THE MINIMAL PRIME SPECTRUM OF A REDUCED RING

BY

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Introduction

Throughout this discussion $R$ will be a commutative ring with 1. We say $R$ is a reduced ring if it has no nilpotent elements other than 0. Of course, this is equivalent to saying that the intersection of the minimal prime ideals of $R$ is 0. The purpose of this paper is to study $\text{min } R$, the minimal prime spectrum of $R$, in order to obtain information about $Q(R)$, the classical ring of quotients of $R$; and $E(R)$, the injective envelope of $R$; as well as other properties of $R$.

Some of this information is already known. Thus in order to present more detailed results, a good deal of background information has to be used, imposing a severe strain on the general reader unfamiliar with the subject. Further compounding the problem is that much of the information is scattered wholesale about the literature. An even deeper difficulty is that this information, while relatively elementary in character, is usually thrown off as pieces of debris from general construction in the theory of non-commutative rings, or category and sheaf theory, so that no easy route to the subject is available.

In order to overcome these problems we shall present statements and proofs of most relevant facts about a reduced ring and its minimal prime spectrum including folklore and elementary exercises, as well as the work of other authors, giving attributions only for the deeper results.

In §1 we give some of the necessary background material. Because we are interested only in commutative rings, and specifically reduced rings, much of this material has been greatly simplified. We conclude §1 with an interesting contrast between reduced rings and non-reduced Noetherian rings.

The notion of a Von-Neumann regular ring, VNR, plays a key role in the subject. The definition that we use (among the many possible equivalent definitions) is that every principal ideal is a direct summand of the ring. This definition (in contrast to the definition of a semi-simple ring as a ring in which every ideal is a direct summand of the ring) shows that it is the set of principal ideals that matters. This definition gives rise immediately

Received June 1, 1981.
to two successively weaker ones: a commutative ring is a PIP if every principal ideal is projective; and it is a PIF if every principal ideal is flat. All of these rings are reduced rings.

In §2 we review the list of known equivalent conditions for a PIF to be a PIP, and add a few of our own. This problem dates back to the problem of finding necessary and sufficient conditions for a ring of weak global dimension 1 to be a semi-hereditary ring; and has been worked on by Hattori, Endo, Vasconcelos, and Quentel. One of the surprising results we prove is that a PIF ring $R$ is a PIP if and only if $C_M$ is a divisible $R_M$-module for all injective $R$-modules $C$ and maximal ideals $M$ of $R$. This provides new examples of rings whose injective modules do not localize.

Looking at the problem one ideal at a time, we find necessary and sufficient conditions for a principal ideal of a reduced ring (or a PIF) to be a projective ideal. Since a principal ideal is projective if and only if its annihilator is a direct summand of the ring, we analyze the conditions for an ideal to be a direct summand of $R$ in terms of properties of subsets of $\min R$.

If $R$ is a reduced ring, one of the known theorems about $E(R)$ is that it is a self-injective VNR. If $P \in \min R$, then $E(P)$ is a field; and hence $E(R)$ is a direct summand of $\Pi P_P$ ($P \in \min R$). We prove in §3 that $E(R)$ is a subdirect product of the $E(P)$’s. There are examples where $E(R) = \Pi P_P$ ($P \in \min R$); and others where $E(R) = \Pi P_{P}$ ($P_{P} \in \Gamma \subseteq _{\pi} \min R$). Thus it is clear that the structure of $E(R)$ can be quite complex.

If $R$ is a reduced ring and $\min R = \{P_{1}, \ldots, P_{n}\}$ is finite, then

$$Q(R) = R_{P_{1}} \oplus \cdots \oplus R_{P_{n}} \simeq E(R).$$

There are two interesting generalizations of this theorem. On the one hand Quentel and others have shown (see Proposition 1.16) that $\min R$ is compact if and only if $E(R)$ is flat. On the other hand we show that $\min R$ is totally disconnected if and only if $E(R) = \Pi P_P$ ($P \in \min R$) (see Proposition 3.5).

In §3 we prove that if $\{P_{P}\}, \beta \in B$, is a subset of $\min R$, then $E(R) = \Pi P_{P}$ ($P \subseteq B$) if and only if the $P_{P}$’s are distinct, every $P_{P}$ is a non-essential ideal of $R$, and $\cap P_{P} = 0$. In this case $\{P_{P}\}, \beta \in B$, is the set of all non-essential minimal prime ideals of $R$. Hence the decomposition (if it exists) is unique.

In the general case, if $\{P_{P}\}, \beta \in B$, is the set of all distinct non-essential minimal prime ideals of $R$, and we let $I = \cap P_{P}$, then $E(R/I) = \Pi P_{P}$; $E(R/I)$ is a ring direct summand of $E(R)$; the complementary summand is $E(R/K)$, where $K = \text{ann}_{R} I$; and $R/K$ is a reduced ring with no non-essential minimal prime ideals. Furthermore, there is a 1-1 correspondence between the non-essential minimal prime ideals of $E(R)$ and those of $R$ given by contraction; and the corresponding localizations are isomorphic.

In §4 we give a number of examples to show that the general theorems of this paper provide efficient methods of deciding whether or not a ring is a PIF, or a PIP, and also of computing $E(R)$. In particular, we produce an example of a ring $R = K[[x]]$ (where $K$ is a hereditary VNR) that is
not a PIF; a fortiori, not a ring of w.gl.dim 1. Nevertheless, $Q(R)$ is a VNR, $E(R)$ is a direct product of copies of $k((x))$ where $k$ is a field, and $R$ is an essential extension of a semi-hereditary ring.

1. Preliminaries

**Definition.** Let $R$ be a commutative ring and $A$ an $R$-module. We shall let $E(A)$ denote the injective envelope of $A$.

**Proposition 1.1.** Let $R$ be a reduced ring, and let $\{P_\alpha\}, \alpha \subseteq \mathcal{A}$, be the set of minimal prime ideals of $R$.

1. $R_{P_\alpha}$ is the quotient field of $R/P_\alpha$, and hence is an injective $R$-module.
2. $E(R)$ is a direct summand of $\Pi_{\alpha}R_{P_\alpha}$.
3. $\bigcup_\alpha P_\alpha$ is the set of all zero divisors of $R$.

**Proof.** (1) Let $O_\alpha = \{r \in R|ur = 0 \text{ for some } u \in R - P_\alpha\}$. Then $O_\alpha$ is an ideal of $R$ and $O_\alpha \subseteq P_\alpha$. Since $P_\alpha R_{P_\alpha}$ is the only prime ideal of $R_{P_\alpha}$, every element of $P_\alpha R_{P_\alpha}$ is nilpotent. Thus if $p \in P$, there exists $u \in R - P_\alpha$ and $n > 0$ such that $up^n = 0$. Hence $(up)^n = 0$, and since $R$ is reduced, $up = 0$. Thus $O_\alpha = P_\alpha$, and hence $P_\alpha R_{P_\alpha} = 0$. Therefore, $R_{P_\alpha}$ is the quotient field of $R/P_\alpha$, and since $R_{P_\alpha}$ is a flat $R$-module, $R_{P_\alpha}$ is an injective $R$-module.

(2) It follows from (1) that $\Pi_{\alpha}R_{P_\alpha}$ is an injective $R$-module; and that the canonical map $R \rightarrow \Pi_{\alpha}R_{P_\alpha}$ has kernel equal to $\cap P_\alpha = 0$, and hence is a monomorphism. Thus the canonical map extends to a monomorphism: $E(R) \rightarrow \Pi_{\alpha}R_{P_\alpha}$.

(3) It follows from (1) that every element of $\bigcup_\alpha P_\alpha$ is a zero divisor in $R$. Conversely, let $x \in R$, $x \neq 0$ be a zero divisor in $R$. Then there exists $y \in R$, $y \neq 0$ such that $xy = 0$. Since $\cap P_\alpha = 0$, there exists $P_\beta$ such that $y \notin P_\beta$, and hence $x \in P_\beta$.

**Definition.** Let $A$ be a subset of an $R$-module $B$. Then define $\text{Ann}_R A = \{r \in R|rA = 0\}$.

**Proposition 1.2.** Let $R$ be a reduced ring.

1. A prime ideal $P$ of $R$ is a minimal prime ideal of $R$ iff for all $x \in P$, $\text{Ann}_R x \not\subseteq P$.
2. Let $J$ be a finitely generated ideal of $R$. Then $J$ is contained in a minimal prime ideal of $R$ if and only if $\text{Ann}_R J \neq 0$.
3. If $x \in R$ and $y \in \text{Ann}_R x$, then $\text{Ann}_R (Rx + Ry) = 0$ iff $x - y$ is not a zero divisor in $R$.

**Proof.** (1) If $P$ is a minimal prime ideal of $R$ and $x \in P$, then by Proposition 1.1(1), there exists $u \in R - P_\alpha$ such that $ux = 0$. Conversely,
suppose that for all $x \in P$, $\text{Ann}_R x \not\subset P$. Suppose $P_1$ is a prime ideal of $R$ and $P_1 \subset P$. Then there exists $x \in P - P_1$, and hence $\text{Ann}_R x \subset P_1 \subset P$. This contradiction shows that $P$ is a minimal prime ideal of $R$.

(2) Let $J = Ra_1 + \cdots + Ra_n$, and let $I = \text{Ann}_R J$. Suppose that $J \subset P$, a minimal prime ideal of $R$. Then by Proposition 1.1(1), there exist elements $u_i \in R - P$ such that $u_ia_i = 0$ for all $i = 1, \ldots, n$. Let $u = u_1u_2 \ldots u_n$; then $u \not\in P$ and $u \in I$. Conversely, suppose that $I \neq 0$. Then there is a minimal prime ideal $P$ of $R$ such that $I \not\subset P$, and hence $J \subset P$.

(3) Assume that $\text{Ann}_R (Rx + Ry) = 0$, and suppose that $t \in R$ and $t(x - y) = 0$. Then $tx = ty$ and hence $(tx)^2 = 0$. Therefore, $tx = 0 = ty$ and hence $t \in \text{Ann}_R (Rx + Ry) = 0$. Thus $x - y$ is not a zero divisor in $R$. The converse assertion is trivial.

**DEFINITION.** A commutative ring $R$ is said to be a Von-Neumann regular ring (VNR) if every principal ideal of $R$ is a direct summand of $R$; i.e., is generated by an idempotent of $R$.

**PROPOSITION 1.3 [8, Theorem 1.16].** Let $R$ be a commutative ring. Then $R$ is a VNR iff $R$ is reduced and every prime ideal of $R$ is minimal. In this case every ideal of $R$ is an intersection of prime ideals of $R$.

**Proof.** Assume that $R$ is a VNR. Let $I$ be an ideal of $R$ and $x$ an element of $R$ such that $x^n \in I$ for some $n > 0$. Since $Rx = Re$, where $e^2 = e$, we have $x \in I$, showing that $I$ is an intersection of prime ideals of $R$. In particular, taking $I = 0$, we see that $R$ is reduced. Now let $I = P$ be a prime ideal of $R$. Then $1 - e \in \text{Ann}_R x$ and $1 - e \not\in P$. Hence by Proposition 1.2(1), $P$ is a minimal prime ideal of $R$.

Conversely, suppose that $R$ is reduced and that every prime ideal of $R$ is minimal. Let $0 \neq x \in R$ and $I = \text{Ann}_R x$. Since $R$ is reduced, $Rx \cap I = 0$; and by Proposition 1.2(1), $Rx + I$ is not contained in any minimal prime ideal of $R$. Therefore, $Rx + I = R$, $Rx$ is a direct summand of $R$; and hence $R$ is a VNR.

**DEFINITION.** Let $R$ be a commutative ring and let $S$ be the set of non-zero divisors of $R$. Then $R_S$ is the classical ring of quotients of $R$, and we shall denote it by $Q(R)$.

**PROPOSITION 1.4 [12, Proposition 9].** The following statements are equivalent:

1. $Q(R)$ is a VNR.
2. If $I$ is an ideal of $R$ contained in the union of the minimal prime ideals of $R$, then $I$ is contained in one of them.
3. If $J$ is a finitely generated ideal of $R$, then there exist $b \in J$ and $a \in \text{Ann}_R J$ such that $a + b$ is not a zero divisor in $R$. 

**Proof.**
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(4) If \( b \in R \) then there exists \( a \in \text{Ann}_R b \) such that \( \text{Ann}_R(Ra + Rb) = 0 \).

Proof. \( (1) \Rightarrow (2) \) Suppose that \( I \) is contained in the union of the minimal prime ideals of \( R \). Then by Proposition 1.1(3), every element of \( I \) is a zero divisor in \( R \). Thus \( QI \neq Q \), and so \( QI \subseteq \mathcal{P} \), a maximal ideal of \( Q \); and \( I \subseteq \mathcal{P} \cap R \). By Proposition 1.3, \( \mathcal{P} \) is a minimal prime ideal of \( Q \); and since \( Q \) is a localization of \( R \), \( \mathcal{P} \cap R \) is a minimal prime ideal of \( R \).

(2) \( \Rightarrow (3) \) Let \( J = Rb_1 + \cdots + Rb_n \) be a finitely generated ideal of \( R \), and \( I = \text{Ann}_R J \). Suppose that there do not exist elements \( b \in J \) and \( a \in I \) such that \( a + b \) is a non-zero divisor in \( R \). Then \( I + J \) is contained in the union of the minimal prime ideals of \( R \), and hence by hypothesis, there exists a minimal prime \( P \) of \( R \) such that \( I + J \subseteq P \). By Proposition 1.2(1), there exists \( c_i \in \text{Ann}_R b_i \) with \( c_i \notin P \). Let \( c = c_1c_2 \cdots c_n \); then \( c \in I \) and \( c \notin P \). This contradiction proves that there exists \( b \in J \) and \( a \in I \) where \( a + b \) is a non-zero divisor in \( R \).

(3) \( \Rightarrow (4) \) This follows from Proposition 1.2(3).

(4) \( \Rightarrow (1) \) Let \( q \in Q = Q(R) \); then there exists \( b \in R \) with \( Qq = Qb \).

Proof. \( (P_1)_S = \cdots = (P_n)_S \) is the set of all prime ideals of \( R_S \), and each of them is both maximal and minimal in \( R_S \). Moreover, \( (R_S)_S \approx R_{P_i} \) for \( i = 1, \ldots, n \). Thus without loss of generality we can assume that \( \{P_1, \ldots, P_n\} \) is the set of all prime ideals of \( R \), and that each of them is both maximal and minimal in \( R \).

Let \( O_i = \{r \in R \mid \text{there exists } u \in R - P_i \text{ with } ur = 0\} \). Since \( P_iR_{P_i} \) is the only minimal prime ideal of \( R_{P_i} \), every element of \( P_i \) is nilpotent modulo \( O_i \). Thus \( P_i \) is the only prime ideal of \( R \) containing \( O_i \). Therefore \( R/O_i = R_{P_i}, \) \( i = 1, \ldots, n \); and \( O_i + O_j = R, \) \( i \neq j \). The annihilator of an element of \( \bigcap_{i=1}^n O_i \) is not contained in any maximal ideal of \( R \) and thus \( \bigcap_{i=1}^n O_i = 0 \). Hence by the Chinese Remainder Theorem,

\[
R \approx R/O_1 \oplus \cdots \oplus R/O_n = R_{P_1} \oplus \cdots \oplus R_{P_n}.
\]

PROPOSITION 1.6. \( R \) is a reduced ring with only a finite number of minimal prime ideals \( \{P_1, \ldots, P_n\} \). Then \( Q(R) \approx R_{P_1} \oplus \cdots \oplus R_{P_n} \approx E(R) \). Hence \( Q(R) \) is a self-injective VNR (in fact a semi-simple ring), and \( E(R) \) is a flat \( R \)-module.
Proof. By Proposition 1.5, $\mathbb{Q}(R) = R_{P_1} \oplus \cdots \oplus R_{P_n}$. Thus by Proposition 1.1 $R_{P_i}$ is a field, and $\mathbb{Q}(R)$ is an injective $R$-module. Since $\mathbb{Q}(R)$ is an essential extension of $R$, we have $\mathbb{Q}(R) \cong E(R)$.

**Proposition 1.7.** If $R$ is a reduced, self-injective ring, then $R$ is a VNR. In this case if $J$ is any ideal of $R$ and $I = \text{Ann}_R J$, then $R = E(J) \oplus I$.

**Proof.** Since $R$ is reduced, $I \cap J = 0$; in addition, $J \oplus I \subset R$ is an essential extension. Thus $R = E(R) = E(J \oplus I) = E(J) \oplus E(I)$. Now $E(I)J \subset E(I) \cap E(J) = 0$, and thus $E(I) \subset I$. Hence $I = E(I)$, and $R = E(J) \oplus I$.

Now suppose that $J = Ra$, $a \in R$. Then by the preceding paragraph we have $I = Re$, where $e^2 = e$; and hence $Ra \subset E(Ra) = R(1 - e)$. Since $\text{Ann}_R (1 - e) = Re = \text{Ann}_R a$, there is an $R$-homomorphism $f : Ra \to R(1 - e)$ with $f(a) = 1 - e$. Since $R$ is self-injective, $f$ extends to an $R$-homomorphism from $R$ into $R$. Thus there exists $t \in R$ such that $1 - e = f(a) = ta$. Therefore, $R(1 - e) \subset Ra$, and so $R(1 - e) = Ra$. Thus $Ra$ is a direct summand of $R$ and hence $R$ is a VNR.

**Proposition 1.8.** Let $R$ be a VNR and let $\{x_\gamma\}$, $\gamma \in \Gamma$, be a set of generators for an ideal $I$ of $R$. If $f : I \to R$ is an $R$-homomorphism, then there exists a set of elements $\{a_\gamma\}$, $\gamma \in \Gamma$, in $R$ such that $f(x_\gamma) = a_\gamma x_\gamma$ for $\gamma \in \Gamma$; and the system of congruences

$$y \equiv a_\gamma \mod(\text{Ann}_R x_\gamma), \quad \gamma \in \Gamma,$$

is finitely solvable. Conversely, if the system is finitely solvable, then there is an $R$-homomorphism $f : I \to R$ so that $f(x_\gamma) = a_\gamma x_\gamma$, $\gamma \in \Gamma$.

