar coordinates, one can separate the radial from the angular movement.

4.4 A very nice generalization of this to higher dimensions is due to P. Kustaanheimo and E. Stiefel (cf. [16]). They noticed that Levi-Civita's construction could be considered as an orthogonal multiplication, the existence of which is known to be strongly related to the existence of an underlying field. The next case they considered is some kind of *quaternionic ramified map* of $\mathbf{H} \simeq \mathbf{R}^4$ over $\mathbf{R}^3$.

The fact that no map from $\mathbf{R}^4$ to $\mathbf{R}^4$ analogous to Levi-Civita's regularization exists is related to the non-commutativity of the field $\mathbf{H}$ of quaternions (the tensor product of two copies of the fundamental representation of Sp_1 ($= SU_2$) is a real representation which decomposes into two representations: a 3-dimensional one and a trivial 1-dimensional one). Their generalization can be described as follows: a point ζ in $\mathbf{H}$ ($\zeta = (x_1 + ix_2) + j(y_1 + iy_2)$ where x_1, x_2,

y_1 and y_2 are real numbers) is mapped by π to the point $z = (u, v, w)$ in $\mathbf{R}^3$ given by

$$u = x_1^2 - x_2^2 - (y_1^2 - y_2^2)$$
$$v = 2(x_1 x_2 - y_1 y_2)$$
$$w = 2(x_1 y_1 + x_2 y_2).$$

In this case again, the origin is the only singular point, the map π squares distances to the origin and, on spheres of radii 1, π reduces to the ordinary Hopf fibration. As before, by transfering the equations of a Kepler motion to the ζ-space one regularizes the motion. (Notice, however, that in order to do so one has to lift the dynamical vector field horizontally for the natural distribution of the Hopf fibration).

4.5 In [4], a connection between the radial Schrödinger equations of the hydrogen atom and isotropic harmonic oscillators of various dimensions is pointed out. As a consequence, the energy levels of the nonrelativistic hydrogen atom can be deduced from one-dimensional harmonic oscillators. This is obtained by taking a one-dimensional quadratic map, i.e., by considering a real two-sheeted ramified covering. This can be viewed merely as a change of variables and does not require any geometric understanding of the situation, unlike the preceding example.

4.6 A generalization of this to the Coulomb problem *in 2 and 3 dimensions* is due to M. Boiteux (cf. [6]) and requires the use of the quadratic maps introduced by Levi-Civita and P. Kustaanheimo and E. Steifel together with our Theorem 2.5. As a consequence, *the bound state Coulomb problem is shown to be equivalent to an harmonic oscillator inverse problem* (with potential $1/r$ where r is the distance to the origin).

5. The canonical variation of a Riemannian submersion

5.1 Keeping the same assumptions as before (i.e., $\pi\colon M \to B$ is a Riemannian submersion with totally geodesic fibres), the metric g on M has a *canonical variation* associated with the submersion, namely:

DEFINITION. For each positive real number t, let g_t be the unique Riemannian metric on M such that

(i) $g_t \restriction V_m M \times H_m M = 0$;
(ii) $g_t \restriction V_m M = t^2 g \restriction V_m M$;
(iii) $g_t \restriction H_m M = g \restriction H_m M$.

We immediately get the following result.

5.2 PROPOSITION. *The map π is a Riemannian submersion from (M, g_t) to (B, j) with totally geodesic fibres isometric to $(F, t^2 k)$.*

Later we *denote by* M_t the Riemannian manifold (M, g_t) and its Laplacian *by* Δ_t^M. We now compute Δ_t^M in terms of the operators Δ^M, Δ_v, Δ_h corresponding to g.

