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PERFECT, u-ADDITIVE MEASURES AND STRICT TOPOLOGIES

BY

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Let X be a completely regular space, and let $C(X)$ denote the space of bounded continuous real-valued functions on X. The finest locally convex topology on $C(X)$ which coincides on the supremum-norm bounded sets with the compact-open topology is denoted by β_0 (see [19]). This topology has been examined by several authors and it is well known that the dual of $(C(X), \beta_0)$ is the space $M_t(X)$ of tight measures on X. If X is locally compact, β_0 coincides with the strict topology of Buck [1].

If βX denotes the Stone-Cech compactification of X, then for every $Q \subset \beta X - X$ the spaces $C(\beta X - Q)$ and $C(X)$ are isomorphic. So the topology β_0 on $C(\beta X - Q)$ can be regarded as a topology on $C(X)$, which is denoted by β_Q. We think of a strict topology on $C(X)$ as an inductive limit of topologies β_Q for some family of sets $Q \subset \beta X - X$ not necessarily compact. Such topologies on $C(X)$ are the topologies β_1 and β studied by Sentilles [19], which yield as duals the spaces $M_\sigma(X)$ and $M_\tau(X)$ of σ-additive and τ-additive measures.

This paper deals with the spaces $M_p(X)$ and $M_u(X)$ of perfect [18] and u-additive [20] measures. In Sections 2 and 3, strict topologies β_p and β_u are defined, so that $(C(X), \beta_p)' = M_p(X)$ and $(C(X), \beta_u)' = M_u(X)$. The topology β_u is the same as that introduced by Wheeler [23]; but the different definition leads to simple proofs of some known results. In Section 4, characterizations of the spaces X for which $M_p(X) \subset M_u(X)$ are given. One such characterization, which contains an extension of Shirota's theorem, is the fairly weak condition that certain closed discrete subsets of X have non-(Ulam-) measurable cardinal. Using this condition one can decide whether a perfect measure is u-additive essentially avoiding the set theoretical difficulties which appear in the general case of a σ-additive measure. Section 1 contains preliminaries and generalities about strict topologies.

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1. Preliminaries and the general strict topology

All topological spaces X are assumed to be completely regular (and Hausdorff). Basic reference for the theory of measures on topological spaces is

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[22]. A zero set in X is a set of the form $f^{-1}(\{0\})$ for some $f \in C(X)$. A cozero set is a complement of a zero set. The Baire sets in X are the members of the σ-algebra $\mathscr{B}(X)$ generated by the zero sets. We are primarily concerned with Baire measures. For the definition of the spaces of measures $M(X) \supset M_\sigma(X) \supset M_\tau(X) \supset M_t(X)$ see [22]. For $H \subset M(X)$ we denote by $|H|$ the set $|H| = \{|\mu|: \mu \in H\}$. According to the Aleksandrov representation theorem [22, p. 165], $M(X)$ can be identified with the dual of $C(X)$ under the (supremum-)norm. All topological statements about $M(X)$ will be relative to the weak topology $\sigma(M(X), C(X))$.

Our principal references for results about perfect and u-additive measures are [18] and [20] respectively. A countably additive measure μ on a measurable space $(S, \mathscr{A})$ is said to be *perfect*, if for every $\mathscr{A}$-measurable function $g\colon S \to \mathbf{R}$ there is a Borel set B in $\mathbf{R}$ such that $B \subset g(S)$ and $|\mu|(g^{-1}(B)) = |\mu|(S)$. If X is a completely regular space, $M_p(X)$ denotes the space of all perfect measures defined on the σ-algebra $\mathscr{B}(X)$ of Baire sets. We have $M_\sigma(X) \supset M_p(X) \supset M_t(X)$, and $M_p(X) = M_t(X)$ if X is separable metrizable [18].

A partition of unity for a completely regular space X is a family $(f_\alpha)_{\alpha \in A}$ of positive functions in $C(X)$ such that

$$\sum_{\alpha \in A} f_\alpha \equiv \sup \left\{ \sum_{\alpha \in F} f_\alpha \colon F \text{ finite, } F \subset A \right\} = 1$$

and $\{x \in X\colon f_\alpha(x) > 0\}_{\alpha \in A}$ is locally finite. A measure μ on X is *u-additive* if

$$\sum_{\alpha \in A} \mu(f_\alpha) = \mu(1)$$

for every partition of unity $(f_\alpha)_{\alpha \in A}$. If $M_u(X)$ denotes the space of u-additive measures on X, we have $M_\sigma(X) \supset M_u(X) \supset M_\tau(X)$ [20].

The topology β_0 on $C(X)$ is defined to be the finest locally convex topology which coincides on the norm bounded sets with the compact-open topology (that is, the topology of uniform convergence on compact sets). The dual $(C(X), \beta_0)'$ of $C(X)$ endowed with this topology is the space $M_t(X)$ [19].