**Proof.** Let $f : I \to R$ be an $R$-homomorphism; and let $x \in I$. Since $R$ is a VNR, we have $Rx = Rx^2$, and thus there is an $a \in I$ with $f(x) = ax$. In particular there is a set of elements $\{a_\gamma\}$, $\gamma \in \Gamma$, in $I$ with $f(x_\gamma) = a_\gamma x_\gamma$, $\gamma \in \Gamma$. Let $\{x_{\gamma_1}, \ldots, x_{\gamma_n}\}$ be any finite subset of the generators $\{x_\gamma\}$. Then there is an $x \in I$ with $Rx_{\gamma_1} + \cdots + Rx_{\gamma_n} = Rx$; and hence there are elements $s_i \in R$ with $x_{\gamma_i} = s_i x$, $i = 1, \ldots, n$. Since $f(x) = ax$, we have

$$a_{\gamma_i} x_{\gamma_i} = f(x_{\gamma_i}) = s_i f(x) = a s_i x = ax_{\gamma_i}.$$

Therefore $a = a_{\gamma_i} \mod(\text{Ann}_R x_{\gamma_i})$, and the system of congruences $y \equiv a_\gamma \mod(\text{Ann}_R x_\gamma)$ is finitely solvable.

Conversely, suppose that the system is finitely solvable, and define $f : I \to R$ by $f(x_\gamma) = a_\gamma x_\gamma$, and extend $f$ linearly to all of $I$. With the notations of the preceding paragraph, suppose that $\Sigma_{i=0}^n r_i x_{\gamma_i} = 0$, where $r_i \in R$. Since $a_\gamma x_{\gamma_i} = a x_{\gamma_i}$, we have $\Sigma r_i a_\gamma x_{\gamma_i} = a \Sigma r_i x_{\gamma_i} = 0$. Thus, $f$ is a well defined $R$-homomorphism.
Definition. Let $R$ be a commutative ring with $1$, and
\[ y \equiv a_\gamma \pmod{I_\gamma}, \quad \gamma \in \Gamma, \]
a system of congruences where the $a_\gamma$'s are elements of $R$ and the $I_\gamma$'s are principal ideals of $R$. If every such system that is finitely solvable has a simultaneous solution, we shall say that $R$ is linearly compact on principal ideals.

Proposition 1.9. Let $R$ be a reduced ring. Then $R$ is self-injective iff $R$ is a VNR and is linearly compact on principal ideals.

Proof. By Proposition 1.7 we can assume that $R$ is a VNR. Then the principal ideals of $R$ are exactly the annihilators of elements of $R$. Now $R$ is self-injective iff every $R$-homomorphism from an ideal $I$ of $R$ can be realized by multiplication by an element $a \in R$. By Proposition 1.9 the $R$-homomorphisms from $I$ into $R$ arise from finitely solvable systems of congruences $y \equiv a_\gamma \pmod{\text{Ann}_R x_\gamma}$ where the $a_\gamma$'s are in $R$ and the $x_\gamma$'s generate $I$. It is immediate that $f$ is multiplication by an element $a \in R$ iff $a$ is a simultaneous solution of the system of congruences. Therefore, $R$ is self-injective iff $R$ is linearly compact on principal ideals.

Definition. Let $R$ be a commutative ring with $1$, $E = E(R)$, and $H = \text{Hom}_R(E, E)$. Let
\[ \mathcal{J} = \{ f \in H | f(1) = 0 \}. \]
Then $\mathcal{J}$ is a left ideal of $H$. Define $\phi : H \to E$ by $\phi(h) = h(1)$, $h \in H$. Then $\phi$ is an $H$-homomorphism of $H$ onto $E$ with $\text{Ker} \phi = \mathcal{J}$. Thus $E = H/\mathcal{J}$ is a cyclic $H$-module with generator $1 \in E$. We have $\phi(1) = 1$, where $I$ is the identity map on $E$. Since $E$ is a faithful $R$-module we have a canonical injection $R \subset H$ sending $1$ to $I$. If $Q = Q(R)$, then it is readily seen that $E$ is the $Q$-injective envelope of $Q$; and that the injection $R \subset E$ extends to an injection $Q \subset E$.

Proposition 1.10 [10, Proposition 3, p. 95]. Let $R$ be a commutative ring. Then the following statements are equivalent:

1. $\mathcal{J}$ is a two-sided ideal of $H$.
2. $\mathcal{J} = 0$.
3. $H \cong E$ as $H$-modules.
4. $E$ is a projective $H$-module.
5. $H$ is a commutative ring.

Proof. The implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (4), and (5) $\Rightarrow$ (1) are trivial. For (2) $\Rightarrow$ (3) we observe that $\phi$ is then an isomorphism. And for (1) $\Rightarrow$...
(2), let \( f \in \mathcal{I} \) and \( x \in E \). Since there is an \( h \in H \) with \( h(1) = x \), and \( fh \in \mathcal{I} \), we have \( f(x) = 0 \). Thus \( f = 0 \) and hence \( \mathcal{I} = 0 \).

(4) \( \Rightarrow \) (2) Since \( \phi \) is onto and \( E \) is projective, there is an \( H \)-homomorphism \( \lambda : E \to H \) with \( \phi \lambda = I \). Let \( g = \lambda(1) \); then \( I(1) = \phi \lambda(1) = \phi(g) = g(1) \). Thus \( \ker g \cap R = 0 \), and since \( E \) is an essential extension of \( R \), we have \( \ker g = 0 \). But then \( \text{Im } g \) is an injective \( R \)-module containing \( R \), and hence \( \text{Im } g = E \). Thus \( g \) is an isomorphism. Hence \( I = g^{-1}g = g^{-1}(\lambda(1)) = \lambda(g^{-1}(1)) \) and thus \( I \in \text{Im } \lambda \). Since \( \text{Im } \lambda \) is a left ideal of \( H \), we have \( \text{Im } \lambda = H \). Thus, if \( f \in \mathcal{J} = \ker \phi \), then there is an \( x \in E \) with \( f(x) = \lambda(x) \), and hence \( 0 = \phi(f) = \phi \lambda(x) = x \). Therefore, \( f = \lambda(0) = 0 \), and so \( \mathcal{J} = 0 \).

(2) \( \Rightarrow \) (5) Let \( h \in H \) and \( h(1) = x \in E \). Define \( h_{x} : E \to E \) as follows: if \( y \in E \), then there is a unique \( g \in H \) with \( g(1) = y \) because \( \mathcal{J} = 0 \). We define \( h_{x}(y) = g(x) \). It is obvious that \( h_{x} \) is an Abelian group homomorphism of \( E \) into \( E \). Let \( k \in H \); then \( kg \) is the unique element of \( H \) such that \( (kg)(1) = k(y) \). Hence \( h_{x}(ky) = kg(x) = kh_{x}(y) \). Thus \( h_{x} = kh_{x}, k \in H \). Therefore \( h_{x} \in H \); and in fact \( h_{x} \) is in the center of \( H \). Now \( h_{x}(1) = I(x) = x = h(1) \). Hence \( h_{x} - h \in \mathcal{J} = 0 \). Thus \( h_{x} = h \), and so \( h \) is in the center of \( H \). Thus \( H \) is a commutative ring.

**Definition.** Let \( B \) be a submodule of an \( R \)-module \( A \), and let \( x \in A \). Then we define

\[ (B : x) = \{ r \in R | rx \in B \}. \]

We say that \( B \) is an essential submodule of \( A \) (or \( A \) an essential extension of \( B \)) if every non-zero submodule of \( A \) has a non-zero intersection with \( B \).

**Proposition 1.11.** Let \( R \) be a commutative ring, \( E = E(R) \), and \( H = \text{Hom}_{R}(E, E) \). Let \( x \in E \); then \( (R : x) \) is an essential ideal of \( R \). Moreover, \( \exists f \in H \) \( f(1) = 0 \) and \( f(x) \neq 0 \) if \( \text{Ann}_{R}(R : x) \neq 0 \).

**Proof.** Let \( r \in R \); if \( rx = 0 \), then \( r \in (R : x) \); while if \( rx \neq 0 \), then there is a \( t \in R \) with \( 0 \neq trx \in R \), and hence \( 0 \neq tr \in (R : x) \). Therefore \( (R : x) \) is an essential ideal of \( R \).

If there is an \( f \in H \) with \( f(1) = 0 \) and \( f(x) \neq 0 \), then there is an \( a \in R \) such that \( af(x) = s \in R \) and \( s \neq 0 \). Clearly \( s \in \text{Ann}_{R}(R : x) \). Conversely, if there is \( 0 \neq s \in \text{Ann}_{R}(R : x) \), then there is an \( R \)-homomorphism \( g : R + Rx \to Rs \) with \( g(1) = 0 \) and \( g(x) = s \). Because \( E \) is injective, \( g \) extends to an element of \( f \in H \).

**Proposition 1.12 [10, Proposition 1, p. 102].** Let \( R \) be a reduced ring and \( E = E(R) \). Then \( E \) is a commutative, self-injective VNR, and \( \text{Hom}_{R}(E, E) = \text{Hom}_{E}(E, E) = E \).
Proof. Let \( H = \text{Hom}_R(E, E) \), and suppose that there is an \( f \in H \) and \( x \in E \) with \( f(1) = 0 \) and \( f(x) \neq 0 \). By Proposition 1.11, \( \text{Ann}_R(R : x) \neq 0 \) and \((R : x)\) is an essential ideal of \( R \). But then \((R : x) \cap \text{Ann}_R(R : x) \neq 0\), contradicting the fact that \( R \) is a reduced ring. Hence the map \( \phi : H \to E \) defined by \( \phi(h) = h(1), h \in H \), is an \( H \)-isomorphism. Therefore, by Proposition 1.10, \( H \) is a commutative ring extension of \( R \). Since \( \phi \) is the identity on \( R \), we can use \( \phi \) to give \( E \) the structure of a commutative ring extension of \( R \) such that \( \text{Hom}_R(E, E) = \text{Hom}_R(E, E) \).

Let \( x \in E, x \neq 0 \), and suppose that \( x^2 = 0 \). Since \( E \) is an essential extension of \( R \) there exists \( r \in R \) with \( 0 \neq rx \in R \). But \((rx)^2 = r^2x^2 = 0\); and this contradiction shows that \( E \) is a reduced ring.

Let \( \mathscr{I} \) be an ideal of \( E \) and \( f : \mathscr{I} \to E \) and \( E \)-homomorphism. Because \( E \) is \( R \)-injective, \( f \) extends to an element \( g \in H \). But then \( g \) is an \( E \)-homomorphism, and hence \( E \) is a self-injective ring. Thus by Proposition 1.9, \( E \) is a VNR.

Remarks. Let \( R \) be a reduced ring and \( E = E(R) \). Since \( \text{Hom}_R(E, E) = \text{Hom}_R(E, E) \) \( E \) is a commutative ring extension of \( R \), it follows readily that if \( A \) is another injective envelope of \( R \) with a commutative ring structure extending that of \( R \), and if \( \theta : E \to A \) is an \( R \)-homomorphism that is the identity on \( R \), then \( \theta \) is a ring isomorphism.

**Proposition 1.13.** Let \( R \) be a reduced ring and \( E = E(R) \). Suppose that \( A \) and \( B \) are \( R \)-submodules of \( E \) such that \( E = A \oplus B \). Then \( A \) and \( B \) are ideals of \( E \), and \( \text{Hom}_R(A, B) = 0 \).

**Proof.** Let \( f \) be the element of \( \text{Hom}_R(E, E) \) that is 0 on \( A \) and the identity on \( B \). Then by Proposition 1.12, \( f \) is multiplication by \( e \in E \) and \( e^2 = e \). Thus \( B = Ee \) and \( A = E(1 - e) \); and hence \( A \) and \( B \) are ideals of \( E \). If \( g \in \text{Hom}_R(A, B) \), define \( h \in \text{Hom}_R(E, E) \) to be \( g \) on \( A \) and 0 on \( B \). Then \( h \) is multiplication by \( y \in E \), and hence \( g(A) = yA \subseteq A \cap B = 0 \). Thus \( g = 0 \).

**Definition.** Let \( R \) be a reduced ring and let \( \text{min} R \) be the minimal prime spectrum of \( R \). If \( x \in R \), define \( D(x) = \{P \in \text{min} R | x \notin P \} \). Then the sets of the form \( D(x) \) form a basis for the Zariski topology on \( \text{min} R \). When we say that \( \text{min} R \) is compact, we mean that it is compact in this topology.

**Proposition 1.14** [12, Lemma 1]. Let \( R \) be a reduced ring, and let \( A \) be a commutative ring extension of \( R \).

1. If every prime ideal of \( A \) contracts to a minimal prime ideal of \( R \), then \( \text{min} R \) is compact.

2. Assume that \( A \) is a VNR. Then \( A \) is a flat \( R \)-module iff every prime ideal of \( A \) contracts to a minimal prime ideal of \( R \). Hence in this case \( \text{min} R \) is compact.
Proof. (1) Suppose that we have an open cover of \( \text{min } R \). Then without loss of generality we can assume it is of the form \( \text{min } R = \bigcup_{\lambda} D(x_{\lambda}), \lambda \in \Lambda, x_{\lambda} \in R \). Let

\[ D_A(x_{\lambda}) = \{ \mathcal{P} \in A | x_{\lambda} \notin \mathcal{P} \}; \]

then as a consequence of our hypothesis we have \( \text{spec } A = \bigcup_{\lambda} D_A(x_{\lambda}) \). Since the spec of any commutative ring is compact, there exist \( x_{\lambda_1}, \ldots, x_{\lambda_n} \) such that

\[ \text{spec } A = \bigcup_{i=1}^n D_A(x_{\lambda_i}). \]

Let \( P \in \text{min } R \); since \( R_P \subset A_P \), and \( R_P \) is a field, it is easily seen that there is a prime ideal \( \mathcal{P} \) of \( A \) with \( \mathcal{P} \cap R = P \). It follows from this that \( \text{min } R = \bigcup_{i=1}^n D(x_{\lambda_i}) \). Thus \( \text{min } R \) is compact.

(2) Assume that \( A \) is a flat \( R \)-module. Let \( \mathcal{P} \) be a prime ideal of \( A \), and let \( P = \mathcal{P} \cap R \), and suppose that there is a prime ideal \( P_1 \) of \( R \), \( P_1 \subset \mathcal{P} \). Then there is a \( p \in P - P_1 \), and we have an exact sequence

\[ 0 \to R/P \to A/P \to A/P_1. \]

Since \( A \) is flat over \( R \), we have an exact sequence

\[ 0 \to A/P_1 \to A/P_1A. \]

However, since \( A \) is a VNR, there exists \( u \in A - \mathcal{P} \) with \( pu = 0 \). This contradiction shows that \( P \) is a minimal prime ideal of \( R \).

Conversely, assume that if \( \mathcal{P} \) is a prime ideal of \( A \), then \( \mathcal{P} \cap R = P \) is a minimal prime ideal of \( R \). Then since \( R \) is reduced, \( R_P \) is a field. Now \( A_{\mathcal{P}} \) is an \( R_P \)-module, and hence \( A_{\mathcal{P}} \) is flat over \( R_P \), and thus over \( R \). Thus \( \Sigma \oplus A_{\mathcal{P}}, \mathcal{P} \) maximal in \( A \), is a flat \( R \)-module. Since \( \Sigma \oplus A_{\mathcal{P}} \) is a faithfully flat \( A \)-module, it follows that \( A \) is flat over \( R \). (See [1, Proposition 7, Chapter 1, §4].)

Proposition 1.15 [12, Proposition 9]. Let \( R \) be a reduced ring. Then the following statements are equivalent:

1. \( Q = Q(R) \) is a VNR.
2. \( \text{min } R \) is compact; and if a finitely generated ideal is contained in the union of the minimal prime ideals of \( R \), then it is contained in one of them.

Proof. (1) \( \Rightarrow \) (2) Since \( Q \) is a localization of \( R \), it is a flat \( R \)-module. Thus \( \text{min } R \) is compact by Proposition 1.14(2). The latter part of (2) follows immediately from Proposition 1.4(2).

(2) \( \Rightarrow \) (1) Suppose that \( Q \) is not a VNR. Then by Proposition 1.3, \( Q \) has a maximal ideal \( \mathcal{P} \) that is not a minimal prime ideal of \( Q \). Then \( \mathcal{P} \cap \)
$R$ is not a minimal prime ideal of $R$, because $Q$ is a localization of $R$. Hence $\mathcal{P} \cap R \supsetneq P$, where $P \in \text{min } R$. Choose $b \in (\mathcal{P} \cap R) - P$; and then 

$$\text{min } R = D(b) \cup \{ \cup D(a) | a \in P \}.$$ 

Since min $R$ is compact, there exist $a_1, \ldots, a_n \in P$ with min $R = D(b) \cup D(a_1) \cup \cdots \cup D(a_n)$. Let $J = Rb + Ra_1 + \cdots + Ra_n$; then $J$ is not contained in any minimal prime ideal of $R$. Hence by hypothesis $J$ is not contained in the union of the minimal prime ideals of $R$; and thus $J$ contains an element that is not a zero. Hence $QJ = Q$. But $QJ \subset \mathcal{P}$; and this contradiction shows that $Q$ is a VNR.

**Remarks.** Quentel has produced an example of a reduced ring $R$ where min $R$ is compact, but $Q(R)$ is not a VNR. Thus the latter part of statement (2) in Proposition 1.15 is not redundant.

The following proposition is also in part due to Quentel although with a proof that depends on considerable machinery.

**Proposition 1.16** [12, Proposition 3]. Let $R$ be a reduced ring. Then the following statements are equivalent:

1. min $R$ is compact.
2. If $b \in R$, then there is a finitely generated ideal $J \subset \text{Ann}_R b$ with $\text{Ann}_R(Rb + J) = 0$.
3. $\Pi R_P$, $P \in \text{min } R$, is a flat $R$-module.
4. $E(R)$ is a flat $R$-module.
5. If $P$ is a prime ideal of $E(R)$, then $\mathcal{P} \cap R \in \text{min } R$.

**Proof.** (1) $\Rightarrow$ (2) Let $b \in R$; then by Proposition 1.2(1), $Rb + \text{Ann}_R b$ is not contained in any minimal prime ideal of $R$. Thus 

$$\text{min } R = D(b) \cup \{ \cup D(a) | a \in \text{Ann}_R b \}.$$ 

Since min $R$ is compact, there exist $a_1, \ldots, a_n \in \text{Ann}_R b$ such that 

$$\text{min } R = D(b) \cup D(a_1) \cup \cdots \cup D(a_n).$$

Thus if $J = Ra_1 + \cdots + Ra_n$, then $Rb + J$ is not contained in any minimal prime ideal of $R$. Hence by Proposition 1.2(2), $\text{Ann}_R(Rb + J) = 0$.