5.3 PROPOSITION. *The following formulas hold:*

$$\Delta_t^M = t^{-2}\Delta_v + \Delta_h = t^{-2}\Delta^M + (1 - t^{-2})\Delta_h.$$

Proof. It suffices to show that between the vertical Laplacian Δ_v^t and the horizontal Laplacian Δ_h^t of g_t and Δ_v and Δ_h the following relations hold: $\Delta_v^t = t^{-2}\Delta_v$ and $\Delta_h^t = \Delta_h$. Let $(X_i)_{i=1,\dots,p}$ and $(U_j)_{j=1,\dots,n-p}$ be the same moving frames that we used in the proof of Theorem 2.5. For g_t, (X_i) is still an orthonormal basis of H_mM and $(t^{-1}U_j)$ is an orthonormal basis of V_mM. Therefore

$$\Delta_v^t = t^{-2} \sum_{j=1}^{n-p} (U_j \circ U_j - D_{U_j}U_j) = t^{-2}\Delta_v$$

and

$$\Delta_h^t = \Delta_h. \quad \blacksquare$$

5.4 A common eigenfunction of Δ_v and Δ_h is an eigenfunction of Δ_t^M *for each* t. More precisely, since, as topological spaces, $L^2(M) = L^2(M_t)$, one has the next result.

5.5 COROLLARY. $\mathbf{H}^g(b, \phi) = \mathbf{H}^{g_t}(b, t^{-2}\phi)$.

Proof. If $\Delta_h f = bf$ and $\Delta_v f = \phi f$, then $\Delta_v^t f = t^{-2}\Delta_v f = t^{-2}\phi f$ and in particular $\Delta_t^M f = (b + t^{-2}\phi)f$. $\blacksquare$

5.6 *Remark.* There exists a Hilbert basis of $L^2(M)$ which consists of eigenfunctions for each Δ_t^M, but corresponding to *varying* eigenvalues. In particular the ordering of the eigenvalues may change as t varies (see 6.6 for an example).

6. H. Urakawa's example revisited

6.1 We now describe the particular situation of H. Urakawa's example.

The Hopf fibration $\pi: S^3 \to S^2$ is a Riemannian submersion from a metric with constant curvature 1 on S^3 to a metric with constant curvature 4 on S^2 (in fact S^2 stands there for the complex projective line CP^1). This fibration is the fibration between homogeneous spaces

$$U_2/U_1 \to U_2/U_1 \times U_1 = SO_3/SO_2$$

corresponding to the inclusion of U_1 as first summand of $U_1 \times U_1$. Moreover, any U_2-invariant metric on $S^3 = U_2/U_1$ coincides with one of the metrics g_t of the canonical variation introduced in Section 5. These metrics, sometimes called the *Berger metrics*, have provided counter-examples to various geometric conjectures (cf. [12], page 230, and [27]).

6.2 H. Urakawa was interested in the *first* nonzero eigenvalue of $\Delta_t^{S^3}$. For the canonical metric c, $\mu_1 = 3$ with multiplicity $m_1 = 4$ and $\mu_2 = 8$ with multiplicity $m_2 = 9$ (see [3], page 162). Now, one easily sees that, in the decomposition of the Hilbert space $L^2(S^3)$ into the $\mathbf{H}^c(b, \phi)$'s, $\mu_1 = b_1 + \phi_1$ with $b_1 = 2$, $\phi_1 = 1$ (the restriction to any great circle of a first spherical harmonic is a first spherical harmonic of the circle). Now μ_2 has two decompositions as sum of a b and a ϕ, say

$$\mu_2 = b_2' + \phi_2' \quad \text{with } b_2' = 8,\ \phi_2' = 0$$

and

$$\mu_2 = b_2'' + \phi_2'' \quad \text{with } b_2'' = 6,\ \phi_2'' = 2.$$

The eigenfunctions in $\mathbf{H}^c(8, 0)$ come from the base (they are the polynomials of degree 2 in z and $\bar{z}$, where z is a complex coordinate in $\mathbf{C}^2$, which are invariant under conjugation and orthogonal to the hermitian norm $z \cdot \bar{z}$).