If X is locally compact, then β_0 coincides with the strict topology of Buck [1], that is, β_0 is determined by the seminorms $p_\phi(f) = \| f \cdot \phi \|$, as ϕ runs through the space $C_0(X)$ of continuous functions vanishing at infinity (see [4, p. 119] or [19, Theorem 2.3]).

The following characterization of β_0-equicontinuity will be used.

1.1 PROPOSITION [4], [19]. *A subset H of $M_t(X)$ is β_0-equicontinuous if and only if* (a) *H is norm bounded, and* (b) *for every $\varepsilon > 0$ there is a compact set $K \subset X$ such that $|\mu|^*(K) > |\mu|(X) - \varepsilon$ for all $\mu \in H$ (where $|\mu|^*$ denotes the outer measure).*

A space X is said to be a Prohorov space if every compact subset of $M_t^+(X)$ is β_0-equicontinuous. It is well known that locally compact spaces are Prohorov (cf. [19]).

For every $Q \subset \beta X - X$, the spaces $C(X)$ and $C(\beta X - Q)$ are isomorphic since $\beta(\beta X - Q) = \beta X$. So the topology β_0 on $C(\beta X - Q)$ can be regarded as a topology on $C(X)$, which is denoted by β_Q. If $Q_1 \subset Q_2$ then obviously $\beta_{Q_1} \supset \beta_{Q_2}$. If α is a family of subsets of $\beta X - X$ we define $\beta_\alpha = \text{Lin}\,\{\beta_Q: Q \in \alpha\}$ as the inductive limit of $(\beta_Q)_{Q \in \alpha}$, i.e., the finest locally convex topology which is contained in each β_Q. By the term strict topology we mean any locally convex topology on $C(X)$ of the form β_α for some non-empty family α of subsets of $\beta X - X$.

Clearly $\beta_0 = \beta_{(\beta X - X)}$ and $\|\cdot\| = \beta_\phi$, where $\|\cdot\|$ denotes the norm topology of $C(X)$. So β_0 and $\|\cdot\|$ are strict topologies and every strict topology β_α lies between them, hence $M(X) \supset (C(X), \beta_\alpha)' \supset M_t(X)$. Since β_0 is Hausdorff, every strict topology is Hausdorff. A number of significant properties of the strict topologies (see [19]) can now be deduced from the properties of the topology β_0. Here we shall use only the following.

1.2. Proposition. *Let α be a family of subsets of $\beta X - X$ and β_α the corresponding strict topology on $C(X)$. Then*

(i) $(C(X), \beta_\alpha)' = \bigcap_{Q \in \alpha} M_t(\beta X - Q)$; *and*

(ii) *if α contains only compact sets, then β_α is the topology of uniform convergence on the compact subsets $(C(X), \beta_\alpha)'$ consisting of positive measures.*

The proof follows from standard duality arguments [17, pp. 79, 80]. We only note that to show (ii) one uses the fact that β_α is an inductive limit of topologies β_0 for locally compact, hence Prohorov, spaces, and that β_α-equicontinuity of any $H \subset (C(X), \beta_\alpha)'$ is equivalent to β_α-equicontinuity of $|H|$, since this is true for the topology β_0 (Proposition 1.1).

For every $\mu \in M(X)$ we denote by $\bar{\mu}$ the corresponding regular Borel measure on βX (via the isometry of $C(X)$ and $C(\beta X)$). Then $\mu(f) = \bar{\mu}(\bar{f})$ for every $f \in C(X)$, where $\bar{f}$ denotes the continuous extension of f to βX. The measure μ is σ-additive (resp. τ-additive) if and only if $|\bar{\mu}|(Q) = 0$ for all zero (resp. compact) sets $Q \subset \beta X - X$; also $\mu \in M_t(X)$ if and only if $|\bar{\mu}|^*(\beta X - X) = 0$ (Knowles [12]). Therefore if we define the strict topologies β_1 and β by

$$\beta_1 = \text{Lin}\,\{\beta_Q: Q \text{ zero set, } Q \subset \beta X - X\}$$

and

$$\beta = \text{Lin}\,\{\beta_Q: Q \text{ compact, } Q \subset \beta X - X\}$$

then Proposition 1.2(i) yields $(C(X), \beta_1)' = M_\sigma(X)$ and $(C(X), \beta)' = M_\tau(X)$ (Sentilles [19, Theorem 4.4.]). Topologies equivalent to β_1 and β have been introduced independently by Fremlin, Garling and Haydon [4].

2. Perfect measures

Every Baire measurable function f: $X \to Y$ induces a map f_*: $M_\sigma(X) \to M_\sigma(Y)$ defined by $f_*(\mu)(B) = \mu(f^{-1}(B))$ for all $B \in \mathscr{B}(Y)$. It is easily seen that a σ-additive measure μ on X is perfect if and only if $g_*(\mu) \in M_t(g(X))$ for every Baire measurable function g: $X \to g(X) \subset \mathbf{R}$. This means that f_* preserves perfect measures. Moreover, using the corresponding properties of M_t we can show that $M_p(X)$ is a band, norm closed, vector subspace of $M(X)$.