(2) $\Rightarrow$ (3) Let $\Pi = \Pi R_{P_\alpha}$, $P_\alpha \in \text{min } R$; and let $I \neq 0$ be an ideal of $R$. In order to prove that $\Pi$ is flat it is sufficient to prove that $\text{Tor}_1^R(R/I, \Pi) = 0$. But $\text{Tor}_1^R(R/I, \Pi)$ is isomorphic to the kernel of the canonical map $\theta : I \otimes_R \Pi \to \Pi$. Thus it is sufficient to prove that $\text{Ker } \theta = 0$.

Let $b \in I$ and $\langle x_\alpha \rangle \in \Pi$, where $x_\alpha \in R_{P_\alpha}$. We will show that we can write $b \otimes \langle x_\alpha \rangle$ in the form where $x_\alpha = 0$, for all $\alpha$ such that $b \in P_\alpha$. By hypothesis there is a finitely generated ideal 

$$J = Ra_1 + \cdots + Ra_n$$

with $J \subset \text{Ann}_R b$ and $\text{Ann}_R(Rb + J) = 0$.

By Proposition 1.2, $\text{min } R = D(b) \cup D(a_1) \cup \cdots \cup D(a_n)$. Thus we can
take subsets $D(a_i) \subseteq D(a_i)$ such that we have the disjoint union
\[ \min R = D(b) \cup D(a_1) \cup \cdots \cup D(a_n). \]

Consider a fixed integer $i$, $1 \leq i \leq n$. If $P_a \in D(a_i)$, then $a_i$ is a unit in $R_{P_a}$, and we can write $x_\alpha = a_i(y_\alpha(i))$, where $y_\alpha(i) \in R_{P_a}$. If $P_a \notin D(a_i)$, we put $y_\alpha(i) = 0$. If $P_a \in D(b)$, we let $y_\alpha(0) = x_\alpha$; and if $P_a \notin D(b)$ we let $y_\alpha(0) = 0$. It is then clear that
\[ \langle x_\alpha \rangle = \langle y_\alpha(0) \rangle + a_1\langle y_\alpha(1) \rangle + \cdots + a_n\langle y_\alpha(n) \rangle. \]
Since $a_i \in \text{Ann}_b b$, we have $b \otimes \langle x_\alpha \rangle = b \otimes \langle y_\alpha(0) \rangle$. Thus without loss of generality we can assume that $x_\alpha = 0$ if $b \in P_a$.

Now if $b \otimes \langle x_\alpha \rangle \in \text{Ker} \theta$, then $x_\alpha = 0$ if $b \in P_a$, and $bx_\alpha = 0$ if $b \notin P_a$. But if $b \notin P_a$, then $b$ is a unit in $R_{P_a}$, and so $x_\alpha = 0$ for all $\alpha$. Therefore $b \otimes \langle x_\alpha \rangle = 0$. In general, suppose that $x \in \text{Ker} \theta$ and
\[ x = (b_1 \otimes \langle x_\alpha(1) \rangle) \cdots + (b_k \otimes \langle x_\alpha(k) \rangle) \]
where $b_i \in I$ and $x_\alpha(i) \in R_{P_a}$.

We shall prove that $x = 0$ by induction on $k$, the case $k = 1$ having already been proved.

As we have demonstrated, we can assume that $x_\alpha(1) = 0$ if $b_1 \in P_a$. Now
\[ 0 = \theta(x) = \langle b_1x_\alpha(1) \rangle + \cdots + b_kx_\alpha(k), \]
and hence $b_1x_\alpha(1) + \cdots + b_kx_\alpha(k) = 0$, for all $\alpha$. For all $\alpha$ such that $b_1 \notin P_a$, $b_1$ is a unit in $R_{P_a}$, and hence if $i > 1$ we can write $x_\alpha(i) = -b_1y_\alpha(i)$, where $y_\alpha(i) \in R_{P_a}$. Thus
\[ b_1[x_\alpha(1) - b_2y_\alpha(2) - \cdots - b_ky_\alpha(k)] = 0. \]
But since $b_1$ is a unit in $R_{P_a}$, we have $x_\alpha(1) = b_2y_\alpha(2) + \cdots + b_ky_\alpha(k)$. It is now clear that by substitution we can write $x$ as
\[ x = (b_2 \otimes \langle z_\alpha(2) \rangle) \cdots + (b_k \otimes \langle z_\alpha(k) \rangle). \]
Hence $x = 0$ by induction on $k$. Thus $\text{Ker} \theta = 0$, and so $\Pi_{P_a}$ is flat.

(3) $\Rightarrow$ (4) Since $E(R)$ is a direct summand of $\Pi_{P_a}$ by Proposition 1.1(1), $E(R)$ is also a flat $R$-module.

(4) $\Rightarrow$ (5) By Proposition 1.12, $E(R)$ is a commutative VNR containing $R$. Hence by Proposition 1.14(1), the prime ideals of $E(R)$ contract to minimal prime ideals of $R$.

(5) $\Rightarrow$ (1) This is an immediate consequence of Proposition 1.14(1).

Remarks. (1) If $R$ is a reduced coherent ring, then $\min R$ is compact. For one of the definitions of a coherent ring is that every direct product of flat $R$-modules is flat. Thus $\Pi_{P_a}$ is flat, and hence $\min R$ is compact by Proposition 1.16.

(2) If $R$ is a reduced ring, then $\text{Hom}_R(E(R), E(R))$ is always a commutative ring; but $E(R)$ is flat iff $\min R$ is compact. However, if $R$ is a Noetherian
ring (not necessarily reduced) then min $R$ is always finite (hence compact); but as we shall show in Proposition 1.18, $E(R)$ is flat iff $\text{Hom}_R(E(R), E(R))$ is a commutative ring.

**Definition.** We shall say that an ideal is irreducible if it is not the intersection of two properly larger ideals.

Portions of the following two propositions are contained in [3, Theorem 3].

**Proposition 1.17.** Let $R$ be a commutative, Noetherian, local ring with maximal ideal $M$. Then the following statements are equivalent:

1. $R$ is self-injective.
2. $M$ is the only prime ideal of $R$ and $0$ is an irreducible ideal of $R$.
3. $R \cong E(R/M)$.
4. $E(R/M)$ is a flat $R$-module.

**Proof:** (1) $\Rightarrow$ (2) Let $0 = Q_1 \cap \cdots \cap Q_n$ be an irredundant decomposition of $0$ in $R$, where $Q_i$ is an irreducible $P_i$ primary ideal. Then by [11, Theorem 2.3],

$$E(R) = E(R/P_1) \oplus \cdots \oplus E(R/P_n).$$

But $R \cong E(R)$ and $R$ is indecomposable. Hence $n = 1$ and $R \cong E(R/P_1)$. Thus every element of $R - P_1$ is a unit in $R$. Therefore, $P_1 = M$ and $0 = Q_1$ is irreducible and $M$-primary. Hence $M$ is the only prime ideal of $R$.

(2) $\Rightarrow$ (3) Since $0$ is an irreducible $M$-primary ideal of $R$, we have $E(R) = E(R/M)$. Now $R$ has finite length and $L(R) = L(\text{Hom}_R(R, E(R/M))) = L(E(R/M))$. Since $R \subset E(R/M)$, we have $R \cong E(R/M)$.

(3) $\Rightarrow$ (4) Trivial.

(4) $\Rightarrow$ (1) Let $E = E(R/M)$ and let $I$ be an ideal of $R$. Then by [2, Ch. VI, Proposition 5.3],

$$\text{Hom}_R(\text{Ext}_R^1(R/I, R), E) = \text{Tor}_1^R(\text{Hom}_R(R, E), R/I) = 0$$

because $E$ is flat. Therefore, $\text{Ext}_R^1(R/I, R) = 0$ showing that $R$ is self-injective.

**Proposition 1.18.** Let $R$ be a commutative, Noetherian ring, and let $\{P_1, \ldots, P_n\}$ be the prime ideals belonging to $0$ in $R$. Let $O_i = \{r \in R \mid ur = 0 \text{ for some } u \in R - P_i\}$. Then the following statements are equivalent:

1. Every $P_i$ is a minimal prime ideal of $R$, and $O_i$ is an irreducible ideal of $R$.
2. $E(R) \cong R_{P_1} \oplus \cdots \oplus R_{P_n}$.
3. $E(R) \cong Q(R)$.
(4) Hom_R(E(R), E(R)) is a commutative ring.
(5) E(R) is a flat R-module.

Proof. (1) \( \Rightarrow \) (2) \( O_i \) is a \( P_i \)-primary ideal because \( P_i \) is minimal; and 
\( 0 = O_1 \cap \cdots \cap O_n \) is a normal, irreducible decomposition of 0 in \( R \). Thus 
\[
E(R) = E(R/P_1) \oplus \cdots \oplus E(R/P_n)
\]
by [11, Theorem 2.3]. Since 0 is an irreducible ideal of \( R_{P_i} \), and \( P_iR_{P_i} \) is 
the only prime ideal of \( R_{P_i} \), it follows from Proposition 1.17 that 
\[
E(R/P_i) = E(R_{P_i}/P_iR_{P_i}) = R_{P_i}.
\]
Therefore \( E(R) = R_{P_1} \oplus \cdots \oplus R_{P_n} \).

(2) \( \Rightarrow \) (3) Since every \( R_{P_i} \) is a self-injective ring, it follows from Proposition 
1.17 that every \( P_i \) is a minimal prime ideal of \( R \). Since \( S = R - \bigcup_{i=1}^n P_i \) 
is the set of non-zero divisors of \( R \), we have by Proposition 1.5 that 
\[
Q(R) = R_S = R_{P_1} \oplus \cdots \oplus R_{P_n}.
\]
Therefore, \( E(R) = Q(R) \).

(3) \( \Rightarrow \) (4) Hom_R(E(R), E(R)) = Hom_R(S, S) = S is a commutative 
ing.

(4) \( \Rightarrow \) (5) By Proposition 1.10, \( E(R) = \text{Hom}_R(E(R), E(R)) \). Let \( I \) be an 
ideal of \( R \). By [2, Ch. VI, Proposition 5.2] we have 
\[
\text{Tor}_1^R(\text{Hom}_R(E(R), E(R)), R/I) = \text{Hom}_R(\text{Ext}_R^1(R/I, E(R)), E(R)) = 0.
\]
Therefore, Hom_R(E(R), E(R)) is a flat R-module.

(5) \( \Rightarrow \) (1) Let \( E_i = E(R/P_i) \); then \( E(R) = E_1^{k_1} \oplus \cdots \oplus E_n^{k_n} \). Hence \( E_i \) is 
a flat \( R \)-module, \( i = 1, \ldots, n \). Since \( E_i \) is an \( R_{P_i} \)-module, \( E_i \) is a flat 
\( R_{P_i} \)-module. Hence by Proposition 1.17, \( P_iR_{P_i} \) is the only prime ideal of 
\( R_{P_i} \), and 0 is irreducible in \( R_{P_i} \). Therefore, \( P_i \) is a minimal prime ideal of \( R \) 
and \( O_i \) is an irreducible ideal of \( R \).

2. PIF Rings

Definitions. Let \( R \) be a commutative ring. We shall say that \( R \) is a PIF 
if every principal ideal of \( R \) is flat; and we shall say that \( R \) is a PIP if every 
principal ideal of \( R \) is projective. If \( P \) is a prime ideal of \( R \) we shall define 
\[
O_P = \{ r \in R \mid \text{there is } u \in R - P \text{ with } ur = 0 \}.
\]

In this section we shall give necessary and sufficient conditions for a PIF 
to be a PIP. Then we shall focus on a single ideal and give necessary and 
sufficient conditions for it to be a direct summand of \( R \) in terms of the 
properties of subsets of \( \text{min } R \). From these considerations we shall be able 
to give necessary and sufficient conditions for a principal ideal of a PIF to 
be a projective ideal.
The following proposition characterizes PIF rings as those rings that are locally integral domains.

**Proposition 2.1.** Let $R$ be a commutative ring. Then the following statements are equivalent:

1. $R$ is a PIF.
2. $R_M$ is an integral domain for all maximal ideals $M$ of $R$.
3. $R$ is reduced; and a maximal ideal $M$ of $R$ contains only one minimal prime ideal $P$ of $R$.

In this case, $P = O_M$; and $R_P = Q(R_M)$, the quotient field of $R_M$.

**Proof.** (1) $\Rightarrow$ (2) Let $M$ be a maximal ideal of $R$. Then every principal ideal of $R_M$ is flat, hence free over $R_M$. Thus $R_M$ has no zero divisors.

(2) $\Rightarrow$ (1) Let $a \in R$; then either $R_{Ma} = 0$ or $R_{Ma}$ is $R_{M}$-free for maximal ideals $M$ of $R$. Hence $\text{w.dim}_R Ra = \sup \text{w.dim}_{R_{Ma}} R_{Ma} = 0$. Thus $Ra$ is a flat ideal of $R$.

(2) $\Rightarrow$ (3) Let $M$ be a maximal ideal of $R$. Then $0_M$ is contained in every prime ideal of $R$ contained in $M$. On the other hand $R/O_M \subset R_M$, and hence $O_M$ is a prime ideal of $R$. Thus $O_M$ is the unique minimal prime ideal of $R$ contained in $M$. The annihilator of an element of $\cap O_M$ (where $M$ ranges over all of the maximal ideals of $R$) is not contained in any maximal ideal of $R$. Hence $\cap O_M = 0$ and thus $R$ is reduced. If $O_M = P$, then $R_P$ is a field by Proposition 1.1, and clearly $R_P = Q(R_M)$.

(3) $\Rightarrow$ (2) Let $M$ be a maximal ideal of $R$. Then $R_M$ is reduced and has only one minimal prime ideal. Therefore, $R_M$ is an integral domain.

**Proposition 2.2.** Let $R$ be a commutative ring with only a finite number of minimal prime ideals. Then the following statements are equivalent:

1. $R$ is a PIF.
2. $R$ is a PIP.
3. $R$ is a finite direct sum of integral domains.

**Proof.** (2) $\Rightarrow$ (1) is trivial.

(1) $\Rightarrow$ (3) Let $\{P_1, \ldots, P_n\}$ be the minimal prime ideals of $R$. Then $R$ is reduced and so $\cap_{i=1}^n P_i = 0$. By Proposition 2.1, $P_i + P_j = R$, $i \neq j$. Thus by the Chinese Remainder Theorem, $R = R/P_1 \oplus \cdots \oplus R/P_n$.

(3) $\Rightarrow$ (2) If $R$ is a finite direct sum of integral domains, then every principal ideal of $R$ is a direct sum of principal ideals over each of these domains. Hence it is obvious that principal ideals of $R$ are projective.

**Proposition 2.3.** Let $R$ be a reduced ring and $J$ a finitely generated flat ideal of $R$. If $\exists$ elements $a \in J$ and $b \in \text{Ann}_R J$ such that $a + b$ is not a zero divisor in $R$, then $J$ is a projective ideal of $R$. 
Proof. Let \( I = Rb + J \); then \( I = Rb \oplus J \) because \( R \) is reduced and \( b \in \text{Ann}_R J \). Let \( P \) be a prime ideal of \( R \). If \( J_P \neq 0 \), then \( J_P \) is a free \( R_p \)-ideal of rank 1 and \( bJ_P = 0 \). Therefore, \( R_pb = 0 \), and so \( I_P = J_P \) is free of rank 1 over \( R_p \). On the other hand suppose that \( J_P = 0 \). Then \( I_P = R_pb = R_p(a + b) \) is free of rank 1 over \( R_p \) because \( a + b \) is not a zero divisor in \( R_p \). Thus \( I_P \) is free of rank 1 if \( P \) is a prime ideal of \( R \). Hence by [1, Ch. II, §5, Theorem 2] \( I \) is a projective ideal of \( R \). Since \( J \) is a direct summand of \( I \), \( J \) is also a projective ideal of \( R \).

Proposition 2.4. Let \( R \) be a reduced ring such that \( Q(R) \) is a VNR. Then every finitely generated flat ideal of \( R \) is projective.

Proof. This is an immediate consequence of Propositions 1.4(3) and 2.3.

Proposition 2.5. Let \( R \) be a commutative ring and \( I \) an ideal of \( R \). Then the following statements are equivalent:

(1) \( R/I \) is a flat \( R \)-module.
(2) \( I \cap K = IK \) for any ideal \( K \) of \( R \).
(3) If \( a \in I \), then there exists \( c \in I \) with \( (1 - c)a = 0 \).
(4) If \( J \) is a finitely generated ideal of \( R \), \( J \subseteq I \), then there is a \( c \in I \) with \( (1 - c)J = 0 \).
(5) \( I_M = 0 \) or \( R_M \) for any maximal ideal \( M \) of \( R \).

If \( I \) and \( K \) are ideals of \( R \) such that \( R/I \) and \( R/K \) are flat, then \( R/(I + K) \) is also flat.

Proof. (1) \( \Rightarrow \) (2) If \( A \) is an \( R \)-module, then \( A \otimes_R R/I \cong A/IA \). Let \( K \) be an ideal of \( R \); then since \( R/I \) is flat we have a commutative diagram with exact rows and vertical isomorphisms:

\[
\begin{array}{cccccc}
0 & \rightarrow & K/IK & \rightarrow & R/I & \rightarrow & R/(I + K) & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & (I + K)/I & \rightarrow & R/I & \rightarrow & R/(I + K) & \rightarrow & 0.
\end{array}
\]

Since the kernel of the canonical map \( K/IK \rightarrow (I + K)/I \) is \( (I \cap K)/IK \), we have \( I \cap K = IK \).

(2) \( \Rightarrow \) (3) Let \( a \in I \); then \( Ra = Ia \). Hence there is a \( c \in I \) with \((1 - c)a = 0 \).

(3) \( \Rightarrow \) (4) Let \( J = Ra_1 + \cdots + Ra_n \), where \( a_i \in I \). By hypothesis, there are \( c_i \in I \) with \((1 - c_i)a_i = 0 \), \( i = 1, \ldots, n \). Then \((1 - c_1)(1 - c_2) \cdots (1 - c_n) = 1 - c \), where \( c \in I \) and \((1 - c)J = 0 \).