All the corresponding eigenfunctions remain eigenfunctions for the operators $\Delta_t^{S^3}$, but associated with the eigenvalues $2 + t^{-2}$ for the elements of $\mathbf{H}^c(2, 1)$, 8 for the elements of $\mathbf{H}^c(8, 0)$ and $6 + 2t^{-2}$ for the elements of $\mathbf{H}^c(6, 2)$.

Therefore, as t decreases to 0 the eigenvalues of the type $b + t^{-2}\phi$ (with $\phi \neq 0$) increase. Hence, for $6^{-1/2} < t$, $2 + t^{-2}$ is still the first nonzero eigenvalue of (M, g_t) with multiplicity 4. For $t = 6^{-1/2}$ the first eigenvalue is 8 with multiplicity $4 + 3 = 7$. For $t < 6^{-1/2}$, 8 stays the first eigenvalue but only with multiplicity 3.

6.3 COROLLARY (cf. [24]). *There exists a Riemannian metric on S^3 whose first nonzero eigenvalue has multiplicity* 7 (*the latter for the canonical metric is* 4).

6.4 As we saw, if $\mu_1(t)$ denotes the first nonzero eigenvalue of Δ_t^M, we have

$$\mu_1(t) = 8 \qquad \text{for } t \leq 6^{-1/2}$$
$$\mu_1(t) = 2 + t^{-2} \quad \text{for } t \geq 6^{-1/2}.$$

H. Urakawa used this fact to study the expression $\mu_1^q(\mathrm{vol}\,(S^3, g))^{2/3}$ which is invariant under homothetical changes of the metric on 3-dimensional manifolds. For the canonical variation of the Hopf fibration, one finds

$$\mathrm{vol}\,(S^3, g_t) = t\ \mathrm{vol}\,(S^3, g)$$

(by integration along the fibres), so that

$$\psi(t) = \mu_1(t)(\mathrm{vol}\,(S^3, g_t))^{2/3} = \begin{cases} 8t^{2/3}(2\pi^2)^{2/3} & \text{for } t \leq 6^{-1/2} \\ (2 + t^{-2})t^{2/3}(2\pi^2)^{2/3} & \text{for } t \geq 6^{-1/2}. \end{cases}$$

In particular, one sees that $\psi(t)$ goes to $+\infty$ as t goes to $+\infty$ and $\psi(t)$ goes to 0 as t goes to 0.

6.5 COROLLARY (cf. [24]). *The function $\mu_1^q(\mathrm{vol}\,(S^3, g))^{2/3}$ has no universal bound when g varies among metrics on S^3.*

6.6 *Remarks.* (i) The only new information is really that the function is not bounded from above, since it is known (see [3], page 188, for a proof) that on any manifold this function can be made arbitrarily small (this was J. Cheeger's starting point in introducing an isoperimetric constant in [9]). Corollary 6.5 contrasts sharply with what happens on the sphere S^2. There, by J. Hersch's theorem (cf. [14]), the function μ_1^g vol (S^2, g) is bounded from above by 8π, i.e., the value it achieves for the standard metric. (For a more detailed discussion of this point, see the exposé n°IX of the Séminaire Goulaouic-Schwartz 1979–1980 by the second author.)

(ii) The canonical variation of the Hopf fibration allows us also to exhibit an example of the following phenomenon. The first eigenspace for the standard metric on S^3 consists of functions with only two nodal domains. These functions remain eigenfunctions for all the metrics g_t, but associated with the eigenvalue $2 + t^{-2}$. As t goes to 0, $2 + t^{-2}$ goes to $+\infty$. However, the spectrum of (S^3, g_t) contains that of (S^2, c) which is fixed as we noticed earlier. Therefore $2 + t^{-2}$ appears arbitrarily far away in the spectrum as t goes to 0.

6.7 COROLLARY. *For any integer l, there exists a Riemannian metric on S^3 such that there are eigenfunctions of an order larger than l with only two nodal domains.*

6.8 Recall that, by the extension of Courant's nodal line theorem to compact Riemannian manifolds, the number of nodal domains of an eigenfunction is bounded from above by the order of the associated eigenvalue plus one (0 is considered as the 0th eigenvalue). We just showed that for certain manifolds such as S^3 no nontrivial lower bound exists (for a different point of view, see [18]).