In general, the dual of $C(X)$ under any strict topology has the above mentioned properties. So a natural question is whether $M_p(X)$ is the dual of $C(X)$ under a strict topology. Indeed this is the case. Towards this purpose we use the distinguishable sets of Frolik: a subset G of a completely regular space Y is distinguishable if there is a continuous function ϕ from Y onto a separable metric space such that $G = \phi^{-1}(\phi(G))$. We denote by $\mathscr{D}(Y)$ the family of distinguishable sets in Y. $\mathscr{D}(Y)$ is a σ-algebra containing the σ-algebra of Baire sets (see [5, p. 408]).

We start with a relation between perfect measures and distinguishable sets.

2.1. THEOREM. *For a measure $\mu \in M(X)$ the following are equivalent*:

(i) *μ is perfect*;
(ii) $|\bar{\mu}|^*(G) = 0$ *for all $G \in \mathscr{D}(\beta X)$, $G \subset \beta X - X$.*

We will use the following.

2.2 LEMMA. *If $\mu \in M_\sigma^+(X)$, then μ is perfect if and only if for every continuous function f from X onto a separable metric space, $f_*(\mu)$ is a tight measure.*

Proof. If μ is perfect then the measure $f_*(\mu)$ is a perfect measure on a separable metric space, hence tight.

Conversely, assume that $f_*(\mu)$ is perfect for every continuous function f from X onto a separable metric space. Let g: $X \to \mathbf{R}$ be a Baire measurable function and $\mathscr{B} = g^{-1}(\mathscr{B}(\mathbf{R}))$. It suffices to show that the restriction $\mu|_{\mathscr{B}}$ of μ to $\mathscr{B}$ is a perfect measure.

Let $\{V_n\}$ be a countable base for the topology of $\mathbf{R}$ and $A_n = g^{-1}(V_n)$. Since each A_n is a Baire set, there is a continuous function f from X onto a separable metric space Y such that $A_n = f^{-1}(B_n)$, where each B_n is a Baire set in Y [5, p. 408]. Now $f^{-1}(\mathscr{B}(Y))$ is a σ-algebra containing all A_n. Therefore $f^{-1}(\mathscr{B}(Y)) \supset \mathscr{B}$ and it is enough to show that $\mu|_{f^{-1}(\mathscr{B}(Y))}$ is perfect.

We consider the family

$$\mathscr{K} = \{f^{-1}(K)\colon K \text{ compact, } K \subset Y\}.$$

$\mathscr{K}$ is "compact" and, since $f_*(\mu)$ is tight, $\mathscr{K}$ approximates the measure μ from within on every element of $f^{-1}(\mathscr{B}(Y))$. It follows from [18, Theorem 2] that $\mu|_{f^{-1}(\mathscr{B}(Y))}$ is perfect.

Proof of Theorem 2.1. Without loss of generality we assume that μ is positive.

(i) $\Rightarrow$ (ii) Let $G \in \mathscr{D}(\beta X)$, $G \subset \beta X - X$ and let ϕ be a continuous function from βX onto a (compact) metric space Y such that $G = \phi^{-1}(\phi(G))$. If f is the restriction of ϕ to X, then $f_*(\mu)$ is a tight measure on $f(X)$. Let L be a σ-compact subset of $f(X) = \phi(X)$ such that $\mu(f^{-1}(L)) = \mu(X)$. The set $B = \phi^{-1}(L)$ is a Baire set in βX, $B \cap G = \emptyset$, and we have

$$\bar{\mu}(B) = \mu(B \cap X) = \mu(f^{-1}(L)) = \mu(X) = \bar{\mu}(\beta X),$$

where the first equality follows from the fact that $\bar{\mu}(A) = 0$ for all Baire sets $A \subset \beta X - X$ since μ is σ-additive (Section 1). Therefore $\bar{\mu}^*(G) = 0$.

(ii) $\Rightarrow$ (i) Since every zero set of βX is distinguishable, (ii) implies that μ is σ-additive (Section 1). Let f be a continuous function from X onto a separable metric space Y. By Lemma 2.2, it suffices to show that $f_*(\mu)$ is tight. Let $\bar{Y}$ be a metrizable compactification of Y and $\bar{f}$: $\beta X \to \bar{Y}$ the continuous extension of f. The set $G = \bar{f}^{-1}(\bar{Y} - Y)$ is distinguishable and, by (ii), $\bar{\mu}^*(G) = 0$. Let L be a σ-compact subset of $\beta X - G$ such that $\bar{\mu}(L) = \bar{\mu}(\beta X)$. Then $\bar{f}(L)$ is a σ-compact subset of Y and it is easy to see that $f_*(\mu)(\bar{f}(L)) = f_*(\mu)(Y)$. Therefore $f_*(\mu)$ is tight.

We denote by vX the realcompatification of X. We have $X \subset vX \subset \beta X$ and $X = vX$ if and only if X is realcompact (see [2] or [6]).