(4) \( \Rightarrow \) (5) Let \( M \) be a maximal ideal of \( R \). If \( I \nsubseteq M \), then \( I_M = R_M \).

Hence assume that \( I \subseteq M \). If \( a \in I \), then there is a \( c \in I \) with \((1 - c)a = 0 \); and since \( 1 - c \nsubseteq M \), it follows that the image of \( a \) in \( R_M \) is 0. Therefore, \( I_M = 0 \).
(5) ⇒ (1) Since \((R/I)_M = R_M\) or 0, it follows that \((R/I)_M\) is a flat \(R_M\)-module for all maximal ideals \(M\) of \(R\). Therefore because flatness is determined locally, \(R/I\) is a flat \(R\)-module.

Assume that \(I\) and \(K\) are ideals of \(R\) such that \(R/I\) and \(R/K\) are flat \(R\)-modules. Let \(x = a + b\), where \(a \in I\) and \(b \in K\). Then there are \(c \in I\) and \(d \in K\) such that \((1 - c)a = 0\) and \((1 - d)b = 0\). Thus \((1 - c)(1 - d)x = 0\), and \((1 - c)(1 - d) = 1 - y\), where \(y \in I + K\). Thus \(R/(I + K)\) is a flat \(R\)-module by (3).

The following proposition is due to Vasconcelos [14, Proposition 3.4].

**Proposition 2.6.** Let \(R\) be a reduced ring such that min \(R\) is compact. Then every principal flat ideal of \(R\) is projective.

**Proof.** Let \(Rb\) be a flat ideal of \(R\) and let \(I = \text{Ann}_{Rb}\). By Proposition 1.16, there is a finitely generated ideal \(J \subseteq I\) such that \(\text{Ann}_{R}(J + Rb) = 0\). Since \(R/I \cong Rb\) is flat, there exists \(a \in I\) with \((1 - a)J = 0\) by Proposition 2.5. Thus \(J \subseteq Ra\) and hence \(\text{Ann}_{R}(Ra + Rb) = 0\). By Proposition 1.2(3), \(a + b\) is not a zero divisor in \(R\). Thus by Proposition 2.3, \(Rb\) is a projective ideal of \(R\).

**Definition.** A commutative ring \(R\) is said to be semi-hereditary if every finitely generated ideal of \(R\) is projective.

 Portions of the next proposition are due to Hattori [9], Endo [6], Vasconcelos [13], [14], and Quentel [12].

**Proposition 2.7.** Let \(R\) be a PIF; then the following statements are equivalent:

1. \(R\) is a PIP.
2. Every finitely generated flat ideal of \(R\) is projective.
3. \(Q(R)\) is a semi-hereditary ring.
4. \(Q(R)\) is a VNR.
5. min \(R\) is compact.
6. \(E(R)\) is a flat \(R\)-module.

**Proof.** (4) ⇒ (2) is Proposition 2.4; and (2) ⇒ (1) is trivial. (4) ⇒ (5) follows from Proposition 1.15; and (5) ⇒ (1) is Proposition 2.6. (5) ⇔ (6) is Proposition 1.16.

1. ⇒ (4) Let \(a \in R\) and \(I = \text{Ann}_{R}a\); then \(I\) is a direct summand of \(R\), and hence \(I = Re\), where \(e^2 = e\). Now \(\text{Ann}_{R}(Re + Ra) = 0\) by Proposition 1.2, and so \(Q(R)\) is a VNR by Proposition 1.4.

4. ⇒ (3) is elementary.

3. ⇒ (4) Since principal ideals of \(Q(R)\) are projective \(Q(R)\)-modules, and \(Q(R)\) is its own ring of quotients, \(Q(R)\) is a VNR by (1) ⇒ (4).
**Remarks.** The assumption in Proposition 2.7 that $R$ is a PIF, is necessary. There exist many examples where $R$ is a reduced ring and $Q(R) = E(R)$ is a VNR, but $R$ is not a PIF. The easiest example is the following. Let $R$ be a quasi-local reduced ring with only a finite number of minimal prime ideals but such that $R$ is not a domain. (Take any Noetherian local ring that is not a domain and factor out the intersection of the minimal prime ideals.) By Proposition 1.6, $Q(R) = E(R)$ is a semi-simple ring. If $R$ were a PIF, then by Proposition 2.2, $R$ would be a finite direct sum of integral domains. But since $R$ is quasi-local, it is indecomposable. This contradiction shows that $R$ is not a PIF.

**Definitions.** Let $R$ be a commutative ring. The weak global dimension of $R$ (w.gl.dim $R$) is defined to be the smallest non-negative integer $n$ (if any such exist) such that $\text{Tor}^R_{n+1} = 0$ is the 0-functor. Otherwise, w.gl.dim $R = \infty$.

Thus w.gl.dim $R = 0$ iff every $R$-module is flat. It is not hard to see that w.gl.dim $R = 0$ iff $R$ is a VNR. It also follows from the definitions that w.gl.dim $R \leq 1$ iff every submodule of a flat $R$-module is flat iff every ideal of $R$ is flat.

**Remarks.** (1) Let $R$ be a commutative ring such that w.gl.dim $R \leq 1$. Then each of the 6 conditions of Proposition 2.7 is equivalent to $R$ being a semi-hereditary ring. Thus Proposition 2.7 is a generalization of the results of Hattori [9], Endo [6], Vasconcelos [13] and Quentel [12].

(2) It is well known (see [7, Corollary 11.30]) that a finitely generated flat ideal is projective iff it is finitely presented. Since a coherent ring is defined to be a ring such that finitely generated ideals are finitely presented, it follows that if w.gl.dim $R \leq 1$, then $R$ is semi-hereditary iff $R$ is a coherent ring. One would therefore expect that the conditions for a PIF to be a PIP would involve some weakened form of coherence. Since a ring $R$ is coherent iff a direct product of flat $R$-modules is flat [4, Theorem 2.1], we have a better understanding of why a PIF is a PIP iff min $R$ is compact (Proposition 2.7). For as we have seen in Proposition 1.16, if $R$ is a reduced ring, then min $R$ is compact iff $\text{II}_{R_P} (P \in \text{min } R)$ is a flat $R$-module.

In a recent paper (Commutative coherent rings, Canad. J. Math., vol. 34 (1982), pp. 1240–1244) we have shown that a commutative ring $R$ is a coherent ring iff $\text{Hom}_R (B, C)$ is flat for injective $R$-modules $B$ and $C$. Thus condition (4) of the next proposition is seen to be related to coherence, since flat modules over integral domains are torsion-free. Furthermore, it follows from Proposition 2.7 that if w.gl.dim $R \leq 1$, then each of the four conditions of the next proposition is equivalent to $R$ being a semi-hereditary ring.

**Proposition 2.8.** Let $R$ be a PIF. Then the following statements are equivalent:
(1) $R$ is a PIP.
(2) $Q(R)_M$ is the quotient-field of $R_M$ for any maximal ideal $M$ of $R$.
(3) If $C$ is an injective $R$-module, then $C_M$ is a divisible $R_M$-module for any maximal ideal $M$ of $R$.
(4) If $C$ is an injective $R$-module, then $\text{Hom}_R(C, E(R/M))$ is a torsion-free $R_M$-module for any maximal ideal $M$ of $R$.

Proof: (1) $\Rightarrow$ (2) Let $Q = Q(R)$ and $S$ the set of non-zero divisors in $R$ so that $Q = R_S$. Let $M$ be a maximal ideal of $R$, $P$ the unique minimal prime ideal of $R$ contained in $M$, and $\overline{S}$ the image of $S$ in $R/P$. Then $Q/PQ = (R/P)\overline{S} \subset R_P$ because $R_P$ is the quotient field of $R/P$. But $Q$ is a VNR by Proposition 2.7, and so $Q/PQ$ is a field. Thus $Q/PQ = R_P$. But by Proposition 2.1, $R_P$ is the quotient field of $R_M$ and $P_M = 0$. Therefore, $(PQ)_M = 0$, and we have $R_P = (P_M)_M = Q_M/(PQ)_M = Q_M$.

(2) $\Rightarrow$ (3) Let $C$ be an injective $R$-module. For each $x \in C$, there is an $R$-homomorphism $f : Q \to C$ with $f(1) = x$. Hence there is an $R$-module $F$ that is a direct sum of copies of $Q$ and an $R$-surjection $g : F \to C \to 0$. Let $M$ be a maximal ideal of $R$. Then we have an $R_M$-surjection $g_M : F_M \to C_M \to 0$. But by hypothesis $F_M$ is a direct sum of copies of the quotient field of $R_M$. Therefore, $C_M$ is a divisible $R_M$-module.

(3) $\Rightarrow$ (4) Let $C$ be an injective $R$-module and $M$ a maximal ideal of $R$. Then $E(R/M)$ (the injective envelope of $R/M$) is an $R_M$-module; and hence

$$\text{Hom}_R(C, E(R/M)) = \text{Hom}_{R_M}(C_M, E(R/M))$$

Now $\text{Hom}_{R_M}(C_M, E(R/M))$ is $R_M$-torsion-free because $C_M$ is a divisible $R_M$-module.

(4) $\Rightarrow$ (1) Let $a \in R$, and suppose that $Ra$ is not a projective $R$-module. Then $hhd_R(R/Ra) > 1$; and thus there is a $B$ (a homomorphic image of an injective $R$-module) such that

$$\text{Ext}^1_R(R/Ra, B) \neq 0.$$ 

Hence there is a maximal ideal $M$ of $R$ such that $\text{Hom}_R(\text{Ext}^1_R(R/Ra, B), D) = 0$ where $D = E(R/M)$. Because $R/Ra$ is finitely presented, there is a canonical surjection

$$\text{Tor}^1_R(\text{Hom}_R(B, D), R/Ra) \to \text{Hom}_R(\text{Ext}^1_R(R/Ra, B), D).$$

Thus $\text{Tor}^1_R(\text{Hom}_R(B, D), R/Ra) \neq 0$. Because $D$ is an $R_M$-module, we have

$$\text{Tor}^1_R(\text{Hom}_R(B, D), R/Ra) \simeq \text{Tor}^1_{R_M}(\text{Hom}_{R_M}(B_M, D), R_M/R_M a).$$

But $R_M a$ is a flat $R_M$-module; and $\text{Hom}_{R_M}(B_M, D)$ is $R_M$-torsion-free by hypothesis. Thus $\text{Tor}^1_{R_M}(\text{Hom}_{R_M}(B_M, D), R_M/R_M a) = 0$. This contradiction proves that $Ra$ is a projective ideal of $R$. 


Remarks. Let $R$ be a PIF that is not a PIP (there is an example due to Vasconcelos of such a ring that we shall reproduce in §4). Then injective $R$-modules do not localize. In fact, according to Proposition 2.8, there is an injective $R$-module $C$ and a maximal ideal $M$ of $R$ such that $C_M$ is not even a divisible $R_M$-module. Again this is related to a lack of coherence. For further results on this subject and its relation to coherence see [5].

We now turn our attention to the problem of finding out when a single principal ideal is projective, or when an ideal is a direct summand of the ring.

**Definition.** Let $R$ be a reduced ring and $\mathcal{C}$ a subset of $\min R = \{P_\alpha\}$. Let $\mathcal{C}'$ denote the complement of $\mathcal{C}$ in $\min R$, and let $J_{\mathcal{C}} = \bigcap P_\sigma, P_\sigma \in \mathcal{C}$. It is immediate that $\mathcal{C}$ is a closed subset of $\min R$ iff $P_\alpha \supset J_{\mathcal{C}}$ implies that $P_\alpha \in \mathcal{C}$. If $\mathcal{C}$ is closed, it is also easy to verify that $\Ann_{R,J_{\mathcal{C}}} = J_{\mathcal{C}}$. We shall say that $\mathcal{C}$ is a good subset of $\min R$ if $J_{\mathcal{C}} \subseteq P_\gamma, P_\gamma \in \mathcal{C}$.

**Proposition 2.9.** Let $R$ be a reduced ring.

1. $\mathcal{C}$ is a good subset of $\min R$ iff there exists $a \in R$ with $\mathcal{C} = D(a)$. In this case $\mathcal{C}$ is both open and closed in $\min R$; $\Ann_{R,a} = J_{\mathcal{C}}$; and $\Ann_{R,\Ann_{R,a}} = J_{\mathcal{C}}$.

2. $\mathcal{C}$ good implies that $\mathcal{C}'$ is good for any subset $\mathcal{C}$ of $\min R$ iff $Q(R)$ is a VNR.

**Proof.** (1) $\mathcal{C}$ is a good subset of $\min R$ iff $J_{\mathcal{C}} \subseteq P_\sigma, P_\sigma \in \mathcal{C}$ iff there exists $a \in J_{\mathcal{C}}, a \not\subseteq P_\gamma, P_\gamma \in \mathcal{C}$ iff there exists $a \in R$ with $D(a) = \mathcal{C}$. Suppose that $\mathcal{C} = D(a)$, then $\mathcal{C}$ is open in $\min P$. By Proposition 1.1, $\mathcal{C}$ is the set of minimal prime ideals of $R$ that contain $\Ann_{R,a}$. Thus $\mathcal{C}$ is closed in $\min R$ and $J_{\mathcal{C}} \supset \Ann_{R,a}$. On the other hand, $J_{\mathcal{C}} \cap (J_{\mathcal{C}} \cap J_{\mathcal{C}'} = 0$, and so $\Ann_{R,a} = J_{\mathcal{C}}$. Since $\mathcal{C}$ is closed, $\Ann_{R,J_{\mathcal{C}}} = J_{\mathcal{C}}$.

(2) Let $\mathcal{C}$ be a good subset of $\min R$. Then there exists $a \in J_{\mathcal{C}}, a \not\subseteq P_\gamma, P_\gamma \in \mathcal{C}$, and hence $\Ann_{R,a} = J_{\mathcal{C}}$. Now $\mathcal{C}'$ is a good subset of $\min R$ iff there exists $b \in J_{\mathcal{C}}$ such that $b \not\subseteq P_\delta, P_\delta \in \mathcal{C}'$, iff there exists $b \in \Ann_{R,a}$ such that $a + b$ is not in any minimal prime ideal of $R$. Hence by Proposition 1.4(3), $\mathcal{C}$ good implies $\mathcal{C}'$ good for any subset $\mathcal{C}$ of $\min R$ iff $Q(R)$ is a VNR.

**Proposition 2.10.** Let $R$ be a reduced ring and $J$ an ideal of $R$ such that $R/J$ is a flat $R$-module. Then $R/J$ is a reduced ring and $J = J_{\mathcal{C}}$, where $\mathcal{C}$ is a closed subset of $\min R$. If $R$ is a PIF, or a PIP, then so is $R/J$.

**Proof.** Let $x \in J$; then by Proposition 2.5 there is a $b \in J$ such that $(1 - b)x = 0$. Thus if $P$ is any prime ideal of $R$ containing $J$, then $x$ is contained in every minimal prime ideal of $R$ contained in $P$. Thus if $\mathcal{C}$ is the set of minimal prime ideals of $R$ containing $J$, then $\mathcal{C}$ is a non-empty
closed subset of \( \text{min } R \) and \( J \subseteq J_\mathcal{E} \). Now suppose that \( y \in R \) and \( y^n = x \in J \); then \( (1 - b)y^n = 0 \), and so \((1 - b)y^n = 0 \). Therefore \((1 - b)y = 0 \), and hence \( y \in J \). Therefore, \( J = J_\mathcal{E} \), and \( R/J \) is a reduced ring.

Assume that \( R \) is a PIF; let \( M \) be a maximal ideal of \( R \) containing \( J \); and let \( P \) be the unique minimal prime ideal of \( R \) contained in \( M \). As we have seen, \( J \subseteq P \), and so \( P/J \) is the unique minimal prime ideal of \( R/J \) contained in \( M/J \). Hence \( R/J \) is a PIF by Proposition 2.1.

Let \( r \in R \), and \( c \in (J : r) \), and \( x = r \in J \). Then there is a \( b \in J \) with \((1 - b)c r = 0 \) and so

\[
(1 - b)c = a \in \text{Ann}_R r.
\]

Thus \((J : r) = J + \text{Ann}_R r\). Let \( I = \text{Ann}_R r \) and assume that \( Rr \) is a projective \( R \)-module. Then \( I = Re \), where \( e^2 = e \); and hence \((J : r)/J = (Re + J)/J \) is generated by an idempotent element of \( R/J \). Since \((J : r)/J \) is the annihilator in \( R/J \) of \( r + J \), we see that \( r + J \) generates a projective ideal of \( R/J \). Hence if \( R \) is a PIP, then so is \( R/J \).

**Proposition 2.11.** Let \( R \) be a PIF and \( \mathcal{E} \) a finite subset of \( \text{min } R \). Then \( R/J_\mathcal{E} \) is a flat \( R \)-module.

**Proof.** Let \( M \) be a maximal ideal of \( R \) and let \( P \) be the unique maximal prime ideal of \( R \) contained in \( M \). If \( P \notin \mathcal{E} \), then \( J_\mathcal{E} \subseteq P \), and so \((J_\mathcal{E})_M \subseteq P_M = 0 \). If \( P \notin \mathcal{E} \), and \( J_\mathcal{E} \subseteq M \), then there is a \( P' \in \mathcal{E} \) with \( P' \subseteq M \), and hence \( P' = P \). This contradiction shows that \( J_\mathcal{E} \mathcal{E} \subseteq M \) and hence \((J_\mathcal{E})_M = R_M \). Thus \((J_\mathcal{E})_M = 0 \) or \( R_M \), and hence \( R/J_\mathcal{E} \) is a flat \( R \)-module by Proposition 2.5.

**Remarks.** If a principal ideal of a commutative ring \( R \) is a projective \( R \)-module, then its annihilator is a direct summand of \( R \). In the next proposition we characterize the direct summands of a reduced ring \( R \) in terms of the subsets of \( \text{min } R \). We note that if \( \mathcal{E} \) is a subset of \( \text{min } R \) and \( \overline{\mathcal{E}} = \{P \in \text{min } R \mid P \supset J_\mathcal{E}\} \), then \( \overline{\mathcal{E}} \) is closed and \( J_\mathcal{E} = J_{\overline{\mathcal{E}}} \). Thus the restriction in (1) of the next proposition that \( \mathcal{E} \) be closed is no restriction at all on the ideal \( J_\mathcal{E} \). Moreover, by Proposition 2.10, every direct summand of \( R \) is of the form \( J_\mathcal{E} \), where \( \mathcal{E} \) is a closed subset of \( \text{min } R \).