6.9 So far, we have been interested in the metrics g_t only in connection with the Hopf fibration $\pi: S^3 \to S^2$. We can also use them in connection with the covering $\pi: S^3 \to \mathbf{R}P^3$ (which can also be thought of as a Hopf fibration). The metrics g_t are indeed invariant under the antipodal map: we denote by $\bar{g}_t$ the metrics induced on $\mathbf{R}P^3$. The previous discussion shows that, for $t < 6^{-1/2}$, the first nonzero eigenvalues of the Laplacians $\Delta_t^{S^3}$ and $\Delta_t^{\mathbf{R}P^3}$ coincide. Indeed the elements of $\mathbf{H}^c(8, 0)$ which come from functions on S^2 are of course invariant under the antipodal map and give rise to functions on $\mathbf{R}P^3$.

This again contrasts sharply with the two-dimensional situation, where one can prove that the first nonzero eigenvalue of a metric on $\mathbf{R}P^2$ cannot be the first nonzero eigenvalue of the lifted metric on S^2.

7. On the first eigenvalue of the canonical variation

7.1 In this section we consider the general case, i.e., where $\pi: M \to B$ is a Riemannian submersion with totally geodesic fibres with M compact and connected. We analyze how H. Urakawa's observation can be generalized.

Using direct calculations, such an extension has been given independently by S. Tanno (cf. [23]) for the Hopf fibration $\pi: S^{2q+1} \to \mathbf{C}P^q$ and by H. Urakawa and H. Muto (cf. [21]) for certain homogeneous spaces. Here we state a generalization which covers these cases without detailing explicit examples.

7.2 PROPOSITION. *Suppose that the first eigenvalue μ_1 of Δ^M satisfies the following "unique decomposition" property: there exists a unique pair (b_1, ϕ_1) such that $\mu_1 = b_1 + \phi_1$ with $\phi_1 \neq 0$ where b_1 is an eigenvalue of the horizontal Laplacian and ϕ_1 an eigenvalue of the vertical Laplacian.*

Then there exists a positive number t such that the Laplacian Δ_t^M of the metric g_t of the canonical variation has a first nonzero eigenvalue with multiplicity larger than that of Δ^M. Moreover for any integer l, there exists a Riemannian metric on M with eigenfunctions of an order larger than l which have only two nodal domains.

Proof. The proof is the same as in H. Urakawa's example. ∎

7.3 Some remarks on the case of the Hopf fibration $S^{2q+1} \to \mathbf{C}P^q$ are in order. One can realize the canonical variation of the submersion $\pi: S^{2q+1} \to \mathbf{C}P^q$ for $t < 1$ (resp. $1 < t$) as the induced metrics on the distance spheres in $\mathbf{C}P^{q+1}$ (resp. $(\mathbf{C}P^{q+1})^*$, the dual symmetric space with negative curvature). This is detailed in [8], page 85. The exceptional multiplicity appears for $t = (2(q+2))^{-1/2}$, i.e., for the sphere of radius r in $\mathbf{C}P^{q+1}$ (with diameter normalized to be $\pi/2$) determined by $\tan^2 r = 2q + 3$. This value of the radius does not seem to correspond to any other geometric property of the metric (for example, the distance sphere is minimally embedded for $\tan^2 r = 2q + 1$).

7.4 Let us mention one case where Proposition 7.2 does not apply: the Hopf fibration $\mathbf{C}P^{2r+1} \to \mathbf{H}P^r$ for which one easily checks that the standard metric has the largest multiplicity in the canonical variation.