2.3. Corollary. $M_p(X) = M_p(vX)$ *(as subsets of* $M(X)$*).*

Proof. The spaces X and vX have the same Stone-Cech compactification. So $M_p(X)$ and $M_p(vX)$ can be considered as subsets of $M(\beta X) = M(X)$. The conclusion follows from Theorem 2.1 since every distinguishable set G in βX which is contained in $\beta X - X$ doesn't meet vX. Indeed, this is well known when G is a zero set and for the general case it is enough to observe that G is a union of zero sets.

Next we define the strict topology β_p by

$$\beta_p = \text{Lin}\,\{\beta_G \colon G \in \mathscr{D}(\beta X),\ G \subset \beta X - X\}.$$

Since every zero set is distinguishable, we have $\beta_1 \supset \beta_p \supset \beta_0$. Theorem 2.1 and Proposition 1.2(i) yield the following.

2.4 Corollary. $(C(X), \beta_p)' = M_p(X)$.

2.5. Proposition. (i) $\beta_1 = \beta_p$ *if and only if* $M_\sigma(X) = M_p(X)$ *and every compact subset of* $M_p^+(X)$ *is* β_p*-equicontinuous*; (ii) $\beta_p \subset \beta$ *if and only if* $M_p(X) \subset M_\tau(X)$; (iii) *if* X *is a Prohorov space and* $M_p(X) = M_t(X)$ *then* $\beta_0 = \beta_p$.

Proof. We give a proof only for the "if" part of (ii) since the rest is similar (see also [19, Theorem 5.8]). So we assume that $M_p(X) \subset M_\tau(X)$ and we show that every β_p-equicontinuous subset H of $M_p(X)$ is β-equicontinuous. By

Proposition 1.1, there is no loss of generality to assume that H consists of positive measures. Then $H \subset M_p^+(X) \subset M_\tau^+(X)$ is relatively compact and Proposition 1.2(ii) implies that H is β-equicontinuous.

Since β_p-equicontinuity is involved in Proposition 2.5 we mention the following.

2.6. PROPOSITION. *A subset H of $M_p(X)$ is β_p-equicontinuous if and only if* (a) *H is norm bounded, and* (b) *for every continuous function f from X onto a separable metric space Y and every $\varepsilon > 0$, there is a compact set $K \subset Y$ such that $|\mu|(X - f^{-1}(K)) < \varepsilon$ for all $\mu \in H$.*

Proof. H is β_p-equicontinuous if and only if H is β_G equicontinuous for every $G \in \mathscr{D}(\beta X)$, $G \subset \beta X - X$. The result follows then easily from Proposition 1.1.

Remarks. 1. Every compact distinguishable set is a zero set. In general, there are non-compact distinguishable sets in βX which are contained in $\beta X - X$. It follows that β_p, unlike β_1 and β, is not determined by a family of compact subsets of $\beta X - X$. Moreover, any strict topology $\bar{\beta}$ with $(C(X), \bar{\beta})' = M_p(X)$ cannot in general be determined by a family of compact sets. Indeed, if this happened, then we should have $M_\tau \subset M_p$. But this fails even when X is a separable metric space [18, p. 248].

2. If $X \subset Y \subset \beta X$, then $Y \in \mathscr{D}(\beta X)$ if and only if Y admits a perfect function onto a separable metric space (cf. [11, Remark D]) or, equivalently, Y is a Lindelöf M-space in the terminology of [15]. Therefore the topology β_p on $C(X)$ is the inductive limit of the topologies β_0 on $C(Y)$ for all Lindelöf M-spaces Y with $X \subset Y \subset \beta X$.

3. If Y is a Lindelöf M-space, then by [10, Corollary 9] every compact countable subset of $M_t^+(Y)$ is β_0-equicontinuous. Therefore Remark 2 implies that for any space X, every compact countable subset of $M_p^+(X)$ is β_p-equicontinuous.

4. The conclusion of Proposition 2.5(iii) may hold without X being Prohorov. Indeed, if X is a Lindelöf M-space then $\beta_0 = \beta_p$ by Remark 2. However, X is not necessarily Prohorov since there are separable metric spaces which are not Prohorov (see [16]). On the other hand, the assumption that X is Prohorov cannot be dropped. Indeed Varadarajan [22, p. 225] gives an example of a space Y which is countable and a convergent sequence $\{\mu_n\}$ in $M_t^+(Y)$ which is not β_0-equicontinuous. Then $M_p(Y) = M_t(Y)$ but $\beta_0 \neq \beta_p$ since the sequence $\{\mu_n\}$ is β_p-equicontinuous by Remark 3.

3. u-additive measures

In this section a strict topology on $C(X)$ is defined, which yields $M_u(X)$ as dual space. The idea to define such a strict topology using the notion of paracompactness is discussed in [20, p. 495] where a natural straightforward approach is proved to fail. However, using a property involved in a characterization [2, Theorem 4.4-c] of paracompactness, it is possible to define such a strict topology. The family of compact subsets of $\beta X - X$ which determines this topology is specified in the following lemma by several equivalences.