**Proposition 2.12.** Let \( R \) be a reduced ring and \( \mathcal{E} \) a subset of \( \text{min } R \). Then the following statements are equivalent:

1. \( \mathcal{E} \) is closed and \( J_\mathcal{E} \) is a direct summand of \( R \).
2. \( \mathcal{E} \) is a good subset of \( \text{min } R \) and \( R/J_\mathcal{E} \) is flat.
3. \( \mathcal{E} \) is both open and closed in \( \text{min } R \) and \( R/J_\mathcal{E} \) and \( R/J_{\overline{\mathcal{E}}} \) are flat.

In this case both \( \mathcal{E} \) and \( \overline{\mathcal{E}} \) are good subsets of \( \text{min } R \). If \( R \) is a PIF, then this condition is equivalent to the other three.
Proof. (1) ⇒ (2) \( J_\epsilon = Re \), where \( e^2 = e \). Since \( \epsilon \) is closed, if \( P \in \epsilon \), then \( J_\epsilon \subseteq P \), and so \( e \notin P \). Thus \( J_\epsilon \subseteq P \subseteq \epsilon \), and so \( \epsilon \) is a good subset of \( R \). Since \( \epsilon \) is closed, \( J_\epsilon = \text{Ann}_R J_\epsilon = R(1 - e) \). Moreover, since \( \epsilon' \) is closed by Proposition 2.9, the same argument we have just used shows that \( \epsilon'' = \epsilon \) is a good subset of \( \text{min} \ R \). Finally, \( R/J_\epsilon = Re \) is \( R \)-projective.

(2) ⇒ (3) By Proposition 2.9, \( \epsilon \) is both open and closed in \( \text{min} \ R \); \( \epsilon = D(a) \) for some element \( a \in R \); \( \text{Ann}_R a = J_\epsilon \) and \( \text{Ann}_R J_\epsilon = J_\epsilon ' \). Therefore \( a \in J_\epsilon ' \); and hence by Proposition 2.5, there exists \( b \in J_\epsilon ' \) with \( (1 - b)a = 0 \). But then \( 1 - b \in \text{Ann}_R a = J_\epsilon \) and so \( J_\epsilon + J_\epsilon ' = R \). Since \( J_\epsilon \cap J_\epsilon ' = 0 \), \( J_\epsilon \) and \( J_\epsilon ' \) are direct summands of \( R \); and \( R/J_\epsilon \approx J_\epsilon \) and \( R/J_\epsilon ' \approx J_\epsilon \) are \( R \)-projective.

(3) ⇒ (1) By Proposition 2.5, \( R/(J_\epsilon + J_\epsilon ') \) is a flat \( R \)-module. Suppose that \( J_\epsilon + J_\epsilon ' \neq R \); then by Proposition 2.10, \( J_\epsilon + J_\epsilon ' = J_\varnothing \), where \( \varnothing \) is a non-empty subset of \( \text{min} \ R \). Let \( P \in \mathcal{D} \); then \( P \supset \mathcal{D} \supset J_\varnothing \) and since \( \epsilon \) is closed \( P \subseteq \epsilon \). Similarly \( P \subseteq \epsilon ' \). This contradiction shows that \( J_\epsilon + J_\epsilon ' = R \). Since \( J_\epsilon \cap J_\epsilon ' = 0 \), \( J_\epsilon \) is a direct summand of \( R \).

In the course of proving (1) ⇒ (2) we showed that both \( \epsilon \) and \( \epsilon ' \) are good subsets of \( \text{min} \ R \). Conversely, suppose that \( R \) is a PIF and that both \( \epsilon \) and \( \epsilon ' \) are good subsets of \( \text{min} \ R \). By Proposition 2.9, there exists \( b \in R \) such that \( D(b) = \epsilon ' \) and \( J_\epsilon = \text{Ann}_R b \). Therefore \( R/J_\epsilon \approx Rb \); and since \( R \) is a PIF, \( R/J_\epsilon \) is a flat \( R \)-module. Thus we have proved (2).

Proposition 2.13. Let \( R \) be a PIF; \( a \in R \); \( I = \text{Ann}_R a \); and \( J = \text{Ann}_R I \). Then the following statements are equivalent:

(1) \( Ra \) is a projective ideal of \( R \).
(2) \( \text{Hom}_R(I, R) \) is a flat \( R \)-module.
(3) \( R/J \) is a flat \( R \)-module.
(4) There exists \( b \in R \) such that \( J = \text{Ann}_R b \).
(5) If \( \{P_\gamma \} = \mathcal{D} \) is a subset of \( \text{min} \ R \) and \( I \subseteq \cup_\gamma P_\gamma \), then there is a \( P_\gamma \subseteq \mathcal{D} \) with \( I \subseteq P_\gamma \).

Proof. (1) ⇒ (2) \( I \) is a direct summand of \( R \), and hence \( \text{Hom}_R(I, R) \) is a projective \( R \)-module.

(2) ⇒ (3) We have an exact sequence

\[
0 \to \text{Hom}_R(R/I, R) \to R \to \text{Hom}_R(I, R).
\]

Since \( \text{Hom}_R(R/I, R) = \text{Ann}_R I = J \), we have an embedding \( R/J \subseteq \text{Hom}_R(I, R) \). Let \( M \) be a maximal ideal of \( R \). Then \( R_M/J_M \subseteq \text{Hom}_R(I, R)_M \). Now \( \text{Hom}_R(I, R)_M \) is a flat \( R_M \)-module; and \( R_M \) is an integral domain by Proposition 2.1. Thus \( \text{Hom}_R(I, R)_M \) is a torsion-free \( R_M \)-module; and hence so is \( R_M/J_M \). Therefore \( J_M = 0 \) or \( R_M \); and thus by Proposition 2.5, \( R/J \) is a flat \( R \)-module.

(3) ⇒ (4) Let \( \epsilon = D(a) \); then by Proposition 2.9, \( \epsilon \) is a good subset of \( \text{min} \ R \); \( J_\epsilon = I \); and \( J_\epsilon ' = J \). By hypothesis, \( R/J_\epsilon \) is flat; and thus by
Proposition 2.12, $J_\mathcal{C}$ is a direct summand of $R$. Therefore, $J_\mathcal{C} = Re$, where $e^2 = e$; and hence $J_\mathcal{C} = \text{Ann}_R e$.

(4) $\Rightarrow$ (5) Clearly $I = \text{Ann}_R J$, and since $J = \text{Ann}_R b$, we have $b \in I$.

Suppose that $\{P_\alpha\} = \mathcal{D}$ is a subset of $\text{min} \, R$ and that $I \subset \bigcup \alpha \, P_\alpha$. Then there exists $P_{\gamma_0} \in \mathcal{D}$ such that $b \in P_{\gamma_0}$. Hence by Proposition 1.2(1), $J \not\subset P_{\gamma_0}$. However, since $IJ = 0$, we have $I \subset P_{\gamma_0}$.

(5) $\Rightarrow$ (1) By Proposition 2.9, $I = J_\mathcal{C}$, where $\mathcal{C}$ is a good subset of $\text{min} \, R$. If $J_\mathcal{C} \subset \bigcup \alpha \, P_\alpha$, $P_\alpha \in \mathcal{C}$; then by hypothesis there is a $P_{\alpha_0} \in \mathcal{C}$ with $J_\mathcal{C} \subset P_{\alpha_0}$. But $\mathcal{C}$ is closed, and hence $P_{\alpha_0} \in \mathcal{C}$. This contradiction shows that $\mathcal{C}$ is a good subset of $\text{min} \, R$. Hence by Proposition 2.12, $I$ is a direct summand of $R$, and hence $Ra$ is a projective ideal of $R$.

The following proposition is a generalization of Proposition 2.2.

**Proposition 2.14.** Let $R$ be a PIF and $a \in R$; and suppose that $a$ is an element of only a finite number of primes $P_1, \ldots, P_n$ in $\text{min} \, R$.

1. $Ra$ is a projective ideal of $R$.
2. $R = (\cap_{i=1}^n P_i) \oplus (\Sigma_{i=1}^n R/P_i)$.
3. Every $P_i$ is a direct summand of $R$.

**Proof.** (1) Let $\mathcal{C} = D(a)$; then $J_\mathcal{C} = \text{Ann}_R a$ by Proposition 2.9(1). Now $\mathcal{C}' = \{P_1, \ldots, P_n\}$ and $J_\mathcal{C} \not\subset P_i$ for all $i$ since $\mathcal{C}$ is closed. Therefore $J_\mathcal{C} \not\subset \bigcup_{i=1}^n P_i$, and hence $\mathcal{C}'$ is a good subset of $\text{min} \, R$. Therefore, by Proposition 2.12, $J_\mathcal{C} \oplus J_\mathcal{C}' = R$. Thus $Ra = R/J_\mathcal{C} = J_\mathcal{C}'$ is a projective ideal of $R$.

(2) By Proposition 2.1, every maximal ideal of $R$ contains a unique minimal prime ideal of $R$. Thus $P_i + P_j = R$, $i \neq j$. Hence by the Chinese Remainder Theorem,

$$R/J_\mathcal{C} = \sum_{i=1}^n \oplus R/P_i.$$  

Since $J_\mathcal{C} = \cap_{i=1}^n P_i$ and $R = J_\mathcal{C} \oplus J_\mathcal{C}'$, we have $R = (\cap_{i=1}^n P_i) \oplus (\Sigma_{i=1}^n R/P_i)$.

(3) Since $R/P_i$ is isomorphic to a direct summand of $R$, there is an idempotent $e_i \in R$ so that $P_i = \text{Ann}_R e_i = R(1 - e_i)$. Hence $P_i$ is a direct summand of $R$, for $i = 1, \ldots, n$.

**Proposition 2.15.** Let $R$ be a commutative ring such that $\text{w.gl.dim} \, R \leq 1$. Then the following statements are equivalent:

1. $R$ is a semi-hereditary ring.
2. $R/J_\mathcal{C}$ is flat for all subsets $\mathcal{C}$ of $\text{min} \, R$.
3. $R/J_\mathcal{C}$ is a semi-hereditary ring for all subsets $\mathcal{C}$ of $\text{min} \, R$.
4. If $\mathcal{C}$ is any subset of $\text{min} \, R$ and $M$ is a maximal ideal of $R$, then $J_\mathcal{C} \subset M$ iff $J_\mathcal{C} \subset O_M$, the unique minimal prime ideal of $R$ contained in $M$.
Proof. (1) ⇒ (2) Let $\mathcal{C} = \{P_\alpha\}$ be a subset of $\min R$. Since $R_{P_\alpha}$ is a flat $R$-module, and $R$ is a coherent ring, $\prod R_{P_\alpha}$ ($P \in \mathcal{C}$) is a flat $R$-module. Since $\text{w.gl.dim } R \leq 1$, and $R/J_\mathcal{C} \subset \prod R_{P_\alpha}$ ($P_\alpha \in \mathcal{C}$), it follows that $R/J_\mathcal{C}$ is a flat $R$-module.

(2) ⇒ (4) Let $\mathcal{C}$ be a subset of $\min R$ and $M$ a maximal ideal of $R \ni J_\mathcal{C} \subset M$. By Proposition 2.5(5), we have $(J_\mathcal{C})_M = 0$. Thus $J_\mathcal{C} \subset O_M$, the unique minimal prime ideal of $R$ contained in $M$.

(4) ⇒ (2) Let $\mathcal{C}$ be a subset of $\min R$ and $M$ a maximal ideal of $R$. If $J_\mathcal{C} \not\subset M$, then $(J_\mathcal{C})_M = R_M$; while if $J_\mathcal{C} \subset M$, then $J_\mathcal{C} \subset O_M$ and hence $(J_\mathcal{C})_M = 0$. Thus by Proposition 2.5, $R/J_\mathcal{C}$ is a flat $R$-module.

(2) ⇒ (1) Let $\mathcal{C}$ be a good subset of $\min R$. Then by Proposition 2.12, $\mathcal{C}$ is also a good subset of $\min R$. Thus by Proposition 2.9, $\mathcal{Q}(R)$ is a VNR. Therefore by Proposition 2.7, $R$ is a semi-hereditary ring.

Remarks. (1) Let $R$ be a commutative semi-hereditary ring, and let $J$ be an ideal of $R$. Then it follows from Propositions 2.12 and 2.15 that $J$ is a direct summand of $R$ iff $J = J_\mathcal{C}$ where $\mathcal{C}$ is an open and closed subset of $\min R$. An equivalent formulation of this fact is that a subset $\mathcal{C}$ of $\min R$ is good : is open and closed in $\min R$.

(2) In order to illustrate how close a PIF is to being an integral domain, we make the following definitions. Let $R$ be a PIF, $\min R = \{P_\alpha\}$, and $C = \prod R_{P_\alpha}$. If $A$ is an $R$-module, we say that $A$ is a torsion $R$-module if $\text{Hom}_R(A, C) = 0$; and we say that $A$ is torsion-free if $A$ has no non-zero torsion submodules. We define $t(A)$ to be the sum of all of the torsion submodules of $A$. Then $t(A)$ is the unique largest torsion submodule of $A$, and $A/t(A)$ is torsion-free.

Since $C$ is an injective $R$-module, submodules as well as factor modules of torsion modules are again torsion modules. Thus $t(A)$ has the usual properties of a torsion-functor. It is easy to verify that $t(A) = \bigcap \text{Ker } f, f \in \text{Hom}_R(A, C)$; and that $A$ is torsion-free iff $A$ can be embedded in a direct product of copies of $C$.

If $M$ is a maximal ideal of $R$, and $P$ is the unique minimal prime ideal of $R$ contained in $M$, then by Proposition 2.1, $R_M$ is an integral domain and $R_P$ is the quotient field of $R_M$. It follows readily from this that $A$ is a torsion $R$-module iff $A_M$ is a torsion $R_M$-module $\forall$ maximal ideals of $R$. As a consequence it can easily be shown that if $A$ is a flat $R$-module, then $A$ is a torsion-free $R$-module.

3. The Injective Envelope of a Reduced Ring

Let $R$ be a reduced ring and $\{P_\alpha\} = \min R$. By Proposition 1.1, $E(R)$ is a direct summand of $\prod R_{P_\alpha}, P_\alpha \in \min R$; and by Proposition 1.12, $E(R)$ is
a self-injective VNR. In this section we shall describe some of the structure of $E(R)$, and also exactly how it sits in $\Pi R_{P_a}$.

**Proposition 3.1.** Let $R$ be a reduced ring and $\{P_a\} = \min R$. Then $E(R)$ is a subdirect product of the $R_{P_a}$'s.

**Proof.** Let $E = E(R)$. By Proposition 1.1 we can identify $E$ with its image in $\Pi R_{P_a}$. Let $e_a$ be the identity element of $R_{P_a}$ so that $1 = (e_a)$. Let $P_\beta \in \min R$; then the projection of $\Pi R_{P_a}$ onto $R_{P_\beta}$ induces an $R$-homomorphism $f_\beta: E \to R_{P_\beta}$ such that $f(1) = e_\beta$. We wish to show that $f_\beta$ is onto.

Let $a \in R - P_\beta$; since $E$ is a VNR, $Ea$ is a direct summand of $E$. Hence there is an ideal $F$ of $E$ with $E = Ea \oplus F$. Thus $F = \text{Ann}_{E_a}a$. Now $a$ is a unit in $R_{P_\beta}$; and hence if $x \in F$, then $0 = f_\beta(ax) = af_\beta(x)$ shows that $f_\beta(x) = 0$. Thus $F \subseteq \text{Ker} f_\beta$, and so $f_\beta(E) = f_\beta(Ea) = af_\beta(E)$. Therefore, $f_\beta(E)$ is an $R/P_\beta$-divisible submodule of $R_{P_\beta}$. Since $R_{P_\beta}$ is the quotient field of $R/P_\beta$ it follows that $f_\beta(E) = R_{P_\beta}$.

This shows that $E$ is a subdirect product of the $R_{P_a}$'s.

**Definition.** Let $R$ be a commutative ring with $1$, $\{P_a\}, \alpha \in \mathcal{A}$, a collection of distinct prime ideals of $R$, and let $A = \Pi_a R_{P_a}$. With componentwise addition and multiplication, $A$ is a commutative ring with identity and we have a canonical ring homomorphism $R \to A$. We identify $\text{Hom}_R(A, A)$ with $A$ in the usual way via left multiplication by elements of $A$. For each $\alpha \in \mathcal{A}$, let

$$O_\alpha = \{r \in R \mid sr = 0 \text{ for some } s \in R - P_\alpha\},$$

and let $J_\alpha = \bigcap O_\beta$, $\beta \neq \alpha$. If $I$ and $J$ are ideals of $R$, we define

$$(I : J) = \{r \in R \mid rJ \subseteq I\}.$$

**Proposition 3.2.** If $(O_\alpha : J_\alpha) = O_\alpha$, $\alpha \in \mathcal{A}$, then $\text{Hom}_R(A, A) = A$.

**Proof.** Let $B_\alpha = \Pi R_{P_a}$, $\beta \neq \alpha$. We shall first prove that $\text{Hom}_R(B_\alpha, R_{P_a}) = 0$, $\alpha \in \mathcal{A}$. Let $f \in \text{Hom}_R(B_\alpha, R_{P_a})$, $x = (x_\beta) \in B_\alpha$, and $a \in J_\alpha$. Since $R_{P_a}O_\beta = 0$ for all $\beta$, we have $ax = \langle ax_\beta \rangle = 0$. Thus $af(x) = 0$. Now $f(x) = r/s$, where $r \in R$ and $s \in R - P_\alpha$. Since $af(x) = 0$, there exists $u \in R - P_\alpha$ with $uar = 0$. Thus $ar \in O_\alpha$ for $a \in J_\alpha$, and so $r \in (O_\alpha : J_\alpha) = O_\alpha$. Therefore, $r/s = 0$, and hence $f = 0$.