7.5 We now come back to the function $\psi(g) = \mu_1^q(\mathrm{vol}\,(M, g))^{2/n}$.
By Fubini's theorem, ones has

$$\begin{aligned} \mathrm{vol}\,(M, g_t) &= \mathrm{vol}\,(B, j)\,\mathrm{vol}\,(F, t^2k) \\ &= \mathrm{vol}\,(B, j)t^{n-p}\,\mathrm{vol}\,(F, k) \\ &= t^{n-p}\,\mathrm{vol}\,(M, g). \end{aligned}$$

It is therefore sufficient in the case of the canonical variation to study the function $t \mapsto \mu_1(t)t^{2-2p/n}$.

We know that any eigenvalue of Δ_t^M may be written as $b + t^{-2}\phi$.

When t goes to 0, $b + t^{-2}\phi$ goes to $+\infty$ if $\phi \neq 0$ and is constant and equal to b if $\phi = 0$.

When t goes to $+\infty$, $b + t^{-2}\phi$ goes to b.

Since there are only a finite number of eigenvalues of the Laplacian on a compact Riemannian manifold smaller than a given number, one sees that, for t small enough, $\mu_1(t)$ is equal to the first eigenvalue of B (appearing as b with $\phi = 0$). So $\mu_1(t)$ is a positive constant for t small enough. Hence, we have the following result.

7.6 PROPOSITION. *For the canonical variation g_t of a Riemannian submersion with totally geodesic fibres, $\psi(t) = \mu_1^{g_t}(\text{vol}\,(M, g_t))^{2/n}$ goes to 0 with t.*

When t goes to $+\infty$, the result depends on the nature of the spectrum of the horizontal Laplacian Δ_h. We have the following, for example

7.7 PROPOSITION. (i) *If there exists a non-constant function f so that $\Delta_h f = 0$, then for the canonical variation g_t of a Riemannian submersion $\psi(t) = \mu_1^{g_t}(\text{vol}\,(M, g_t))^{2/n}$ goes to 0 as t goes to $+\infty$.*

(ii) *If $\Delta_h f = 0$ implies f constant and if the spectrum of Δ_h does not accumulate at 0, then $\psi(t)$ goes to $+\infty$ with t.*

Proof. (i) By assumption, there exists a non-constant function f such that $\Delta_h f = 0$. Since $[\Delta_v, \Delta_h] = 0$, we can suppose that f has been chosen so that $\Delta_v f = \phi f$.

Then $\Delta_t^{M_f} = \Delta_v^t f = t^{-2}\phi f$.

In particular $\mu_1(t) \leq t^{-2}\phi$ and $\psi_1(t) = \mu_1(t)t^{2-2p/n} \leq t^{-2p/n}\phi$. Therefore $\psi(t)$ goes to 0 when t goes to $+\infty$.

(iii) Let f be a common eigenfunction to Δ_t^M and Δ_v^t associated with the first nonzero eigenvalue of Δ_t^M. Since f is not constant, then $\Delta_h^t f = b_1 f$ with $0 < c \leq b_1$ for some c. Therefore $c \leq b_1 + t^{-2}\phi_1 = \mu_1(t)$. In particular $ct^{2-2p/n} \leq \mu_1(t)t^{2-2p/n}$ and $\psi(t)$ goes to $+\infty$ with t. ∎

7.8 As we mentioned in 6.6, Proposition 7.6 and Part (i) of Proposition 7.7 do not exhibit new behaviour of the function ψ. (An instance where the assumption in (i) is fulfilled is developed at the end of 2.2.) On the contrary, Part (ii) of Proposition 7.7 shows that J. Hersch's theorem (cf. [14]) does not generalize under these assumptions for example to the spheres S^{2q+1}. Recently H. Muto proved (cf. [20]) that it does not generalize to S^{2q} $(2 \leq q)$ either.