3.1. LEMMA. *For a compact set $K \subset \beta X - X$ the following are equivalent*:

(i) *There is a cozero cover $(U_\alpha)_{\alpha \in A}$ of X which is* (a) *locally finite,* (b) *σ-locally finite or* (c) *σ-discrete such that*

$$\mathrm{cl}_{\beta X}\, U_\alpha \cap K = \emptyset \text{ for all } \alpha \in A.$$

(ii) *There is a continuous function f from X onto a metric space Y such that $\bar{f}(K) \subset \beta Y - Y$, where $\bar{f}\colon \beta X \to \beta Y$ is the continuous extension of f.*

(iii) *There is a partition of unity $(f_\alpha)_{\alpha \in A}$ for X such that $\bar{f}_{\alpha|K} = 0$ for all $\alpha \in A$.*

(iv) *There is a partition of unity $(f_\alpha)_{\alpha \in A}$ for X and $0 < \varepsilon < 1$ such that*

$$\sum_{\alpha \in A} \bar{f}_\alpha(x) \leq 1 - \varepsilon \quad \text{for all } x \in K.$$

Proof. (i)$\Leftrightarrow$(ii) If X is paracompact, then every compact set $K \subset \beta X - X$ satisfies (i) (a)–(c) (cf. [2, Theorem 4.4-c)]). Since every metric space is paracompact, (ii) implies (i) (a)–(c). The converse follows from [2, Theorem 3.2].

(i)$\Rightarrow$(iii) Let $(U_\alpha)_{\alpha \in A}$ be a locally finite cozero cover of X such that

$$\mathrm{cl}_{\beta X}\, U_\alpha \cap K = \emptyset \quad \text{for all } \alpha \in A.$$

We choose $f_\alpha \in C(X), f_\alpha \geq 0$, such that $U_\alpha = \{x \in X : f_\alpha(x) > 0\}$. Since

$$K \subset \beta X - \mathrm{cl}_{\beta X}\, U_\alpha \subset \mathrm{cl}_{\beta X}\, (X - U_\alpha),$$

we have $\bar{f}_{\alpha|K} = 0$. So $\{f_\alpha \cdot (\sum_{\alpha \in A} f_\alpha)^{-1}\}_{\alpha \in A}$ is the desired partition of unity.

(iii)$\Rightarrow$(iv) Obvious.

(iv)$\Rightarrow$(i) For every finite $F \subset A$, let

$$V_F = \left\{x \in \beta X\colon \sum_{\alpha \in F} \bar{f}_\alpha(x) > 1 - \varepsilon\right\}.$$

Since V_F is a cozero set in βX, V_F is Lindelöf and every cozero set in V_F is also a cozero set in βX. Let $(U_{F,n})_{n \in \mathbf{N}}$ be a cozero cover of V_F with $\mathrm{cl}_{\beta X}\, U_{F,n} \cap K = \emptyset$ and let $\mathscr{V}_n = \{U_{F,n} \cap X\colon F \text{ finite}, F \subset A\}$, $n \in \mathbf{N}$. Since $(f_\alpha)_{\alpha \in A}$ is a partition of unity for X, $\bigcup_{n \in \mathbf{N}} \mathscr{V}_n$ is the desired σ-locally finite cozero cover of X.

Let $\mathscr{U}_X = \mathscr{U}$ be the family of all compact subsets of $\beta X - X$ satisfying any of the equivalent assertions of Lemma 3.1. Then we have:

3.2. THEOREM. *For a measure $\mu \in M(X)$ the following are equivalent:*

(i) *μ is u-additive;*
(ii) *$|\bar{\mu}|(K) = 0$ for all $K \in \mathcal{U}$.*

Proof. Without loss of generality we assume that μ is positive.

(i) $\Rightarrow$ (ii) Let $K \in \mathcal{U}$ and $\varepsilon > 0$. By Lemma 3.1(iii), there is a partition of unity $(f_\alpha)_{\alpha \in A}$ for X such that $\bar{f}_{\alpha|K} = 0$ for all $\alpha \in A$. Since μ is u-additive, there is a finite $F \subset A$ such that $\mu(1) - \varepsilon \leq \sum_{\alpha \in F} \mu(f_\alpha)$. Then

$$\bar{\mu}(\beta X) - \varepsilon = \bar{\mu}(1) - \varepsilon \leq \sum_{\alpha \in F} \bar{\mu}(\bar{f}_\alpha) = \bar{\mu}\left(\sum_{\alpha \in F} \bar{f}_\alpha\right) \leq \bar{\mu}(\beta X - K)$$

since $\sum_{\alpha \in F} \bar{f}_\alpha \leq \chi_{(\beta X - K)}$ (the characteristic function of $\beta X - K$). Therefore $\bar{\mu}(K) = 0$.