Let $i_\alpha: R_{P_a} \to A$ and $\Pi_a: A \to R_{P_a}$ be the canonical inclusion and projection maps, respectively. Let $f \in \text{Hom}_R(A, A)$ and define $f_\alpha: R_{P_a} \to R_{P_a}$ by $f_\alpha = \Pi_a f i_\alpha$; then $f_\alpha$ is multiplication by $q_\alpha \in R_{P_a}$. We let $q = \langle q_\alpha \rangle \in A$, and we shall show that $f$ is multiplication by $q$.

Let $x = (x_\alpha) \in A$, where $x_\alpha \in R_{P_a}$. For each $\alpha \in \mathcal{A}$, we can write $x = i_\alpha(x_\alpha) + y_\alpha$, where $y_\alpha \in B_\alpha$. Let $h_\alpha = f_i B_\alpha$; then $\Pi_a h_\alpha \in \text{Hom}_R(B_\alpha, R_{P_a}) = 0$. Thus

$$\Pi_a f(y_\alpha) = \Pi_a h_\alpha(y_\alpha) = 0.$$
Hence $\Pi_a f(x) = \Pi_a f(x) + \Pi_a f(y) = f(x) + f(y)$, $\alpha \in \mathcal{A}$. Therefore,
$$f(x) = \langle f_a(x) \rangle = \langle q_a x \rangle = q(x) = qx \text{ for all } x \in A.$$ 

Thus $\text{Hom}_R(A, A) = A$.

**PROPOSITION 3.3.** Let $R$ be a commutative ring; with the preceding notation, assume that $\cap \alpha O_\alpha = 0$, and that $R_{P_a}$ is a self-injective ring for $\alpha \in \mathcal{A}$. Then the following statements are equivalent:

1. $J_\alpha \neq 0$ and $J_\alpha \cap \text{Ann}_R J_\alpha = 0$, $\alpha \in \mathcal{A}$.
2. $\text{Ann}_R J_\alpha = O_\alpha$, $\alpha \in \mathcal{A}$.
3. $\text{Hom}_R(A, A) = A$.
4. $R \subset A$ is an essential extension and $J_\alpha \cap \text{Ann}_R J_\alpha = 0$, $\alpha \in \mathcal{A}$.
5. $E(R) = A$, $(O_\beta : O_\alpha) = O_\beta$ for $\beta \neq \alpha$, and $J_\alpha \cap \text{Ann}_R J_\alpha = 0$ for all $\alpha \in \mathcal{A}$.

**Proof.** (1) $\Rightarrow$ (2) Let $I_\alpha = \text{Ann}_R J_\alpha$. Then
$$\text{Ann}_{R_{P_a}}(R_{P_a} J_\alpha) = R_{P_a} I_\alpha.$$ 

In one direction the inclusion is obvious. On the other hand, let
$$x = r/v \in \text{Ann}_{R_{P_a}}(R_{P_a} J_\alpha)$$
where $r \in R$ and $v \in R - P_a$. If $a \in J_a$, then $(ra)/v = 0$ in $R_{P_a}$, and so there exists $u \in R - P_a$ such that $ura = 0$ in $R$. Therefore, $ra \in O_{P_a}$, and hence $rJ_a \subset O_{P_a} \cap J_a = 0$. Therefore, $r \in \text{Ann}_R J_a = I_a$, and thus $x \in R_{P_a} I_a$.

Now $R_{P_a} J_a \cap R_{P_a} J_a = R_{P_a}(J_a \cap J_a) = R_{P_a} 0 = 0$ and
$$R_{P_a} I_\alpha \oplus R_{P_a} J_\alpha = \text{Ann}_{R_{P_a}}(R_{P_a} J_\alpha) \oplus R_{P_a} J_\alpha$$
is an essential $R_{P_a}$-submodule of $R_{P_a}$. Thus, since $R_{P_a}$ is self-injective, we have
$$R_{P_a} = E(R_{P_a} I_\alpha) \oplus E(R_{P_a} J_\alpha).$$

But $R_{P_a}$ is a quasi-local ring, and hence decomposable. If $R_{P_a} J_\alpha = 0$, then $J_a \subset O_{P_a}$, and so $J_a = J_a \cap O_{P_a} = 0$, contrary to hypothesis. Therefore $R_{P_a} J_\alpha = 0$, and so $I_a \subset O_{P_a}$. Since $O_{P_a} J_\alpha = 0$, we have $O_{P_a} \subset I_a$. Thus $O_{P_a} I_\alpha = I_a$, $\alpha \in \mathcal{A}$.

(2) $\Rightarrow$ (3) Since $\cap \alpha O_\alpha = 0$, we have $(O_\alpha : J_\alpha) = \text{Ann}_R J_\alpha$, $\alpha \in \mathcal{A}$. Hence $\text{Hom}_R(A, A) = A$ by Proposition 3.2.

(3) $\Rightarrow$ (4) The kernel of the canonical map $R \rightarrow A$ is $\cap \alpha O_\alpha = 0$, and hence $R \subset A$. Since $A$ is an injective $R$-module we have $E(R) \subset A$. Thus $A = E(R) \oplus X$, where $X$ is an $R$-submodule of $A$ and $1 \in E(R)$. Let $f : A \rightarrow A$ be the $R$-homomorphism that is the identity on $E(R)$ and $O$ on $X$. By hypothesis, $f$ is multiplication by $q \in A$. Hence $q = q \cdot 1 = f(1)$
Let $\beta$ be a fixed index and let $t \in J_\beta \cap \text{Ann}_R J_\beta$. Let $y_\alpha = 0$, $\alpha \neq \beta$, let $y_\beta$ be the image of $t$ in $R_{P_\beta}$, and let $y = \langle y_\alpha \rangle$. Let $x_\alpha$ be the identity of $R_{P_\alpha}$, $\alpha \neq \beta$, let $x_\beta = 0$, and let $x = \langle x_\alpha \rangle$. Then $\text{Ann}_R x = J_\beta$, and $J_\beta \subseteq \text{Ann}_R y$. Thus there is an $R$-homomorphism $f : Rx \to Ry$ such that $f(x) = y$. Since $A$ is an injective $R$-module, $f$ extends to an $R$-homomorphism from $A$ to $A$. By hypothesis, this homomorphism is multiplication by an element $q = \langle q_\alpha \rangle \in A$. Therefore, $qx = y$ and so $0 = q_\beta x_\beta = y_\beta$. Thus $t \in O_\beta$, and since $t \in J_\beta$, we have $t \in O_\beta \cap J_\beta = 0$. Thus $J_\beta \cap \text{Ann}_R J_\beta = 0$.

(4) $\Rightarrow$ (5) Since $A$ is injective, we have $E(R) = A$. Let $\alpha, \beta$ be a fixed pair of indices, $\alpha \neq \beta$, let $s \in (O_\alpha : O_\beta)$ and suppose $s \notin O_\beta$. Let $y_\gamma = 0$, $\gamma \neq \beta$, let $y_\beta$ be the image of $s$ in $R_{P_\beta}$, and let $y = \langle y_\gamma \rangle$. Since $A$ is an essential extension of $R$, there is an $r \in R$ with $ry = a \neq 0 \in R$. We have identified $a$ with the element $\langle a_\gamma \rangle$, where $a_\gamma$ is the image of $a$ in $R_{P_\gamma}$ for all $\gamma$. Therefore $ry_\gamma = a_\gamma$, $\gamma \notin \mathfrak{A}$. Therefore, $a_\gamma = 0$, $\gamma \neq \beta$, and thus $a \in J_\beta$. Since $s \in (O_\beta : O_\alpha)$ we have $O_\alpha s \subseteq O_\beta$. Therefore, $O_\alpha y_\beta = 0$, and hence $O_\alpha y = 0$. Thus $O_\alpha a = rO_\alpha y = r \cdot 0 = 0$. Since $J_\beta \subseteq O_\alpha$, we have $J_\beta a = 0$. Therefore, $a \in J_\beta \cap \text{Ann}_R J_\beta = 0$. This contradiction shows that $(O_\beta : O_\alpha) = O_\beta$.

(5) $\Rightarrow$ (1) Suppose $J_\alpha = 0$. Let $B_\alpha = \Pi R_{P_\beta}, \beta \neq \alpha$. Since $J_\alpha$ is the kernel of the canonical map $R \to B_\alpha$, we have $R \subseteq B_\alpha$. Since $B$ is injective, we have $A = E(R) \subseteq B_\alpha$. Now $R_{P_\alpha} \subseteq A$, and hence there is an $x = \langle x_\beta \rangle \in B_\alpha$ with $\text{Ann}_R x = O_\alpha$. Now there exists $\beta \neq \alpha$ so that $x_\beta \neq 0$, and we have $O_\alpha \subseteq \text{Ann}_R x_\beta$. Since $x_\beta = t/u$ where $t \in R$ and $u \in R - P_\beta$, we have $O_\alpha t \subseteq O_\beta$. Thus by hypothesis, $t \in (O_\beta : O_\alpha) = O_\beta$. But then $x_\beta = 0$. This contradiction shows that $J_\alpha \neq 0$.

Note. It is easy to see directly that $\text{Ann}_R J_\beta = O_\beta$ implies $(O_\beta : O_\alpha) = O_\beta$. For suppose that $t \in (O_\beta : O_\alpha)$. Since $J_\beta \subseteq O_\alpha$, we have $J_\beta t \subseteq O_\beta$. Thus $J_\beta t \subseteq O_\beta \cap J_\beta = 0$. Therefore, $t \in \text{Ann}_R J_\beta = O_\beta$ by assumption.

Definition. We shall let $\text{n-min } R$ denote the set of those minimal prime ideals of $R$ that are not essential ideals of $R$.

Proposition 3.4. Let $R$ be a reduced ring.

(1) Let $P$ be a prime ideal of $R$. Then $P \in \text{n-min } R$ iff $\text{Ann}_R P \neq 0$.

(2) If $P$ is a prime ideal of $R$ and $0 \neq a \in \text{Ann}_R P$, then $P = \text{Ann}_R a$; and $a$ is an element of every prime ideal in $R$ that is not equal to $P$.

Proof. (1) Suppose that $P \in \text{n-min } R$. Then there exists $a \in R$, $a \neq 0$, $Ra \cap P = 0$. Thus $Pa = 0$, and so $\text{Ann}_R P \neq 0$. Conversely, suppose that $\text{Ann}_R P \neq 0$, and let $0 \neq a \in \text{Ann}_R P$. Because $R$ is reduced $Ra \cap P = 0$; and hence $P$ is a non-essential ideal of $R$. Suppose that $P_1$ is a prime
ideal of $R$ and that $P_1 \subseteq P$. Since $a \notin P$, and $Pa = 0 \subseteq P_1$, we have $P \subseteq P_1$. Thus $P \in \text{n-min } R$.

(2) Since $Pa = 0$, we have $P \subseteq \text{Ann}_R a$. But $a \notin P$; and $a \cdot (\text{Ann}_R a) = 0 \subseteq P$ implies $\text{Ann}_R a \subseteq P$. Let $P' \in \text{min } R$ and $P' \neq P$. Then $aP = 0 \subseteq P'$ implies $a \in P'$.

**PROPOSITION 3.5.** Let $R$ be a reduced ring; let $\{P_\alpha\}$, $\alpha \in \mathcal{A}$, be a subset of $\text{min } R$, and let $A = \prod P_\alpha$. Assume that $\cap \mathcal{P}_\alpha = 0$. Then the following statements are equivalent:

1. $\{\cap_\beta P_\beta \mid \beta \neq \alpha\} \neq 0$, $\alpha \in \mathcal{A}$.
2. $\text{Ann}_R P_\alpha \neq 0$, $\alpha \in \mathcal{A}$ (i.e., $P_\alpha \in \text{n-min } R$, $\alpha \in \mathcal{A}$).
3. $\text{Hom}_R(A, A) = A$.
4. $R \subseteq A$ is an essential extension.
5. $(E(R) = A$.

Thus $E(R) = \prod P_\alpha$ where $P_\alpha$ ranges over all elements of $\text{min } R$ if and only if min $R$ is totally disconnected.

**Proof.** Let $O_\alpha = \{r \in R \mid ur = 0 \text{ for some } u \in R - P_\alpha\}$. By Proposition 1.1, $O_\alpha = P_\alpha$ and $R_{P_\alpha}$ is a self-injective ring for all $\alpha$. Thus with the notation of Proposition 3.3, $J_\alpha = \cap_\beta P_\beta$, $\beta \neq \alpha$. Since $R$ is reduced, we have $J_\alpha \cap \text{Ann}_R J_\alpha = 0$ for all $\alpha$. It is also obvious that $(P_\beta : P_\alpha) = P_\beta$, $\beta \neq \alpha$. Thus the equivalence of (1)–(5) is a consequence of Proposition 3.3. The final statement of Proposition 3.5 follows from (1) and the fact that an element $P$ in min $R$ is an open set in min $R$ if and only if there exists an element $x$ of $R$ such that $P$ is the only prime ideal in min $R$ that does not contain $x$.

**PROPOSITION 3.6.** Let $R$ be a reduced ring and $\{P_\beta\}$, $\beta \in \mathcal{B}$, be a subset of $\text{min } R$ so that $E(R) = \prod P_\beta$ $(\beta \in \mathcal{B})$.

1. $\cap P_\beta = 0$; and the $P_\beta$’s are all distinct.
2. $\{P_\beta\} = \text{n-min } R$.

Thus the representation $E(R) = \prod P_\beta$ (if it exists) is unique.

**Proof.** (1) Since $\cap P_\beta$ annihilates $E(R)$, we have $\cap P_\beta = 0$. Suppose that $P_{\beta_1} = P_{\beta_2} = P$ for $\beta_1 \neq \beta_2$. Then there are elements $x$ and $y$ of $E(R)$ such that $\text{Ann}_R x = P = \text{Ann}_R y$ and $Rx \cap Ry = 0$. Now there are $r, t \in R$ with $0 \neq rx = a \in R$ and $0 \neq ty = b \in R$. We have $Pa = 0 = Pb$, and thus $a \notin P$ and $b \notin P$. However, $ab \in Rx \cap Ry = 0 \in P$. This contradiction shows that the $P_\beta$’s are distinct.

(2) If $\beta \in \mathcal{B}$, then there exists nonzero $a_\beta \in R$ with $P_\beta a_\beta = 0$, and hence $P_\beta \in \text{n-min } R$ by Proposition 3.4. Conversely, let $P \in \text{n-min } R$ and $0 \neq a \in \text{Ann}_R P$. Since $a \in \prod P_\beta$, there exists $\beta \in \mathcal{B}$ such that $\text{Ann}_R a \subseteq P_\beta$. Thus $P = P_\beta$, and so $\{P_\beta\} = \text{n-min } R$. 

Proposition 3.7. Let $R$ be a reduced ring such that $\min R$ is compact. Then the following statements are equivalent:

1. $E(R) = \Pi_{P} R_{P}$, where $P$ ranges over all elements of $\min R$.
2. Every minimal prime ideal of $R$ is non-essential.
3. $\min R$ is finite.

Proof. (1) $\Rightarrow$ (2) follows from Proposition 3.6; and (3) $\Rightarrow$ (1) is Proposition 1.6.

(2) $\Rightarrow$ (3) For each $\alpha \in \mathcal{A}$, there exists nonzero $a_{\alpha} \in \text{Ann}_{R} P_{\alpha}$. By Proposition 3.4, $a_{\alpha} \in \cap P_{\beta}$, $\beta \neq \alpha$. Thus $P_{\alpha} = D(a_{\alpha})$ is an open subset of $\min R$. Therefore, $\min R$ is finite.

Proposition 3.8. Let $R$ be a commutative ring and let $\{P_{\alpha}\} = \min R$. Then $\min R$ is finite iff $P_{\alpha} \not\subseteq \{\cup P_{\beta} \mid \beta \neq \alpha\}$ for all $\alpha$.

Proof. If $\min R$ is finite, then the assertion is an elementary and well-known fact. On the other hand, assume that $P_{\alpha} \not\subseteq \{\cup P_{\beta} \mid \beta \neq \alpha\}$ for all $\alpha$. By factoring out $\{\cap P_{\alpha} \mid \alpha \in \mathcal{A}\}$ we can assume without loss of generality that $R$ is reduced. Let $A = \prod R_{P_{\alpha}}$, then $A$ is a commutative ring and $R \subseteq A$.

Suppose that $\min R$ is not finite. Then $\Sigma_{\alpha} \oplus R_{P_{\alpha}}$ is a proper ideal of $A$, and hence is contained in a maximal ideal $\mathcal{M}$ of $A$. Then $\mathcal{M} \cap R$ contains a minimal prime ideal $P_{\gamma}$ of $R$. By hypothesis there is an $a \in P_{\gamma}$ such that $a \not\in \{\cup P_{\beta} \mid \beta \neq \gamma\}$. Let $a_{\alpha}$ be the image of $a$ in $R_{P_{\alpha}}$ for all $\alpha$; then by our identification $R \subseteq A$ we have $a = \langle a_{\alpha} \rangle$. For $\beta \neq \gamma$, $a_{\beta}$ is a unit in $R_{P_{\beta}}$ because $R_{P_{\beta}}$ is a field and $a_{\beta} \neq 0$. Let $u_{\beta} = a_{\beta}^{-1}$ for $\beta \neq \gamma$, let $u_{\gamma} = 0$, and let $u = \langle u_{\alpha} \rangle \in A$. Since $a \in P_{\gamma} \subseteq \mathcal{M}$, we have $ua \in \mathcal{M}$. But $ua$ is the element of $A$ that is the identity at every component $\beta \neq \gamma$ and is 0 at the $\gamma$-component. Since $\Sigma_{\alpha} \oplus R_{P_{\alpha}}$ is also contained in $\mathcal{M}$ we see that $1 \in \mathcal{M}$. This contradiction shows that $\min R$ is finite.

Proposition 3.9. Let $R$ be a reduced ring and $E = E(R)$. There is a 1-1 correspondence of $\text{n-min} E$ onto $\text{n-min} R$ such that if $M \in \text{n-min} E$, then $M \cap R = P \in \text{n-min} R$. In this case $E/M = E_{M} = R_{P}$ and $M$ is the only prime ideal of $E$ contracting to $P$.