7.9 Another function of interest when studying the first eigenvalue is

$$\delta(g) = \mu_1^g(\text{diam}\,(M, g))^2.$$

Estimating the diameter in the case of a Riemannian submersion (even with totally geodesic fibres) is not such an obvious matter. For that purpose we introduce two new quantities: firstly we *denote by* diam (F/G) the diameter of the metric space F/G obtained from the Riemannian manifold (F, k) by dividing by the action of the holonomy group G (which is known to act by isometries in our situation); secondly we *denote by* $\text{diam}_h\, M$ the *horizontal diameter* of M, i.e., the supremum over all pairs of points p and q of M which can be joined by a horizontal curve of the infimum of the lengths of such curves. We then have the following result.

7.10 PROPOSITION. *The following inequalities hold:*
(i) $\operatorname{diam}^2 B + \operatorname{diam}^2 (F/G) \leq \operatorname{diam}^2 M$,
(ii) $\operatorname{diam}^2 M \leq \operatorname{diam}^2 B + \operatorname{diam}^2 F$,
(iii) $\operatorname{diam}^2 M \leq \operatorname{diam}_h^2 M + \operatorname{diam}^2 (F/G)$.

Before giving the proof of Proposition 7.10, we review the consequences that these estimates have for the function $\delta(t) = \delta(g_t)$ (where g_t is the canonical variation of a Riemannian submersion).

7.11 For the metric g_t, we obtain

$$\operatorname{diam}^2 B + t^2 \operatorname{diam}^2 (F/G) \leq \operatorname{diam}^2 (M_t)$$

$$\operatorname{diam}^2 (M_t) \leq \operatorname{diam}^2 B + t^2 \operatorname{diam}^2 F$$

$$\operatorname{diam}^2 (M_t) \leq \operatorname{diam}_h^2 M + t^2 \operatorname{diam}^2 (F/G).$$

Hence *when t goes to* 0, $\operatorname{diam} (M_t)$ goes to $\operatorname{diam} B$ as expected geometrically. *When t goes to* $+\infty$, the situation is more complicated.

If $\operatorname{diam} (F/G) \neq 0$, then $\operatorname{diam} (M_t)$ is asymptotic to $t \operatorname{diam} (F/G)$ by (ii).

If $\operatorname{diam} (F/G) = 0$ and $\operatorname{diam}_h M$ finite (which is ensured for example by G compact or $\pi_1(B)$ finite), then $\operatorname{diam} (M_t)$ remains bounded. (Notice that $\operatorname{diam} (F/G) = 0$ and $\operatorname{diam}_h (M)$ infinite can occur as shown by the example developed in 2.4 with a irrational.)

7.12 PROPOSITION. *If* $\operatorname{diam} (F/G) \neq 0$, *then* $\delta(t)$ *goes to* $+\infty$ *with* t.

7.13 *Remark.* In the case of the Hopf fibrations $\pi: S^{2q+1} \to CP^q$, $G = S^1$ acts transively on the fibres (so that $\operatorname{diam} (F/G, k) = 0$). One even has $\operatorname{diam} (M_t) = \operatorname{diam}_h M = \operatorname{diam} M$ for all t! In this case the function $t \mapsto \delta(t)$ remains bounded in the canonical variation. One can easily see that, as for the function ψ, the function δ is never bounded from below on the space of Riemannian metrics on a compact manifold.

7.14 *Proof of Proposition 7.10.* (i) Let p and q be points realizing the diameter of B. Let y and z be points in the fibre at q so that their classes in the metric space F/G realize the diameter of F/G.

We take the shortest geodesic γ from y to the fibre at p. This geodesic is necessarily horizontal (γ is horizontal when it reaches F_p say at x and this is enough to ensure that it is horizontal all the way). Then, denoting the distance by d, we have

$$\operatorname{diam}^2 B + \operatorname{diam}^2 (F/G) \leq d(x, z)^2.$$

Indeed, let σ be a minimizing geodesic in M from x to z and let s be its projection in B. Let (ϕ_t) be the horizontal transport along the collection of all horizontal liftings of s. It is clear that ϕ_t is an isometry from $F_{s(t)}$ onto the fibre F_q (q is the endpoint of s). Then one can prove that $\zeta(t) = \phi_t(\gamma(t))$ is a geodesic of F_q, that σ makes a constant angle with the fibres (so that s and ζ are parametrized proportionally to arc length) and that $L(\sigma)^2 = L(s)^2 + L(\zeta)^2$ (where L denotes the length of a curve). Since s joins p to q, diam $B \leq L(s)$.