(ii) $\Rightarrow$ (i) Let $(f_\alpha)_{\alpha \in A}$ be a partition of unity for X and $0 < \varepsilon < 1$. For every finite $F \subset A$ we consider the set

$$Z_F = \left\{x \in \beta X\colon \sum_{\alpha \in F} \bar{f}_\alpha(x) \leq 1 - \varepsilon\right\}$$

and let K be the intersection of all Z_F. Then K is a compact subset of $\beta X - X$ and, by Lemma 3.1(iv), $K \in \mathcal{U}$. Therefore, $\inf_F \bar{\mu}(Z_F) = \bar{\mu}(K) = 0$ and

$$\begin{aligned}\mu(1) - \sum_{\alpha \in F} \mu(f_\alpha) &= \bar{\mu}\left(1 - \sum_{\alpha \in F} \bar{f}_\alpha\right)\\ &= \int_{Z_F} \left(1 - \sum_{\alpha \in F} \bar{f}_\alpha\right) d\bar{\mu} + \int_{\beta X - Z_F} \left(1 - \sum_{\alpha \in F} \bar{f}_\alpha\right) d\bar{\mu}\\ &\leq \bar{\mu}(Z_F) + \varepsilon \cdot \bar{\mu}(\beta X).\end{aligned}$$

It follows that $\mu(1) = \sum_{\alpha \in A} \mu(f_\alpha)$; that is, μ is u-additive.

Now the following corollary [8], [14], [20], [23] is an immediate consequence of Theorem 3.2 and the following characterization of paracompactness: X is paracompact if and only if every compact subset of $\beta X - X$ satisfies (i)(a) of Lemma 3.1 (see [2, Theorem 4.4-c]).

3.3. COROLLARY. *$M_u(X) = M_\tau(X)$ whenever X is paracompact.*

Proof. If X is paracompact, then, by the above, $\mathcal{U}$ is the family of all compact subsets of $\beta X - X$. So, if $\mu \in M_u(X)$ then $|\bar{\mu}|(K) = 0$ for all compact $K \subset \beta X - X$ which means that μ is τ-additive (Section 1).

We denote by θX the topological completion of X. We have $X \subset \theta X \subset \nu X$ and $X = \theta X$ if and only if X is topologically complete (see [2] or [6]).

3.4. COROLLARY. $M_u(X) = M_u(\theta X)$.

Proof. Since every $K \in \mathscr{U}$ doesn't meet θX (cf. [2, Theorem 4.4-d]), the conclusion follows from Theorem 3.2.

Next we define the strict topology β_u by

$$\beta_u = \text{Lin}\,\{\beta_K\colon K \in \mathscr{U}\}.$$

We have $\beta_1 \supset \beta_u \supset \beta$ because the corresponding families of subsets of $\beta X - X$ are related in the opposite direction. Theorem 3.2 and Proposition 1.2(i) yield the following.

3.5. COROLLARY. $(C(X), \beta_u)' = M_u(X)$.

In order to show that β_u coincides with the topology studied by Wheeler [23], we need the following.

3.6. PROPOSITION. *A subset H of $M_u(X)$ is β_u-equicontinuous if and only if* (a) *H is norm bounded, and* (b) *for every partition of unity $(f_\alpha)_{\alpha \in A}$ for X and every $\varepsilon > 0$ there is a finite set $F \subset A$ such that*

$$|\mu|\left(1 - \sum_{\alpha \in F} f_\alpha\right) < \varepsilon \textit{ for all } \mu \in H.$$

A proof of this proposition follows using arguments similar to those used in the proof of Theorem 5.2 in [19] and it is omitted. Notice that here we use the coincidence of the strict topology β_0 for a locally compact space with the original strict topology of Buck (determined by the seminorms mentioned in Section 1).

Proposition 3.6 and [20, Theorem 5.2] yield that every relatively countably compact subset of $M_u(X)$ is β_u-equicontinuous, that is, $(C(X), \beta_u)$ is a strong Mackey space. This shows that the topology β_e in [23] coincides with β_u. We note that also Mosiman has shown that β_e can be determined by a family of compact subsets of $\beta X - X$ [3, pp. 124, 139], but no proof of this result has been published.

As it is already mentioned, if X is paracompact then $\mathscr{U}$ is the family of all compact subsets of $\beta X - X$. Therefore the following becomes obvious.

3.7. COROLLARY [23, 3.8]. *If X is paracompact then $\beta = \beta_u$ and consequently $(C(X), \beta)$ is a strong Mackey space.*

It follows from the above corollary that for a paracompact space X the equality $\beta = \beta_0$ implies that $(C(X), \beta_0)$ is strong Mackey. I don't know whether the converse is true; that is, if X is paracompact and $(C(X), \beta_0)$ is strong Mackey, is it then true that $\beta = \beta_0$? Without the assumption that X is paracompact the answer is negative by an example of Haydon [9, 2.5]. However, at least for metric spaces the answer is affirmative.