Proof. Let $M \in \text{n-min} E$ and $P = M \cap R$. Since $(\text{Ann}_{E} M) \cap R \neq 0$, there is a nonzero $t \in R$ with $Mt = 0$. But then $Pt = 0$; and hence by Proposition 3.4, $P \in \text{n-min} R$. If $N$ is any prime ideal of $E$ satisfying $N \cap R = P$, then $t \not\in N$. But $Mt = 0$; and so $M \subseteq N$. Thus $M = N$, since the prime ideals of $E$ are all minimal.

On the other hand, let $P \in \text{n-min} R$. Since $R_{P} \subseteq E_{P}$, and $R_{P}$ is a field, there is a prime ideal $M$ of $E$ with $M \cap R = P$. Now there exists $t \in R$ such that $Pt = 0$. Suppose that $Mt \neq 0$. Then there exist $m \in M$ and
$r \in R$ such that $0 \neq s = rmt \in R$. But then $Ps = 0$ and $s \in P$. This contradiction proves that $Mt = 0$, and hence $M \in n$-min $E$.

Now we have a canonical injection $R/P \subset E/M$; and this is an essential extension. For let $0 \neq x \in E/M$. Since $t \notin M$, and $M$ is a maximal ideal of $E$, there is a $y \in Et$ such that $x = y + M$. Also there exists $r \in R$ with $0 \neq ry \in R$. Since $Pr = 0$, $ry \notin P$. Thus $rx = r(y + M)$ is a non-zero element of $R/P$. Therefore, $R/P \subset E/M$ is an essential extension. But $E/M$ is a field, and thus $E/M$ is the quotient field of $R/P$. Therefore, since $R_P$ is the quotient field of $R/P$, and $E_M = E/M$, we have $R_P \simeq E_M$.

**Proposition 3.10.** Let $R$ be a reduced ring and $J$ an ideal of $R$. Let $I = \text{Ann}_R J$, $K = \text{Ann}_R I$, and $E = E(R)$.

1. $E = E(J) \oplus E(I)$; and $E(J)$ and $E(I)$ are ideals of $E$.
2. $J \subset K$ is an essential extension, and so $E(J) = E(K)$.
3. $E(I) \cap R = I$ and $E(J) \cap R = K$. Thus $R/I$ and $R/K$ are reduced rings.
4. $E(J) = \text{Ann}_E I$; and thus $E(J)$ is a unique submodule of $E$.
5. $E(I) = E(R/K)$ and $E(K) = E(R/I)$.
6. $E(R/I)$ is an $R/I$-module, and as such it is the injective envelope of the ring $R/I$. A similar statement holds for $E(R/K)$.
7. $E = E(R/I) \oplus E(R/K)$ is a ring direct sum decomposition.

**Proof.** (1) Since $R$ is reduced, $J \cap I = 0$. It is easily seen that $I \oplus J$ is an essential ideal of $R$. Thus $E = E(I) \oplus E(J)$. By Proposition 1.13, $E(J)$ and $E(I)$ are ideals of $E$.

2. Since $IJ = 0$, we have $J \subset K$. Let $t$ be a non-zero element of $K$. Since $I \oplus J$ is essential in $R$, there exists $r \in R$ such that $0 \neq rt = a + b$ where $a \in I$ and $b \in J$. Then $a = rt - b \in I \cap K = 0$; hence $rt = b \in J$. Thus $J \subset K$ is an essential extension. Therefore, $E(J) = E(K)$.

3. Of course, $I \subset (E(I) \cap R)$. On the other hand, $J \cdot E(I) \subset E(J) \cap E(I) = 0$, and so $(E(I) \cap R) \subset I$. A similar argument shows that $E(K) \cap R = K$. By (2), $E(J) = E(K)$ and so $E(J) \cap R = K$.

By Proposition 1.3, $E(I)$ is an intersection of some prime ideals of $E$. Thus $I = E(I) \cap R$ is an intersection of some prime ideals of $R$. Therefore, $R/I$ is a reduced ring. Similarly, $R/K$ is a reduced ring.

4. We have $I \cdot E(J) \subset E(I) \cap E(J) = 0$, and so $E(J) \subset \text{Ann}_E I$. Because $E(I)$ is an essential extension of $I$, there is no non-zero element of $E(I)$ that is annihilated by $I$. Hence $E(J) = \text{Ann}_E I$.

5. Since $I \oplus K$ is essential in $R$, it is easy to see that $I$ is isomorphic to an essential $R$-submodule of $R/K$. Thus $E(I) \simeq E(R/K)$. Similarly $E(K) \simeq E(R/I)$.

6. Since $E(K) = E(J) = \text{Ann}_E I$; and $E(K) \simeq E(R/I)$, we see that $E(R/I)$ is annihilated by $I$. Thus $E(R/I)$ is an $R/I$-module. Clearly as such it is injective and essential over $R/I$. By symmetry we have a similar statement for $E(R/K)$.
(7) There is a canonical monomorphism of rings: \( R \to R/I \oplus R/K \); and as \( R \)-modules it is not difficult to verify that this is an essential extension. Hence we have an induced \( R \)-module isomorphism \( \theta : E \cong E(R/I) \oplus E(R/K) \). By (6), \( E(R/I) \oplus E(R/K) \) has a ring structure that is compatible with that of \( R \). By the remarks following Proposition 1.12, \( \theta \) is a ring isomorphism.

**Proposition 3.11.** Let \( R \) be a reduced ring and let \( \{ P_{\gamma} \mid \gamma \in \Gamma \} \) be a non-empty subset of \( n\text{-}\text{min} \; R \). Let \( I = \bigcap_{\gamma} P_{\gamma} \), and let \( J \) be the intersection of the minimal primes of \( R \) that are not in \( \{ P_{\gamma} \mid \gamma \in \Gamma \} \). Let \( \overline{R} = R/I \), \( \overline{P}_\gamma = P_{\gamma}/I \), and \( \overline{E} \) be the injective envelope of \( R \) over \( \overline{R} \).

1. \( \overline{R} \) is a reduced ring; \( \{ \overline{P}_\gamma \mid \gamma \in \Gamma \} = n\text{-}\text{min} \; \overline{R} \); and \( \overline{E} = \prod_{\gamma} \overline{P}_\gamma \).
2. \( I = \text{Ann}_R J \); and \( \overline{E}_P = R_{P_\gamma} \); moreover, \( \overline{E} = E(R/I) \cong \prod P_{\gamma} \) is a direct summand of \( E(R) \).

**Proof.** (1) We have \( \bigcap_{\gamma} \overline{P}_\gamma = 0 \), and so \( \overline{R} \) is a reduced ring. Now there exists \( a_\gamma \notin P_{\gamma} \) such that \( P_{\gamma}a_\gamma = 0 \). But then \( a_\gamma \neq 0 \) and \( \overline{P}_\gamma a_\gamma = 0 \) shows that \( \overline{P}_\gamma \in n\text{-}\text{min} \; \overline{R} \). By Proposition 3.5 we have \( \overline{E} = \prod_{\gamma} \overline{P}_\gamma \). Hence by Proposition 3.6, \( \{ P_{\gamma} \mid \gamma \in \Gamma \} = n\text{-}\text{min} \; \overline{R} \).

(2) \( I \cap J \) is the intersection of all of the minimal prime ideals of \( R \), and thus \( I \cap J = 0 \). Therefore, \( I \subseteq \text{Ann}_R J \). By Proposition 3.4, \( P_{\gamma} = \text{Ann}_R a_\gamma \) and \( a_\gamma \in J \). Now if \( r \in \text{Ann}_R J \), then \( ra_\gamma = 0 \), and hence \( r \in P_{\gamma} \), \( \gamma \in \Gamma \). Thus \( r \in I \). Hence \( I = \text{Ann}_R J \). Thus by Proposition 3.10, \( \overline{E} = E(R/I) \) and \( E(R/I) \) is a direct summand of \( E(R) \). The only thing remaining to be proved is that \( \overline{E}_P = R_{P_\gamma} \). But \( R_{P_\gamma} \) is the quotient field of \( R/P_{\gamma} \), and \( \overline{P}_\gamma \) is the quotient field of \( \overline{R}/\overline{P}_\gamma = R/P_{\gamma} \). Hence \( \overline{E}_P \) and \( R_{P_\gamma} \) are isomorphic \( R \)-modules.

**Definition.** Let \( R \) be a reduced ring; and \( \{ P_{\beta} \mid \beta \in \mathbb{B} \} = n\text{-}\text{min} \; R \); and let \( \{ P_{\delta} \mid \delta \in \Delta \} \) be the set of all essential minimal prime ideals of \( R \). Let \( \mathcal{J}(R) = \cap_{\beta} P_{\beta} \) and \( \mathcal{J}(R) = \cap_{\beta} P_{\delta} \). (By convention we put the intersection of an empty set of ideals equal to \( R \).)

**Proposition 3.12.** Let \( R \) be a reduced ring; \( E = E(R) \); \( I = \mathcal{J}(R) \); \( J = \mathcal{J}(R) \); and \( K = \text{Ann}_RI \).

1. \( I = \text{Ann}_R J \). Thus all of the statements of Proposition 3.10 are true in this case.
2. \( E(R/I) = \prod R_{P_{\beta}} \) (\( \beta \in \mathbb{B} \)) is a direct product of fields.
3. \( E = \prod R_{P_{\delta}} \oplus E(R/K) \).
4. If \( I \neq 0 \), then \( R/K \) is a reduced ring with no non-essential minimal prime ideals and \( E(R/K) \) is the \( R/K \)-injective envelope of \( R/K \) and hence is a self-injective VNR with the same property.

**Proof.** (1) and (2) follow from Proposition 3.11, and (3) follows from Proposition 3.10.
(4) Assume $I \neq 0$. Then by Proposition 3.10, $R/K$ is a reduced ring and $E(R/K)$ is the $R/K$-injective envelope of $R/K$. Thus $E(R/K)$ is a self-injective VNR.

Suppose that $P$ is a prime ideal of $R$ such that $P \supseteq K$ and $P/K \in \text{n-min}(R/K)$. Then there exists $a \in R - K$ with $Pa \subseteq K$. Because $K = \text{Ann}_R I$, we have $PaI = 0$. But since $a \not\in K$, we have $aI \neq 0$. Then $P \in \text{n-min } R$. By Proposition 3.4, we have $aI \subseteq J$; and hence $aI \subseteq I \cap J = 0$. This contradiction shows that $R/K$ has no non-essential minimal prime ideals. Hence by Proposition 3.9, $E(R/K)$ has no non-essential minimal prime ideals.

Remark. Let $R$ be a reduced ring. It is clear from Proposition 3.12 that $E(R)$ is a direct product of fields iff $\mathcal{J}(R) = 0$ iff $\mathcal{J}(R)$ is an essential ideal of $R$. Thus if $R$ has only a finite number (or no) essential minimal prime ideals, then $R$ is a direct product of fields. On the other hand $E(R)$ has no direct summand that is a field iff every minimal prime ideal of $R$ is essential iff $\mathcal{J}(R) = 0$. In general, $E(R)$ is a direct sum of two rings: one of which is a direct product of fields, and the other having no direct summand that is a field. We shall see by the examples in §4 that both kinds of summands can exist.

Proposition 3.13. Let $R$ be a commutative, self-injective VNR. Let $\{A_\beta\}$, $\beta \in \mathcal{B}$, be the set of distinct simple submodules of $R$, and let $P_\beta = \text{Ann}_R A_\beta$. Let $I = \mathcal{J}(R)$, $J = \mathcal{I}(R)$, and $K = \text{Ann}_R I$. Then

1. $\{P_\beta \mid \beta \in \mathcal{B}\}$ is the set of non-essential prime ideals of $R$; and $R_{P_\beta} \simeq A_\beta$.
2. The sum of the $A_\beta$'s is direct and $\Sigma \oplus A_\beta = J$.
3. $K = E(J) = \Pi A_\beta$; and $K$ is the intersection of the essential prime ideals of $R$ that do not contain $I$. Thus $R = I \oplus K = (\cap P_\beta) \oplus \Pi A_\beta$; and if $I \neq 0$, then $I = R/K$ is a self-injective VNR with no non-essential prime ideals.

Proof. We recall that by Proposition 1.3, every prime ideal of $R$ is a minimal prime ideal of $R$.

1. Since $A_\beta$ is simple, $P_\beta$ is a maximal ideal; and by Proposition 3.4, $P_\beta$ is a non-essential prime ideal. We have $A_\beta \simeq R/P_\beta \simeq R_{P_\beta}$. On the other hand let $P$ be a non-essential prime ideal of $R$. By Proposition 3.4, there exists $a \in R$ with $P = \text{Ann}_R a$. Since $P$ is a maximal ideal of $R$, $Ra = R/P$ is a simple $R$-module. Hence $P$ is one of the $P_\beta$'s by definition.

2. Let $A_{\beta_1}$ and $A_{\beta_2}$ be two different simple submodules of $R$. Then $A_{\beta_1} = Re_1$, $e_i^2 = e_i$ for $i = 1, 2$ since $R$ is a VNR. Now $e_1e_2 \in A_{\beta_1} \cap A_{\beta_2} = 0$. And hence $e_1, e_2$ are orthogonal. It follows from this that the sum of the $A_\beta$'s is direct. By Proposition 3.4, $\Sigma \oplus A_\beta \subseteq J$. Since $R$ is a VNR, $\Sigma \oplus A_\beta$ is the intersection of the prime ideals of $R$ that contain it.
no $P_\beta$ can contain $\Sigma \bigoplus A_\beta$, and $J$ is the intersection of the essential prime ideals of $R$, we see that $\Sigma \bigoplus A_\beta = J$.

(3) By Proposition 3.12, $I = \text{Ann}_R J$; and hence by Proposition 1.7, $R = I \oplus E(J)$. Therefore, $E(J) = \text{Ann}_RI = K$. By Propositions 3.10 and 3.12,

$$E(J) = E(R/I) = \Pi R_{P_{\beta}} = \Pi A_\beta;$$

and since $I = \bigcap P_\beta$, we have $R = (\bigcap P_\beta) \oplus \Pi A_\beta$. If $I \neq 0$, then $I \simeq R/K$ is a self-injective VNR with no non-essential minimal prime ideals by Proposition 3.12. Finally, since $K$ is the intersection of the prime ideals of $R$ that contain it, and $R = I \oplus K$, we see that $K$ is the intersection of the prime ideals of $R$ that do not contain $I$, and these are necessarily essential.

PROPOSITION 3.14. Let $R$ be a reduced ring.

(1) There are 1-1 correspondences between the sets of simple submodules $\{A_\beta\}$ of $E(R)$, $n\text{-min } E(R) = \{M_\beta\}$ and $n\text{-min } R = \{P_\beta\}$, given by $P_\beta = \text{Ann}_R A_\beta = M_\beta \cap R$.

(2) $\Sigma \bigoplus A_\beta = \mathcal{H}(E(R))$; and $\Pi A_\beta = E(\mathcal{H}(R)) = E(K)$, where $K = \text{Ann}_RI$.

(3) $E(\mathcal{H}(R)) = \mathcal{H}(E(R))$.

(4) $E(J(R)) \cap R = K$ and $E(\mathcal{H}(R)) \cap R = \mathcal{H}(R)$.

Proof. (1) follows from Propositions 3.9 and 3.13.

(2) By Proposition 3.13 we have $\Sigma \bigoplus A_\beta = \mathcal{H}(E(R))$; and by Proposition 3.12 we have $E(\mathcal{H}(R)) = E(K) = E(R/I) = \Pi R_{P_{\beta}} = \Pi A_\beta$.

(3) Now $\mathcal{H}(E(R)) = \bigcap M_\beta$; and by Proposition 3.12,

$$\mathcal{H}(R) = \bigcap P_\beta = \bigcap M_\beta \cap R = \mathcal{H}(E(R)) \cap R.$$

Thus $\mathcal{H}(E(R))$ is an essential extension of $\mathcal{H}(R)$. By Proposition 3.13, $\mathcal{H}(E(R))$ is a direct summand of $E(R)$ and hence $R$-injective. Thus we have $E(\mathcal{H}(R)) = \mathcal{H}(E(R))$.

(4) follows from Proposition 3.12.

DEFINITION. Let $R$ be a reduced ring and let $P \in \text{min } R$. We shall say that $P$ is irrelevant if $P$ is an essential ideal of $R$ and $P \supset \mathcal{H}(R)$. Otherwise an essential minimal prime will be called relevant.

PROPOSITION 3.15. Let $R$ be a reduced ring such that $\text{min } R$ is compact. Then $R$ has an irrelevant minimal prime ideal iff $n\text{-min } R$ is infinite.

Proof. If $n\text{-min } R$ is finite, then $\mathcal{H}(R)$ is the intersection of finitely many non-essential minimal primes, and hence these are the only minimal primes that can contain $\mathcal{H}(R)$. Conversely, suppose that $\{P_{\beta}\} = n\text{-min } R$ is infinite. Let $A_\beta$ be the simple submodule of $E(R)$ corresponding to $P_{\beta}$. Now $\Pi A_\beta$ is the intersection of the relevant essential prime ideals of $E(R)$; and $\Sigma \bigoplus$
$A_\beta$ is the intersection of all of the essential prime ideals of $E(R)$ by Proposition 3.13. Since $\Sigma \oplus A_\beta \neq \Pi A_\beta$, $E(R)$ has an irrelevant prime ideal $N$. Thus

$$N \cap R \supset \mathcal{J}(E(R)) \cap R = E(\mathcal{J}(R)) \cap R = \mathcal{J}(R)$$

by Proposition 3.12. Since $\text{min } R$ is compact, $N \cap R$ is a minimal prime ideal of $R$ by Proposition 1.6. By Proposition 3.9, $N \cap R$ is an essential prime ideal of $R$. Thus $N \cap R$ is an irrelevant prime ideal of $R$.

4. Examples

In this section we present some examples to illustrate the ideas of this paper.

*Example 1.* Let $\mathcal{A}$ be an infinite index set; for each $\alpha \in \mathcal{A}$ let $K_\alpha$ be a field, and let $K = \Pi K_\alpha (\alpha \in \mathcal{A})$. Then $K$ is a self-injective VNR. Let $e_\alpha$ be the element of $K$ that is the identity of $K_\alpha$ at the $\alpha$-coordinate and 0 elsewhere; and let 1 be the identity of $K$. For each $\alpha$ let $P_\alpha = K(1 - e_\alpha)$; then $P_\alpha$ is a non-essential prime ideal of $K$ and $\cap P_\alpha = 0$. Thus the $P_\alpha$'s are all of the non-essential prime ideals of $K$ and $K = \Pi K_\alpha$ by Propositions 3.5 and 3.6. Since $K_{e_\alpha} = K_\alpha$, this is not surprising.