The geodesic ζ ends at z and originates at a point y_1 which can be joined to y by a horizontal curve. Therefore y_1 lies in the same G-orbit as y so that

$$\operatorname{diam}(F/G) = d_{F/G}(\bar{y}, \bar{z}) = d_{F/G}(\bar{y}_1, \bar{z}) \leq d(y, z).$$

Since the proofs of (ii) and (iii) are not directly used in the article, they are left to the reader (they involve the same ideas as the proof of (i)). ∎

8. More on the horizontal Laplacian

8.1 Proportion 7.7 underlines the importance of the nature of the spectrum of Δ_h.

In this section we look for geometric conditions which ensure the assumptions on Δ_h made in Proposition 7.7. We shall see that the question is intimately related with the transitivity of the holonomy group on the fibre.

We recall that *the holonomy group of a fibre* is the group of all isometries of the fibre induced by horizontal transport along the horizontal lifts of a loop in B based at the projection of the fibre.

8.2 THEOREM. *The holonomy group has no dense orbit if and only if there exists a non-constant function f on M such that $\Delta_h f = 0$.*

Proof. Let $d_h f$ be the horizontal part of the 1-form df. Clearly $d_h f = 0$ implies that f is invariant under horizontal transport.

Suppose that $\Delta_h f = 0$.

We integrate $\Delta_h f$ against f on M. We get

$$\begin{aligned}\int_M f\Delta_h f &= \int_M f\Delta f - \int_M f\Delta_v f\\ &= \int_M |df|^2 - \int_M |d_v f|^2\\ &= \int_M |d_h f|^2,\end{aligned}$$

hence the vanishing of $d_h f$.

If the holonomy group has no dense orbit in the fibre, there exists a non-constant G-invariant smooth function on one fibre. One can then construct a function f which is invariant under horizontal transport and coincides with the given function on the fibre. Then $d_h f = 0$ and $\Delta_h f = 0$. Conversely, if the holonomy group has a dense orbit, if some C^1-function f satisfies $d_h f = 0$, then f is constant on the dense orbit, hence constant everywhere. Since we already proved before that $\Delta_h f = 0$ implies that $d_h f = 0$, we are done. ∎

8.3 COROLLARY. *If the holonomy group has no dense orbit, then for the canonical variation g_t of a Riemannian submersion $\psi(t) = \mu_1^{g_t}(\mathrm{vol}\,(M, g_t))^{2/n}$ goes to* 0 *as t goes to $+\infty$.*

8.4 There are cases where *none of the assumptions in Proposition 7.7 is satisfied.* Indeed it may happen that $\Delta_h f = 0$ implies f constant, while the eigenvalues of Δ_h accumulate at 0. This happens in the example introduced in 2.4 whose eigenvalues are computed in 3.4. In this example one can see that the holonomy group is discrete but when a is irrational has a dense orbit in the fibre.

8.5 Coming back to the general case, we now introduce a condition which ensures that 0 is isolated in the spectrum of Δ_h.

8.6 DEFINITION. We say that the holonomy group is *infinitesimally transitive* if the basic vector fields together with their iterated brackets generate all the tangent space to M at any point.

8.7 Notice that the vertical part of the bracket of two basic vector fields lies in each fibre in the Lie algebra of the holonomy group.

8.8 THEOREM. *If the holonomy group is infinitesimally transitive at every point, then the spectrum of Δ_h is discrete, and hence* 0 *is an isolated point of the spectrum.*

Proof. It is a consequence of a deep result of L. Hörmander (cf. [15], page 149) that under these assumptions Δ_h is hypoelliptic. ∎

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