3.8. PROPOSITION. *If X is a metric space, then $(C(X), \beta_0)$ is a strong Mackey space if and only if $\beta = \beta_0$.*

Proof. Clearly the "if' part follows from Corollary 3.7. Now assume that $(C(X), \beta_0)$ is strong Mackey. We have that $\beta_0 \subset \beta$ and β_0 is the finest locally convex topology on $C(X)$ which yields $M_t(X)$ as dual; so it is enougn to show that $M_\tau(X) = M_t(X)$. Suppose that this is not valid. Then there is $\mu \in M_\tau^+(X)$ with $\mu(\{x\}) = 0$ for all $x \in X$, which is not tight. By the τ-additivity, μ is concentrated on a closed separable set and, using [22, Part II, Theorem 23], we can find a sequence $\{\mu_n\}$ in $M_t(X)$ with $\mu_n \to 0$ and $|\mu_n| \to \mu$. Then $H = \{\mu_n: n = 1, 2, \ldots\}$ is relatively compact in $M_t(X)$ but not β_0-equicontinuous. This is a contradiction since $(C(X), \beta_0)$ is strong Mackey.

4. D_0-spaces

We denote by $\mathscr{D}$ the family of all continuous pseudometrics on a completely regular space X. If $d \in \mathscr{D}$ we set $\bar{x} = \{y \in X: d(x, y) = 0\}$ and $X_d = \{\bar{x}: x \in X\}$. Then X_d is a metric space by defining $\bar{d}(\bar{x}, \bar{y}) = d(x, y)$ and the function π_d: $X \to X_d$ with $\pi_d(x) = \bar{x}$ is continuous onto. A subset Y of X is *d-discrete* if there is an $\varepsilon > 0$ such that $d(x, y) \geq \varepsilon$ for all $x, y \in Y$, $x \neq y$.

Replacing cardinals of measure zero in the definition of D-spaces of Granirer [7] by non-(Ulam-) measurable cardinals, we say that a space X is a *D_0-space* if for every $d \in \mathscr{D}$ all d-discrete subsets of X have non-measurable cardinal. We recall that a cardinal m is (Ulam-) measurable if there is a non-zero $\{0, 1\}$-valued σ-additive measure defined on all subsets of m and vanishing on singletons.

Every D-space is a D_0-space and, under the continuum hypothesis, the two notions are identical by a well-known result of Ulam [21]. However, even the discrete space of cardinality $2^{\aleph_0}$ has not yet been proved to be a D-space without any set theoretical assumption. Such difficulties for D_0-spaces do not appear because of the large size of measurable cardinals.

In [8] Haydon proved that a space X is a D-space if and only if $M_\sigma(X) = M_u(X)$. For D_0-spaces we prove the following theorem which is the main result of this section.

4.1. THEOREM. *For any completely regular space X the following are equivalent*:

(i) $M_p(X) \subset M_u(X)$;
(ii) $vX = \theta X$;
(iii) *X is a D_0-space.*

The equivalence (ii) $\Leftrightarrow$ (iii) is a known strong form of Shirota's theorem (see [6, Theorem 5.21]) which is not used in our proof. In the essential direction (iii) $\Rightarrow$ (i) we use the following.

4.2. THEOREM [13, 2.5]. *Let* $(X, \mathscr{A}, \mu)$ *be a non-zero, positive, perfect measure space and* $\{A_i: i \in I\}$ *a partition of* X *such that* $\mu^*(A_i) = 0$ *for all* $i \in I$ *and the cardinal of* I *is non-measurable. Then there is* $J \subset I$ *such that* $\bigcup_{i \in J} A_i$ *is not* μ*-measurable.*

Proof of Theorem 4.1 (i) $\Rightarrow$ (ii) The non-zero $\{0, 1\}$-valued σ-additive (resp. u-additive) measures on X are precisely the points of vX (resp. θX). Also, every $\{0, 1\}$-valued σ-additive measure is perfect. Therefore, (i) implies that $vX \subset \theta X$, hence $vX = \theta X$.

(ii) $\Rightarrow$ (iii) Let Y be a d-discrete subset of X for some $d \in \mathscr{D}$. Since X_d is topologically complete, there is a continuous extension $\bar{\pi}_d$: $\theta X \to X_d$ of π_d. Now, d can be extended to a continuous pseudometric on θX by defining $\delta(x, y) = \bar{d}(\bar{\pi}_d(x), \bar{\pi}_d(y))$, $x, y \in \theta X$. Then Y is δ-discrete in θX which is realcompact by (ii), so Y is realcompact and discrete; therefore it has non-measurable cardinal [2, Theorem 2.6].