Let $J = \Sigma \oplus K_{e_\alpha} = \Sigma \oplus K_\alpha$; then $J$ is the sum of all of the simple submodules of $K$, and by Proposition 3.13, $J$ is the intersection of all of the essential prime ideals of $K$. It is clear that there are elements $a$ and $b$ in $K - J$ such that $ab = 0$, and thus $J$ is not a prime ideal of $K$. We put $R = K/J$; and then $R$ is a VNR with an infinite number of prime ideals, and they are all essential in $R$. For let $P$ be a prime ideal of $K$ containing $J$; then $P$ is an essential prime ideal of $K$. The problem is to show that $P/J$ is essential in $R$.

Suppose that there is an $e \in K - J$ such that $Pe \subset J$. Without loss of generality we can assume that $e^2 = e$. Since $R/J$ is reduced, $e \notin P$. Thus $P = K(1 - e) + J$. If $x \in K$, we define Supp $x$ to be the set of coordinates $\alpha$ in $\mathcal{A}$, where $x$ is not 0. Thus $J$ is the set of elements $x \in K$ such that Supp $x$ is finite; and $P$ is the set of elements $x \in K$ such that Supp $x \subset$ Supp$(1 - e)$ except for a finite number of coordinates.

Now Supp $e$ and Supp$(1 - e)$ are complementary subsets of $\mathcal{A}$. Supp $e$ is not finite because $e \notin J$, and Supp$(1 - e)$ is not finite because $P \neq J$. Thus we can write each of Supp $e$ and Supp$(1 - e)$ as disjoint unions of two infinite sets:

$$\text{Supp } e = A_1 \cup A_2 \quad \text{and} \quad \text{Supp}(1 - e) = B_1 \cup B_2.$$

We let $c$ be the element of $K$ such that the $\alpha$-coordinate of $c$ is the identity of $K_\alpha$ for $\alpha \in A_1 \cup B_1$ and 0 for $\alpha \in A_2 \cup B_2$; and we let $d$ be the element of $K$ such that the $\alpha$-coordinate of $d$ is the identity of $K_\alpha$ for $\alpha \in A_2 \cup B_2$ and 0 for $\alpha \in A_1 \cup B_1$. Then $c$ and $d$ are not in $P$, but $cd = 0$. This
contradiction shows that every prime ideal of $R$ is essential in $R$, and hence they are infinite in number.

*Remarks.* (1) It is an open question whether or not the ring $R$ of example (1) is self-injective.

(2) Let $R$ be any self-injective VNR that is not a finite direct sum of fields; and let $J$ be the sum of all of the simple submodules of $R$. ($J$ could be 0.) Then $R/J$ is a VNR with an infinite number of prime ideals and they are all essential in $R$. For by Proposition 3.13 the proof can easily be reduced to the case of Example 1. The question of whether or not $R/J$ is self-injective is a generalization of the open question posed by Example 1.

**Example 2.** Let $D$ be an integral domain and $N$ the natural numbers. Let $D_n = D$, $n \in N$; and let 1 be the identity of $\Pi D_n$. We put $R = \Sigma \oplus D_n + D \cdot 1$; i.e., $R$ is the set of sequences in $\Pi D_n$ that are ultimately constant. In the future we shall denote this ring by $D(\infty)$.

We let $e_n$ be the element of $R$ that is the identity of $D$ at the $n$-th coordinate, and 0 elsewhere, and we put $P_n = R(1 - e_n)$. Then $P_n$ is a non-essential prime ideal of $R$ and $\cap P_n = 0$. Hence by Propositions 3.5 and 3.6, the $P_n$'s are all of the non-essential prime ideals of $R$, and $E(R) = \Pi P_n$. Since $R/P_n = D_n = D$, it follows that $R_{P_n} = Q_n$, the quotient field of $D$, and we have $E(R) = \Pi Q_n$.

It is clear that the annihilator of an element of $R$ is generated by an idempotent element of $R$, and thus $R$ is a PIP.

Let $J = \Sigma \oplus Re_n = \Sigma \oplus D_n$. Since $R/J = D$, $J$ is a prime ideal of $R$ and is the only essential minimal prime ideal of $R$. Since $R/P = D$ for every minimal prime ideal $P$ of $R$, and since $P = O_M$ for every maximal ideal $M$ of $R$ that contains $P$, we see that $\text{w.gl.dim } R = \text{w.gl.dim } D$. Since $R$ is a PIP, it follows from Proposition 2.7 that $R$ is semi-hereditary iff $\text{w.gl.dim } D \leq 1$ (i.e., $D$ is a Prüfer domain).

Let $Q$ be the quotient field of $D$; then it is easily seen that $Q(R) = Q(\infty)$. It is of course easy to verify directly that $Q(\infty)$ is a VNR (so that $\text{w.gl.dim } Q(\infty) = 0$). Since $Q(\infty)$ has only a countable number of idempotents, it follows from [8, Corollary 2.15] that every ideal of $Q(\infty)$ is a projective $Q(\infty)$-module. Thus $Q(\infty)$ is a non-Noetherian hereditary ring (i.e., $\text{gl.dim } Q(\infty) = 1$). $Q(R)$ is not a self-injective ring, since $E(Q(\infty)) = E(R) = \Pi Q_n$.

**Example 3.** The following example of a ring $R$ was constructed by Vasconcelos [13, Example 3.2] as an example of a commutative ring of w.gl.dim 1 that is not semi-hereditary. Our chief interest lies in computing $E(R)$ and showing that $E(R) = \Pi P_n$, where $P$ ranges over all of min $R$, even though min $R$ is infinite. The example is a slight modification of Example 2, but the modification produces some interesting consequences.

Let $N$ be the natural numbers, $Z$ the integers, and $A_n = Z/2Z$, $n \in N$. 

Let \( A = \Sigma \oplus A_n \) and define addition and multiplication componentwise in \( A \). We let \( R \) be the ring obtained by adjoining the identity 1 of \( Z \) to \( A \). That is, \( R = Z \times A \), where addition is defined componentwise and multiplication is given by the formula
\[
(m,a)(m',a') = (mm',ma' + m'a + aa').
\]
It is clear that \( R \) is a reduced ring. Let \( P = (O,A) \). Then \( R/P = Z \), and hence \( P \) is a prime ideal of \( R \). Since \( \text{Ann}_R(P) = (2,0) \), \( P \) is a non-essential minimal prime ideal by Proposition 3.4. We have \( R_p = Q \), the field of rational numbers; and the only prime ideals properly containing \( P \) are of the form \( M = (mz,A) \) where \( 0 \neq m \) is a prime integer. \( M \) is a maximal ideal of \( R \), \( O_M = P \), and \( R_M = Z_{mz} \) is a discrete valuation ring.

Let \( e_n \) be the identity of \( A_n \), and let \( P_n = R(1,e_n) \). Then \( R/P_n \cong Z/2Z \) is a field, and hence \( P_n \) is a maximal ideal of \( R \). But \( \text{Ann}_R(P_n) = R(0,e_n) \), and thus \( P \) is a non-essential minimal prime ideal of \( R \). Since it is easily seen that a prime ideal of \( R \) either contains \( P \), or is equal to \( P_n \) for some \( n \in N \), we see that \( \{P_n\} \) is the full set of minimal prime ideals of \( R \), and that they are all non-essential. Thus we have \( E(R) = R_p \times \Pi \text{IRR}_p = Q \times \Pi A_n \) (where \( A_n = Z/2Z \)). Strangely enough, the multiplication in this direct product is not twisted, but is componentwise multiplication. It follows from Proposition 3.7 that \( \text{min} \ R \) is not compact. Since the localizations of \( R \) at the prime ideals of \( R \) are fields or discrete valuation rings, \( \text{w.gl.dim} \ R = 1 \); and, a fortiori, \( R \) is a PIF. But since \( \text{min} \ R \) is not compact, \( R \) is not a PIP by Proposition 2.7. We have \( Q(R) = Z_{2z} \times A \) (with twisted multiplication) and \( Q(R) \) is not a VNR.

**Example 4.** Let \( K \) be a VNR and \( R = K[X] \). Then \( R \) is a PIP. For let \( f(X) \in R \),
\[
f(X) = a_0 + a_1X + \cdots + a_nX^n \quad \text{where} \ a_i \in K;
\]
and let \( I = \text{Ann}_K(a_0, \ldots, a_n) \). Since \( K \) is a VNR, \( (a_0, \ldots, a_n) = Ke \), where \( e^2 = e \in K \). Thus \( I = K \cdot (1 - e) \). Since \( K \) is a reduced ring, it follows that if \( b^2a_i = 0 \), where \( b \in K \), then \( ba_i = 0 \). Using this fact, and an easy calculation, we obtain \( \text{Ann}_R(f(X)) = I[X] = R \cdot (1 - e) \). Thus \( \text{Ann}_R(f(X)) \) is a direct summand of \( R \), and hence \( R \cdot f(X) \) is a projective ideal of \( R \).

Let \( P \) be a prime ideal of \( R \) and \( p = P \cap K \); then \( p[X] \) is a prime ideal of \( R \) contained in \( P \), and hence all of the minimal prime ideals of \( R \) are of the form \( p[X] \). We have \( R/p[X] = (K/p)[X] \); and since \( K/p \) is a field, \( R/p[X] \) is a principal ideal domain. Thus if \( P \) is not a minimal ideal of \( R \), it is a maximal ideal of \( R \); and \( O_p = p[X] \) by Proposition 2.1. Thus we see that \( R_p \) is a discrete valuation ring, or a field \( \forall \) prime ideals \( P \) of \( R \). Hence \( \text{w.gl.dim} \ R = 1 \). Therefore, by Proposition 2.7, \( R \) is a semi-hereditary ring.

Let \( \{p_\alpha\} \) be the set of all non-essential prime ideals of \( K \), and suppose that \( \cap_\alpha p_\alpha = 0 \); then \( \{p_\alpha[X]\} \) is a set of non-essential minimal prime ideals of \( R \) and \( \cap_\alpha p_\alpha[X] = 0 \). Thus by Propositions 3.5 and 3.6, \( \{p_\alpha[X]\} \) is the
Example 5. Let $k$ be a field, and with the notation of Example 2, let $K = k(\infty)$, so that $K$ is a hereditary VNR. Let $R = K[[X]]$.

1. $R$ is reduced but is not a PIF.
2. $Q(R)$ is a VNR, and $E(R) = \Pi_n k_n((X))$, where $k_n = k, n \in N$.
3. $R$ is a flat essential ring extension of $k[[X]](\infty)$; and the latter ring is semi-hereditary.

Proof. Let $e_n$ be the element of $K$ that is the identity of $k$ at the $n$-th coordinate, and 0 elsewhere; then $e^2_n = e_n$, and $p_n = K(1 - e_n)$ is a non-essential prime ideal of $K$ by Example (2). Let $P_n = p_n[[X]] = R(1 - e_n)$; then it is readily verified that $P_n$ is a non-essential prime ideal of $R$. Since $\cap p_n = 0$, we have $\cap P_n = 0$; and thus $R$ is a reduced ring. Hence by Propositions 3.5 and 3.6, the $P_n$'s are all of the non-essential minimal prime ideals of $R$ and $E(R) = \Pi_n R_{P_n}$. Now $R/P_n = Re_n$, and $Re_n = k_n[[X]]$, where $k_n = k$. Therefore $R_{P_n} = k_n((X))$, and $E(R) = \Pi_n k_n((X))$.

Let $M = \Sigma_n \oplus Ke_n$; then $M$ is a maximal ideal of $K$ by example (2). It is readily verified that $M = M + RX$ is a maximal ideal of $R$. Let $O_M = \{r \in R | ur = O$ for some $u \in R - M\}$. We shall prove that $O_M = R.M$. For let $y \in R.M$; then $y = r_1a_1 + \cdots + r_na_n$, where $r_i \in R$ and $a_i \in M$. Since $K$ is a VNR, $Ka_1 + \cdots + Ka_n = Ke$, where $e^2 = e$. Then $(1 - e) \in R - M$ and $(1 - e)y = 0$. Therefore, $y \in O_M$. Conversely, let $y \in O_M \subseteq M$. Then $y = \Sigma_{i=0}^\infty a_iX^i$, where $a_0 \in M$ and $a_i \in K$ for all $i$; and there exists $u = \Sigma_{i=0}^\infty b_iX^i$ such that $b_0 \in K - M$, $b_i \in K$ and $uy = 0$. Therefore, there exist $c \in K - M$ and $d \in M$ such that $1 = cb_0 + d$. Replacing $u$ by $cu \in R - M$, we see that without loss of generality we can assume that $b_0 = 1 - d$, $d \in M$. Now $a_0b_0 = 0$, and so $a_0 = da_0$. Assume that we have proved that $a_j \in Kd, j < i$. Since $a_0b_i + a_1b_{i-1} + \cdots + a_{i-1}b_1 + a_ib_0 = 0$, we see that $a_i \in Kd$. Thus $y \in Rd \subseteq R.M$, and so $O_M = R.M$.

To prove that $R$ is not a PIF, it is sufficient by Proposition 2.1 to prove that $O_M = R.M$ is not a prime ideal of $R$. Let $a_i = e_{2i+1}$ and $b_i = e_{2(i+1)}$, $i \geq 0$; let $y = \Sigma_{i=0}^\infty a_iX^i$ and $z = \Sigma_{i=0}^\infty b_iX^i$; then $yz = 0$. If $y \in R.M$, then there is an $n_0 \in N$ such that $y_{n_0} = 0$, $n \geq n_0$. But this is not the case and so $y \notin R.M$. Similarly $z \notin R.M$. Therefore, $R.M$ is not a prime ideal of $R$.

In order to prove that $Q(R)$ is a VNR, we need to be able to identify the nonzero divisors in $R$. For this purpose we make the following definitions. If $a \in K$, we define

\[\text{Supp } a = \{n \in N | \text{ the } n\text{-th coordinate of } a \text{ is not } 0\}\]
And if \( y = \sum_{i=0}^{\infty} a_i X^i \in R \), we define \( \text{Supp } y = \bigcup_{i=0}^{\infty} \text{Supp } a_i \). Let \( z = \sum_{i=0}^{\infty} b_i X^i \in R \). Then we shall prove that \( yz = 0 \) iff \( \text{Supp } y \cap \text{Supp } z = \emptyset \), the empty set.

If \( \text{Supp } y \cap \text{Supp } z = \emptyset \), then \( \text{Supp } a_i \cap \text{Supp } b_j = \emptyset \) for all \( i, j \). Therefore \( a_i b_j = 0 \) for all \( i, j \) and so \( yz = 0 \). Conversely, suppose that \( yz = 0 \). Let \( I = \text{Ann}_K \bigcup_{i=0}^{\infty} a_i \). Since \( K \) is reduced, if \( a^2 b = 0 \) in \( K \), then \( ab = 0 \). An easy calculation using this fact shows that \( \text{Ann}_R y = I[[X]] \). Thus \( b_j \in I \) for all \( j \), and so

\[
\text{Supp } a_i \cap \text{Supp } b_j = \emptyset \quad \text{for all } i, j.
\]

Hence \( \text{Supp } y \cap \text{Supp } z = \emptyset \). It follows from this fact that \( y \) is not a zero-divisor in \( R \) iff \( y = N \). By Proposition 1.4, to prove that \( Q(R) \) is a VNR it is sufficient to prove that if \( y \in R \), then there exists \( z \in \text{Ann}_R y \) such that \( y + z \) is not a zero divisor in \( R \). But if \( y = \sum_{i=0}^{\infty} a_i X^i \), \( \text{Supp } y = A \neq N \); and \( A' \) is the complement of \( A \) in \( N \), it is not difficult to find elements \( b_i \in K \) with \( \bigcup_{i=0}^{\infty} \text{Supp } b_i = A' \). If we let \( z = \sum_{i=0}^{\infty} b_i X^i \), then \( yz = 0 \) and \( \text{Supp}(y + z) = N \). Hence \( y + z \) is not a zero divisor in \( R \), and therefore, \( Q(R) \) is a VNR.

If \( a \in K \), define \( n(a) \) to be the smallest element of \( N \) such that the coordinates of \( a \) are constant from \( n(a) \) to \( \infty \). If \( y = \sum_{i=0}^{\infty} a_i X^i \in R \), define \( n(y) = \sup n(a_i) \). Let \( B = \{ y \in R \mid n(y) < \infty \} \). Then \( B \) is a subring of \( R \) containing 1, and \( R, M \subset B \). Thus \( R \), as a \( B \)-module, is an essential extension of \( B \). We shall prove that \( B \approx k[[X]](\infty) \), and hence by Example 2, \( B \) is a semi-hereditary ring.

Let \( f_n : K \to k \) be the \( n \)-th coordinate function; and define \( \theta : B \to k[[X]](\infty) \) as follows: if \( y = \sum_{i=0}^{\infty} a_i X^i \in B \), then \( \theta(y) = \langle \sum_{i=0}^{\infty} f_n(a_i) X^i \rangle \), an element of \( \Pi_n k_n[[X]] \), where \( k_n = k, n \in N \). If \( n_0 = n(y) \), then for \( n \geq n_0 \) we have \( f_n(a_i) = f_{n_0}(a_i) \), \( i = 0, 1, \ldots, \infty \). Thus, in fact, \( \theta(y) \in k[[X]](\infty) \). It is readily verified that \( \theta \) is a ring isomorphism.

By Example (2), \( E(B) = \Pi k_n((X)) \); and we have already proved that \( E(R) = \Pi k_n((X)) \). Hence \( E(B) = E(R) \). Since \( E(B) \) is a flat \( B \)-module by Proposition 2.7, and since \( \text{w.gl.dim } B \leq 1 \) and \( R \) is a \( B \)-submodule of \( E(B) \), we see that \( R \) is a flat ring extension of \( B \).

We note that if we extend \( \theta \) with the same definition to a ring homomorphism from \( R \) into \( \Pi k_n[[X]] \), then \( \theta \) remains a monomorphism. It is not onto because the latter ring is a PIP. Thus we have

\[
k[[X]](\infty) \subset R = k(\infty)[[X]] \subset \Pi_n k_n[[X]]
\]
as essential ring extensions.

**References**


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