(iii) $\Rightarrow$ (i) By [20, Theorem 4.1], a measure $\mu \in M_\sigma^+(X)$ is u-additive if and only if for every σ-discrete cozero cover $\mathscr{V}$ of X, there is a countable $\mathscr{W} \subset \mathscr{V}$ such that $\mu(X) = \mu(\cup \mathscr{W})$. This result can also be deduced from Theorem 3.2 and Lemma 3.1 (i)(c). Now assume (for the purpose of a contradiction) that X is a D_0-space and (i) doesn't hold. Then there is a non-zero positive perfect measure μ on X and a cozero cover $\mathscr{V} = \bigcup_{n \in \mathbf{N}} \mathscr{V}_n$ of X such that each $\mathscr{V}_n$ is discrete and $\mu(V) = 0$ for all $V \in \mathscr{V}$.

We fix n and set $\mathscr{V}_n = \{V_i: i \in I\}$. For every $i \in I$ we choose $x_i \in V_i$ and a continuous function f_i: $X \to \mathbf{R}$ such that $f_i \geq 0, f_i(x_i) = 1$ and

$$V_i = \{x \in X : f_i(x) > 0\}.$$

Then $d(x, y) = \sum_{i \in I} | f_i(x) - f_i(y) |$ is a continuous pseudometric on X and $d(x_i, x_j) = 2$ for $i, j \in I$, $i \neq j$. So $\{x_i: i \in I\}$ is d-discrete and, by assumption, the cardinal of I is non-measurable. For every $J \subset I$ the function $f_J = \sum_{i \in J} f_i$ is continuous, so the set $\bigcup_{i \in J} V_i = \{x \in X : f_J(x) > 0\}$ is a cozero set. In particular, the union $\cup \mathscr{V}_n = \bigcup_{i \in I} V_i$ is measurable. Since μ is non-zero and positive we may assume that n has been chosen so that $\mu(\cup \mathscr{V}_n) > 0$.

Now let $v(B) = \mu(B \cap (\bigcup_{i \in I} V_i))$ for all $B \in \mathscr{B}(X)$. Then v is a non-zero positive perfect [18, 1.6] measure on X and

$$\{V_i: i \in I\} \cup \left\{X - \bigcup_{i \in I} V_i\right\}$$

is a partition of X into sets of v-measure zero. By Theorem 4.2, the union of some members of this partition is non-v-measurable. This is a contradiction since every such union is a Baire set.

We note that (ii) and (iii) of Theorem 4.1 are assumption about $\{0, 1\}$-valued measures. The equivalence of them, that is, the strong form of Shirota's theorem, can be stated as follows: every $\{0, 1\}$—valued (perfect) measure on a completely regular space is u-additive if and only if X is a D_0-space. So direction (iii) $\Rightarrow$ (i) can be considered as an extension of Shirota's theorem to real-valued measures.

The equivalence (i) $\Leftrightarrow$ (ii) implies immediately the following.

4.3. COROLLARY. *A topologically complete space X is realcompact if and only if $M_p(X) \subset M_u(X)$.*

Since every paracompact space is topologically complete (cf. [2, Theorem 4.4], the next corollary follows from 3.3 and 4.3.

4.4 COROLLARY [13, 5.11]. *A paracompact space X is realcompact if and only if $M_p(X) \subset M_\tau(X)$.*

Another consequence of the equivalence (i) $\Leftrightarrow$ (ii) of Theorem 4.1 is the following.

4.5. COROLLARY. *For any space X, $M_p(X) \subset M_u(vX)$.*

Proof. For any realcompact space Y, we have $M_p(Y) \subset M_u(Y)$ by Theorem 4.1 (ii) $\Rightarrow$ (i) since $Y = vY = \theta Y$. For $Y = vX$, using Corollary 2.3 we conclude that $M_p(X) = M_p(vX) \subset M_u(vX)$.

We notice that Corollary 4.5 also implies Theorem 4.1 (ii) $\Rightarrow$ (i); for, if $\theta X = vX$ and $M_p(X) \subset M_u(vX)$, then $M_p(X) \subset M_u(\theta X) = M_u(X)$ by Corollary 3.4.

Finally, we give some other characterizations of D_0-spaces.

4.6. PROPOSITION. *For any space X the following are equivalent:*

(i) *X is a D_0-space;*
(ii) *X_d is realcompact for every $d \in \mathscr{D}$;*
(iii) *$\beta_p \subset \beta_u$.*

Proof. (i) $\Rightarrow$ (ii) For every $d \in \mathscr{D}$, X_d is a D_0-space as a continuous image of a D_0-space. Since X_d is also topologically complete, Theorem 4.1 (iii) $\Rightarrow$ (ii) implies that X_d is realcompact.

(ii) $\Rightarrow$ (i) Let Y be a d-discrete subset of X for some $d \in \mathscr{D}$. Then $\pi_d(Y)$ is closed and discrete in the realcompact space X_d; so the cardinal of $\pi_d(Y)$ which is equal to the cardinal of Y is non-measurable.

(i) $\Leftrightarrow$ (iii) As in the proof of Proposition 2.5(ii) we can show that $\beta_p \subset \beta_u$ if and only if $M_p(X) \subset M_u(X)$. So the equivalence (i) $\Leftrightarrow$ (iii) follows from Theorem 4.1 (i) $\Leftrightarrow$ (iii).

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